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Multifunctional Opial-type Integral Inequalities

K. Rauf[1], O. Anthonio[2], A. Abolarinwa[3*], E. E. Aribike[4]

Abstract

Some new estimates on integral inequalities of Opial-type for multifunctions are established through some analytic methods. In special cases, the results derived in this paper yield some recently obtained results on Opial's inequality. Moreover, more results of Opial-type inequality which extend some known results in the literature are also derived.

1. Introduction

This paper is devoted to obtaining general integral inequalities of Opial-type involving multifunctions and their derivatives from which extended and improved versions of several recent results are deduced. Opial-type inequalities and their variant extensions and generalizations have found considerable applications in Mathematical Analysis and Theory of Differential Equations. For instance, application of this class of inequalities to quantitative and qualitative properties of solutions to Ordinary, Partial and Difference equations are discussed extensively in Agarwal and Pang [1].

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[1]Department of Mathematics, University of Ilorin, Ilorin, Nigeria
[2&4]Department of Mathematics, University of Ilorin, Ilorin, Nigeria
[3]Department of Mathematics, University of Lagos, Akoka, Lagos, Nigeria
E-mails of the corresponding author: a.abolarinwa1@gmail.com
ORCID of the corresponding author: 0000-0003-3478-4854

Opial [2] proved that if a function $g \geq 0$ is continuously differentiable on a closed interval $[0,h], h>0$ with $g(0)=0=g(h)$, then

$$\int_0^h |g(t)g'(t)|dt \leq \frac{h}{4}\int_0^h (g'(t))^2 dt, \tag{1}$$

where $\frac{h}{4}$ is the best possible constant. Equality holds in (1) if and only if $g=ct$ for $0\leq t\leq \frac{h}{2}$ and $g=c(h-t)$ for $\frac{h}{2}\leq t\leq h$.
Immediately after Opial's work, many authors derived different approaches which led to interesting extensions and generalizations of Opial's result. Olech [3] observed that positivity of $g(t)$ in (1) is not necessary and it is also valid if $g(t)$ is absolutely continuous in the interval $0<t<h$ and satisfies $g(0)=0=g(h)$, with $\frac{h}{2}$ being the best possible constant. Maroni [4] applied Hölder's inequality with conjugate exponents ν and μ for continuous function $p(t)>0$ and absolutely continuous function $x(t)$ on $[\alpha,\tau]$, $x(\alpha)=0$ with $\int_\alpha^\tau p^{1-\mu}(t)dt<\infty$, where $\mu>1$, and obtained the following inequality:

$$\int_\alpha^\tau \left|x(t)x'(t)\right| dt \leq \frac{1}{2}\left(\int_\alpha^\tau p^{1-\mu}(t)dt\right)^{\frac{2}{\mu}}\left(\int_\alpha^\tau p(t)|x'(t)|^\nu dt\right)^{\frac{2}{\nu}}, \tag{2}$$

where equality is admissible if and only if $x(t)=c\int_\alpha^t p^{1-\mu}(s)ds$, $\alpha\leq t\leq\tau$. Calvert [5] adopted similar method to Olech [3] together with the use of Hölder's inequality to establish the following result: Assume, for $i=1,2$, that

(a) function $x_i(t)$ are absolutely continuous in $[a,b]$ with $x_i(a)=0$; and
(b) function $P_i(t)$ are continuos positive and $\int_a^b P_i(t)^{-2}dt<\infty$.

Then, the following inequality holds:
(3)

$$\int_a^b |x_1(t)x_2'(t)+x_1'(t)x_2(t)|dt \leq \left(\int_a^b P_1(t)^{-2}dt\right)^{\frac{1}{2}}\left(\int_a^b P_1(t)^{-2}|x_1'(t)|^2dt\right)^{\frac{1}{2}} \times\left(\int_a^b P_2(t)^{-2}dt\right)^{\frac{1}{2}}\left(\int_a^b P_2(t)^{-2}|x_2'(t)|^2dt\right)^{\frac{1}{2}}.$$

Moreover, equality in (3) holds if and only if $x_i=c\int_a^t P_i(s)^{-2}ds$ for all $i=1,2$. The authors in [6] generalized the works of [5] and [4] using modified Jensen inequality for convex function while they obtained the following result: Given absolutely continuous function $x(t),\lambda(t)$ and $f(t)$ which are also nondecreasing on $[a,b]$ for $0\leq a\leq b<\infty$ with $f(t)>0$. Let l and ζ be positive real numbers and $R(t)$ be nonnegative and measurable on $[a,b]$ such that

$$|x'(t)|\times f\left(\left|\int_0^t x'(t)R(t)d\lambda(t)\right|\right)\leq \lambda(t)^{l-\zeta}y(t)^\zeta\times R(t)^{-1}\lambda'(t)^{-1}y'(t). \tag{4}$$

It then follows that the inequality

$$\int_a^b \left|x'(t)f(t)\right| dt \le \int_b^b f(y(t))y'(t)dt \tag{5}$$

holds. For further information on Opial type inequalities, its variant generalizations, refinement, applications and alternative proofs, see [7, 8, 9, 10, 11, 12] and the references therein.

Inspired by the above mentioned references, a new class of Opial-type inequalities is derived for multifunctions through repeated applications of some known analytic tools such as Hölder's and the modified Jensen inequalities. As far as we know, this class of Opial-type inequalities obtained here is new. The required modified Jensen inequality as used in [6] is the following: Let $\psi, \varphi \in C([\alpha, \beta])$. Suppose φ is convex, ψ nonnegative and $\lambda(s)$ is nondecreasing. Then

$$\left(\int_\epsilon^t \psi(s)d\lambda(s)\right)^{\varsigma} \le \left(\int_\epsilon^t d\lambda(s)\right)^{\varsigma-\zeta} \left(\int_\epsilon^t \varphi(\psi(s))^{\frac{1}{\zeta}} d\lambda(s)\right)^{\zeta}. \tag{6}$$

The remaining part of the paper is planned as follows. Some new results involving single functions are discussed in Section 2. These results will assist in proving the results for multifunctions that will be discussed in Section 3. Some remarks are given in Section 3 to show special cases of our results.

2. Main Results I

In this section, some results involving single functions are discussed. These results are very vital to the proofs of inequalities for multifunctions that will be discussed in the next section.

We begin with this result.

Theorem 2.1. *Let $\Delta(t)$ be absolutely continuous, $\lambda(t)$ non decreasing functions on $[a, b]$ for $0 \le a \le b < \infty$ with $t > 0$ and convex function $\varphi(\lambda(t)) = \lambda(t)^{\varsigma}$. Let k, ς, ζ be real numbes such that $\varsigma \ge 0$. Suppose $P(s)$ is a nonnegative measurable function on $[a, b]$ and $\nabla(t) = \int_0^t \Delta(s)ds < \infty$ such that*

(7)

$$\Delta'(t) \times \varphi\left(\int_0^t \Delta'(s)P(s)^{-\frac{1}{k-1}}P(s)^{\frac{1}{k-1}}ds\right) \le \lambda(t)^{\varsigma-\zeta}\nabla(t)^{\zeta} \times P(t)^{-\frac{1}{k-1}}P(t)^{\frac{1}{k-1}}\nabla'(t).$$

The following inequality

$$\int_a^b \Delta'(t) \times \varphi\left(\int_0^t \Delta'(s)ds\right) \le \int_a^b \nabla(t)^{\zeta} d\nabla'(t) \tag{8}$$

then holds.

Proof. Applying the Jensen's inequality and letting $\psi(t) = \Delta'(t)R(t)$, where $R(t)$ is a nonnegative measurable function, in (6), we have

$$\frac{\varphi(\int_0^t \Delta'(s)R(s)d\lambda(s))}{\lambda(s)^{\varsigma}} \leq \frac{(\int_0^t \varphi(\Delta'(s)R(s))^{\frac{1}{\zeta}} d\lambda(s))^{\zeta}}{\lambda(s)^{\zeta}}.$$

that is

(9)
$$\varphi\left(\int_0^t \Delta'(s)R(s)d\lambda(s)\right) \leq \lambda(t)^{\varsigma-\zeta}\left(\int_0^t (\Delta'(s)R(s))^{\frac{1}{\zeta}} d\lambda(s)\right)^{\zeta} = \lambda(t)^{\varsigma-\zeta}\nabla(t)^{\zeta}.$$

But

$$\nabla(t) \leq \int_0^t \varphi(\Delta'(s)R(s))^{\frac{1}{\zeta}}\lambda'(s)$$

$$\nabla'(t)^{\zeta} \leq \Delta'(t)^{\zeta}R(t)^{\zeta}\lambda'(t)^{\zeta}, \tag{10}$$

which implies

$$\Delta'(t) \leq R(t)^{-1}\lambda'(t)^{-1}\nabla'(t). \tag{11}$$

Combining both (9) and (11) yields (7). The proof of (8) follows as:
Letting $R(t) = P(t)^{-\frac{1}{k-1}},\ \lambda'(t) = P(t)^{\frac{1}{k-1}}, k \geq 0, \zeta = \varsigma$ then (7) yields

$$\Delta'(t)\times\varphi\left(\int_0^t \Delta'(s)P(s)^{-\frac{1}{k-1}}P(s)^{\frac{1}{k-1}}ds\right) \leq \lambda(t)^{\varsigma-\zeta}\nabla(t)^{\zeta}\times P(t)^{-\frac{1}{k-1}}P(t)^{\frac{1}{k-1}}\nabla'(t).$$

Integrating both sides of the last inequality over $[a, b]$ with the respect to t implies

$$\int_a^b \Delta'(t) \times \varphi\left(\int_0^t \Delta'(s)ds\right) \leq \int_a^b \nabla(t)^{\zeta}\nabla'(t)dt, \tag{12}$$

which after a simple integration yields

$$\int_a^b \Delta'(t) \times \varphi\left(\int_0^t \Delta'(s)ds\right) \leq \frac{\nabla(t)^{\zeta+1}}{\zeta+1}.$$

By using Hölder's inequality with exponents α and β, we obtain

$$\frac{1}{\zeta+1}\nabla(b)^{\zeta+1} = \frac{1}{\zeta+1}\left(\int_a^b \Delta'(t)dt\right)^{\zeta+1} = \frac{1}{\zeta+1}\left(\int_a^b R^{-\frac{1}{\beta}}(t)R^{\frac{1}{\beta}}(t)\Delta'(t)(t)dt\right)^{\zeta+1}$$
$$\leq \frac{1}{\zeta+1}\left(\int_a^b R^{1-\alpha}(t)dt\right)^{\frac{\zeta+1}{\alpha}}\left(\int_a^b R(t)\Delta'(t)^{\beta}dt\right)^{\frac{\zeta+1}{\beta}}.$$

Therefore,

$$\int_a^b \Delta'(t)\varphi(\Delta(t))dt \leq \frac{1}{\zeta+1}\left(\int_a^b R^{1-\alpha}(t)dt\right)^{\frac{\zeta+1}{\alpha}}\left(\int_a^b R(t)\Delta'(t)^{\beta}dt\right)^{\frac{\zeta+1}{\beta}} \tag{13}$$

which gives

$$\int_a^b \Delta'(t)\Delta(t)^{\varsigma}dt \leq \frac{1}{\zeta+1}\left(\int_a^b R^{1-\alpha}(t)dt\right)^{\frac{\zeta+1}{\alpha}}\left(\int_a^b R(t)\Delta'(t)^{\beta}dt\right)^{\frac{\zeta+1}{\beta}}. \tag{14}$$

□

Remark. (1) If $\zeta = 0$, (14) yields

$$\int_a^b \Delta'(t)\Delta(t)^{\varsigma}dt \leq \left(\int_a^b R^{1-\alpha}(t)dt\right)^{\frac{1}{\alpha}}\left(\int_a^b R(t)\Delta'(t)^{\beta}dt\right)^{\frac{1}{\beta}}, \tag{15}$$

or equivalently,

$$\int_a^b \Delta'(t)\varphi(\Delta(t))dt \leq \left(\int_a^b R^{1-\alpha}(t)dt\right)^{\frac{1}{\alpha}}\left(\int_a^b R(t)\Delta'(t)^{\beta}dt\right)^{\frac{1}{\beta}}. \tag{16}$$

(2) If $\zeta = 1$ then (14) reduces to

$$\int_a^b \Delta'(t)\Delta(t)dt \leq \frac{1}{2}\left(\int_a^b R^{1-\alpha}(t)dt\right)^{\frac{2}{\alpha}}\left(\int_a^b R(t)\Delta'(t)^{\beta}dt\right)^{\frac{2}{\beta}}.$$

(3) If $\varsigma = 1 = \alpha = \zeta = R(t)$, $a = 0$ and $\beta = 2$, (14) yields,

$$\int_0^b \Delta'(t)\Delta(t)dt \leq \frac{b}{2}\int_0^b \Delta'(t)^2 dt \tag{17}$$

which is Opial-type inequality in [3]. If we set $1-\delta = \zeta$ and use convex function $\varphi(\nabla(t)) = \nabla(t)^{\delta}$, $\delta > 0$, in (12) yields

$$\begin{aligned}\int_a^b \Delta'(t)\times\varphi\left(\int_a^b \Delta'(t)dt\right)dt &\leq \int_a^b \nabla(t)^{\delta-1+1}\nabla'(t)dt\\ &= \int_a^b \nabla(t)^{\delta}\nabla'(t)dt = \int_a^b \varphi(\nabla(t))d\nabla'(t).\end{aligned} \tag{18}$$

Also, by applying Hölder's inequality with conjugate pair α and β, we have

$$\int_a^b \Delta'(t)dt = \int_a^b R^{-\frac{1}{\beta}}(t)R^{\frac{1}{\beta}}(t)\Delta'(t)dt \leq \left(\int_a^b R^{1-\alpha}(t)dt\right)^{\frac{1}{\alpha}}\left(\int_a^b R(t)^{\beta}\Delta'(t)dt\right)^{\frac{1}{\beta}}.$$

Hence,

$$\int_a^b \Delta'(t)\times\varphi\left(\int_0^t \Delta'(t)dt\right)dt \leq \varphi\left[\left(\int_a^b R^{1-\alpha}(t)dt\right)^{\frac{1}{\alpha}}\left(\int_a^b R(t)^{\beta}\Delta'(t)dt\right)^{\frac{1}{\beta}}\right], \tag{19}$$

that is

$$\int_a^b \Delta'(t)\varphi(\Delta(t))dt \leq \varphi\left[\left(\int_a^b R^{1-\alpha}(t)dt\right)^{\frac{1}{\alpha}}\left(\int_a^b R(t)^{\beta}\Delta'(t)dt\right)^{\frac{1}{\beta}}\right] \tag{20}$$

which is an Opial-type inequality.

Similarly, we need the following theorem to obtain a new Opial-type inequalities by modified Jensen's inequality for the case of convex functions.

Theorem 2.2. *Let $\Delta(t), \lambda(t), \varphi(u), R(t), \zeta$ and k be defined as in Theorem 2.1 such that:*

$$\Delta'(t)\varphi\left(\int_0^t \Delta'(s)R(s)d\lambda(s)\right) \leq \nabla'(t)R(t)^{-1}\lambda'(t)^{-1}\nabla(t)^{\delta-1}. \tag{21}$$

The following inequality

$$\Delta'(t)\varphi\left(\int_0^t \Delta'(s)dt\right) \leq \nabla(t)^{\delta-1}\nabla'(t) \tag{22}$$

then holds.

Proof. By using (6) and if $\zeta = \delta - 1$, (6) becomes

$$\left(\varphi\left(\frac{\int_0^t \Delta'(s)R(s)d\lambda(s)}{\int_0^t d\lambda(s)}\right)\right)^{\frac{1}{(\delta-1)}} \leq \left(\frac{\int_0^t \varphi(\Delta'(s)R(s))^{\frac{1}{\delta-1}}d\lambda(s)}{\int_0^t d\lambda(s)}\right).$$

that is

$$\varphi\left(\int_0^t \Delta'(s)R(s)d\lambda(s)\right) \leq \left(\int_0^t \varphi(\Delta'(s)R(s))^{\frac{1}{\delta-1}}d\lambda(s)\right)^{\delta-1}.$$

but

$$\nabla'(t)R(t)^{-1}\lambda'(t)^{-1} \leq \Delta'(t) \tag{23}$$

then

$$\Delta'(t)\varphi\left(\int_0^t \Delta'(s)R(s)d\lambda(s)\right) \leq \nabla'(t)R(t)^{-1}\lambda'(t)^{-1}\nabla(t)^{\delta-1}.$$

This completes the proof of (21).

However, putting $R(s) = P(s)^{-\frac{1}{k-1}}$, $\lambda'(s) = P(s)^{\frac{1}{k-1}}$ in the last inequality, we get

$$\Delta'(t) \times \varphi\left(\int_0^t \Delta'(s)P(s)^{-\frac{1}{k-1}}P(s)^{\frac{1}{k-1}}ds\right) \leq \nabla'(t)\nabla(t)^{\delta-1}P(t)^{-\frac{1}{k-1}}P(t)^{\frac{1}{k-1}}.$$

Hence,

$$\Delta'(t)\varphi\left(\int_0^t \Delta'(s)ds\right) \leq \nabla(t)^{\delta-1}\nabla'(t).$$

□

Remark. Integrating both side of the last inequality above over $[a,b]$ with respect to t, yields

$$\int_a^b \Delta'(t)\Delta(t)^{\delta-1}dt \le \int_a^b \nabla(t)^{\delta-1}\nabla'(t).$$

Then, apply Hölder's inequality with exponents indices ρ and π, we have

$$\begin{aligned}\int_a^b \nabla(t)^{\delta-1}\nabla'(t) = \frac{1}{\delta}\nabla(b)^{\delta} &\le \frac{1}{\delta}\left(\int_a^b \Delta'(t)P(t)^{-\frac{1}{\rho}}P(t)^{\frac{1}{\rho}}dt\right)^{\delta} \\ &\le \frac{1}{\delta}\left(\int_a^b P(t)^{1-\pi}dt\right)^{\frac{\delta}{\pi}}\left(\int_a^b \Delta'(t)^{\rho}P(t)dt\right)^{\frac{\delta}{\rho}}.\end{aligned} \tag{24}$$

and

$$\int_a^b \Delta'(t)\Delta(t)^{\varsigma}dt \le \frac{1}{\delta}\left(\int_a^b P(t)^{1-\pi}dt\right)^{\frac{\delta}{\pi}}\left(\int_a^b \Delta'(t)^{\rho}P(t)dt\right)^{\frac{\delta}{\rho}}.$$

That is

$$\int_a^b \Delta'(t)\varphi(\Delta(t))dt \le \frac{1}{\delta}\left(\int_a^b P(t)^{1-\pi}dt\right)^{\frac{\delta}{\pi}}\left(\int_a^b \Delta'(t)^{\rho}P(t)dt\right)^{\frac{\delta}{\rho}}$$

which also implies

$$\int_a^b \Delta'(t)\Delta(t)^{\varsigma}dt \le \frac{\chi}{\delta}\left(\int_a^b \Delta'(t)^{\rho}P(t)dt\right)^{\frac{\delta}{\rho}}, \tag{25}$$

where $\chi = \left(\int_a^b P(t)^{1-\pi}dt\right)^{\frac{\delta}{\pi}}$.

□

3. Main Results II

The modified Jensen's inequality is used to proof some new Opial-type inequalities for multifunctions in what follows.

Theorem 3.1. *Let $\Delta_1(t)$ and $\Delta_2(t)$ be absolutely continuous, $\lambda(t)$ be nondecreasing functions on $[a,b]$ for $0 \le a \le b < \infty$ with $t > 0$, and convex function $\Delta_1(t) \le \varphi\left(\int_0^t \Delta_2'(t)dt\right)$. Let $R(t)$ be nonnegative measurable on $[a,b]$ such that*

$$\int_a^b \Delta_2'(t) \times \varphi\left(\int_0^t \Delta_2'(s)ds\right) \le \int_a^b \nabla_1(s)\nabla_2'(s)ds \tag{26}$$

with $\nabla_1(a) = 0 = \nabla_2(a)$. The following inequality

$$\Delta_2'(t) \times \varphi\left(\int_0^t \Delta_2'(s)R(s)d\lambda(s)\right) \le \nabla_2'(t)^{\varsigma-\zeta}\nabla_1(t)^{\zeta} \times R(t)^{-1}\lambda'(t)^{-1} \tag{27}$$

then holds, where $\varsigma, \zeta \ge 0$ are real numbers.

Proof. Assume $\psi(s) = \Delta_2'(s)R(s)$, in Jensen inequality (6), then we have

$$\frac{\varphi(\int_0^t \Delta_2'(s)R(s)d\lambda(s))}{\lambda(t)^\varsigma} \leq \frac{(\int_0^t \varphi(\Delta_2'(s)R(s))^{\frac{1}{\zeta}} d\lambda(s))^\zeta}{\lambda(t)^\zeta}.$$

Further simplification yields

(28)
$$\varphi\left(\int_0^t \Delta_2'(s)R(s)d\lambda(t)\right) \leq \lambda(s)^{\varsigma-\zeta}\left(\int_0^t \varphi(\Delta_2'(s)R(s))^{\frac{1}{\zeta}} d\lambda(s)\right)^\zeta \leq \nabla_2'(t)^{\varsigma-\zeta}\nabla_1(t)^\zeta.$$

Suppose

$$\nabla_2'(t)^{\varsigma-\zeta} = \lambda(t)^{\varsigma-\zeta} \quad \text{and} \quad \nabla_1(t) \leq \int_0^t \varphi(\Delta_2'(s)R(s))^{\frac{1}{\zeta}} d\lambda(s),$$

hence

$$\Delta_2'(t) \leq R(t)^{-1}\lambda'(t)^{-1}\nabla_1'(t) \tag{29}$$

and then

$$\Delta_2'(t) \times \varphi\left(\int_0^t \Delta_2'(s)R(s)d\lambda(s)\right) \leq \nabla_2'(t)^{\varsigma-\zeta}\nabla_1(t)^\zeta \times \nabla_1'(t)R(t)^{-1}\lambda'(t)^{-1} \tag{30}$$

and the proof of the theorem is complete. □

Let l, k and ζ be real numbers such that $\zeta \geq 0$.

Corollary 3.2. *By setting $R(t) = P(t)^{-\frac{1}{k-1}}$, $\lambda'(t) = p(t)^{\frac{1}{k-1}}, k \geq 0, \nabla_1'(t) = 1$, $\zeta = 1$ and $\varsigma = 2$ in (30), then*

$$\Delta_2'(t) \times \varphi\left(\int_0^t \Delta_2'(s)P(s)^{-\frac{1}{k-1}}P(s)^{\frac{1}{k-1}}ds\right) \leq \nabla_2'(t)^{2-1}\nabla_1(t) \times P(t)^{-\frac{1}{k-1}}P(t)^{\frac{1}{k-1}}.$$

Proof. This implies that

$$\int_a^b \Delta_2'(t) \times \varphi\left(\int_0^t \Delta_2'(s)ds\right) ds \leq \int_a^b \nabla_1(t)\nabla_2'(t)dt$$

$$\int_a^b \Delta_2'(t)\Delta_1(t)dt \leq \int_a^b \nabla_1(t)\nabla_2'(t)dt. \tag{31}$$

Similarly, replacing $\Delta_2(t)$ with $\Delta_1(t)$ in (9) yields

$$\frac{\varphi\left(\int_0^t \Delta_1'(s)R(s)d\lambda(s)\right)}{\lambda(s)^\zeta} \leq \frac{\left(\int_0^t \varphi(\Delta_1'(s)R(s))^{\frac{1}{\zeta}} d\lambda(s)\right)^\zeta}{\lambda(t)^\zeta}. \tag{32}$$

Hence

$$\varphi\left(\int_0^t \Delta_1'(s)R(s)d\lambda(s)\right) \leq \nabla_1'(t)^{\varsigma-\zeta}\left(\int_0^t \varphi(\Delta_1'(s)R(s))^{\frac{1}{l}} d\lambda(s)\right)^\zeta = \nabla_1'(t)^{\varsigma-\zeta}\nabla_2(t)^\zeta.$$

Let

$$\nabla_1'(t)^{\varsigma-\zeta} = \lambda(t)^{\varsigma-\zeta} \quad \text{and} \quad \nabla_2(t) \leq \int_0^t \varphi(\Delta_1'(s)R(s))^{\frac{1}{\zeta}}\lambda'(s)d,$$

therefore,

$$\Delta_1'(t) \leq \nabla_2'(t)R(t)^{-1}\lambda'(t)^{-1}. \tag{33}$$

From (32), we then obtain

$$\Delta_1'(t) \times \varphi\left(\int_0^t \Delta_1'(s)R(s)d\lambda(s)\right) \leq \nabla_1'(t)^{\varsigma-\zeta}\nabla_2(t)^{\zeta} \times \nabla_2'(t)R(t)^{-1}\lambda'(t)^{-1} \tag{34}$$

and the proof is complete. □

Corollary 3.3. *By setting* $R(t) = P(t)^{-\frac{1}{k-1}}, \lambda'(t) = p(t)^{\frac{1}{k-1}}, \nabla_2'(t) = 1, \zeta = 1$ *and* $\varsigma = 2$ *in (34) yields*
(35)

$$\Delta_1'(t) \times \varphi\left(\int_0^t \Delta_1'(s)P(s)^{-\frac{1}{k-1}}P(s)^{\frac{1}{k-1}}ds\right) \leq \nabla_1'(t)^{2-1}\nabla_2(t) \times P(t)^{-\frac{1}{k-1}}P(t)^{\frac{1}{k-1}}$$

Proof. Integrating both sides of (35) over $[a, b]$ with the respect to t,

$$\int_a^b \Delta_1'(t) \times \varphi\left(\int_0^t \Delta_1'(s)ds\right)dt \leq \int_a^b \nabla_2(t)\nabla_1'(t)dt. \tag{36}$$

If $\Delta_2(t) \leq \varphi\left(\int_0^t \Delta_1'(t)dt\right)$, then the inequality in (36) becomes

$$\int_a^b \Delta_1'(t)\Delta_2(t)dt \leq \int_a^b \nabla_2(t)\nabla_1'(t)dt. \tag{37}$$

Adding both sides of (31) and (37) yields

$$\begin{aligned}\int_a^b (\Delta_2'(t)\Delta_1(t) + \Delta_1'(t)\Delta_2(t))dtb &\leq \int_a^b (\nabla(t)_1\nabla_2'(t) + \nabla_2(t)\nabla_1'(t))dt \\ &= \nabla_1(b)\nabla_2(b).\end{aligned}$$

Since $\nabla_1(a) = 0$, then

$$\nabla_1(b) = \int_a^b \Delta_1'(t)dt = \int_a^b \Delta_1'(t)P_1(t)^{-1}P_1(t) \leq \left(\int_a^b P_1(t)^{-2}dt\right)^{\frac{1}{2}}\left(\int_a^b P_1(t)^2\Delta_1'(t)^2dt\right)^{\frac{1}{2}}$$

and since $\nabla_2(a) = 0$, then

$$\nabla_2(b) = \int_a^b \Delta_2'(t)dt = \int_a^b \Delta_2'(t)P_1(t)^{-1}P_1(t)dt \leq \left(\int_a^b P_2(t)^{-2}dt\right)^{\frac{1}{2}}\left(\int_a^b P_2(t)^{-2}\Delta_2'(t)^2dt\right)^{\frac{1}{2}}.$$

Hence

(38)

$$\int_a^b (\Delta_1(t)\Delta_2'(t) + \Delta_1'(t)\Delta_2(t))dt$$

$$\leq \left(\int_a^b P_1(t)^{-2}dt\right)^{\frac{1}{2}} \left(\int_a^b P_1(t)^2\Delta_1'(t)^2dt\right)^{\frac{1}{2}} \left(\int_a^b P_2(t)^{-2}dt\right)^{\frac{1}{2}} \left(\int_a^b P_2(t)^{-2}\Delta_2'(t)^2dt\right)^{\frac{1}{2}},$$

which is the desired inequality. □

Remark. (a) In (37) with the fact that $f(t) = t^\varsigma$ it becomes,

$$\int_a^b \Delta_1'(t) \times \left(\int_0^t \Delta_1'(s)ds\right)^\varsigma \leq \int_a^b \nabla_2(t)\nabla_1'(t)dt,$$

that is,

$$\int_a^b \Delta_1'(t)\ \Delta_1(t)^\varsigma dt \leq \int_a^b \nabla_2(t)\nabla_1'(t)dt. \tag{39}$$

(b) We observe that if $\nabla_1(t) = \nabla_2(t) = \nabla(t)$ and $\nabla(a) = 0$ in (39), we obtain

$$\int_a^b \Delta'(t)\ \Delta(t)^\varsigma dt \leq \int_a^b \nabla(t)\nabla'(t)dt \leq \frac{1}{2}\nabla(b)^2 = \frac{b}{2}\int_a^b \nabla'(t)^2dt, \tag{40}$$

that is, if $\varsigma = 1$ in (40), it becomes the result in [3].

Theorem 3.4. *Let $\Delta_n(t)$ and $\Delta_{n+1}(t)$ be absolutely continuous, $\lambda(t)$be non decreasing functions on $[a,b]$ for $0 \leq a \leq b < \infty$ with $t > 0$, $i = n,\ n+1$. Let ς, k and ζ be real numbers such that $\zeta \geq 0$ and also let $R(t)$ be non negative and measurable function on $[a,b]$ such that:*

$$\Delta_{n+1}'(t)\times\varphi\left(\int_0^t \Delta_{n+1}'(s)R(s)d\lambda(s)\right) \leq \nabla_{n+1}'(t)^{\varsigma-\zeta}\nabla_n(t)^\zeta\times R(t)^{-1}\lambda'(t)^{-1}. \tag{41}$$

Then, the following inequality

$$\int_a^b \Delta_{n+1}'(t) \times \varphi\left(\int_0^t \Delta_{n+1}'(s)ds\right) \leq \int_a^b \nabla_n(t)\nabla_{n+1}'(t)dt \tag{42}$$

holds.

Proof. The proof of Theorem 3.4 could be sourced from the proof of Theorem 2.1. □

Theorem 3.5. *Assume all assumptions in Theorem 2.1 hold with*

$$\Delta_n'(t) \times \varphi\left(\int_0^t \Delta_n'(s)R(s)d\lambda(s)\right) \leq \nabla_n'(t)^{\varsigma-\zeta}\nabla_{n+1}(t)^\zeta \times R(t)^{-1}\lambda'(t)^{-1} \tag{43}$$

Then, the following inequality holds:

$$\int_a^b \Delta_n'(t) \times f\left(\int_0^t \Delta_n'(s)ds\right) \leq \int_a^b \nabla_{n+1}(t)\nabla_n'(t)dt \tag{44}$$

Proof. The proof of Theorem 3.5 is similar to the proof of Theorem 2.1. □

Corollary 3.6. *By setting* $f(t) = t^{\varsigma}, R(t) = P(t)^{-\frac{1}{k-1}}, \lambda'(t) = p(t)^{\frac{1}{k-1}}, \zeta = 1$ *and* $\varsigma = 2$ *in Theorem 2.1, then*

(45)

$$\Delta_{n+1}'(t)\times\varphi\left(\int_0^t \Delta_{n+1}'(t)P(t)^{-\frac{1}{k-1}}P(t)^{\frac{1}{k-1}}dt\right) \leq \nabla_{n+1}'(t)^{2-1}\nabla_n(t)P(t)^{-\frac{1}{k-1}}P(t)^{\frac{1}{k-1}}$$

Proof. Integrating both sides of (3.26) over $[a,b]$ with the respect to t, yields

$$\int_a^b \Delta_{n+1}'(t) \times \varphi\left(\int_0^t \Delta_{n+1}'(s)ds\right) \leq \int_a^b \nabla_n(t)\nabla_{n+1}'(t)dt. \tag{46}$$

If $\Delta_n(t) \leq \varphi\left(\int_0^t \Delta_{n+1}'(s)ds\right)$ in (46) becomes

$$\int_a^b \Delta_{n+1}'(t)\Delta_n(t)dt \leq \int_a^b \nabla_n(t)\nabla_{n+1}'(t)dt. \tag{47}$$

□

Corollary 3.7. *By setting* $R(t) = P(t)^{-\frac{1}{k-1}}, \lambda'(t) = p(t)^{\frac{1}{k-1}}, \zeta = 1$ *and* $\varsigma = 2$ *in Theorem 2.1 yields*

(48)

$$\Delta_n'(t)\times\varphi\left(\int_0^t \Delta_n'(s)P(s)^{-\frac{1}{k-1}}P(s)^{\frac{1}{k-1}}ds\right) \leq \nabla_n'(t)^{2-1}\nabla_{n+1}(t)\times P(t)^{-\frac{1}{k-1}}P(t)^{\frac{1}{k-1}}.$$

Proof. Integrating both sides of (48) over $[a,b]$ with respect to t, to get

$$\int_a^b \Delta_n'(t) \times \varphi\left(\int_0^t \Delta_n'(s)ds\right) \leq \int_a^b \nabla_{n+1}(t)\nabla_n'(t)dt. \tag{49}$$

If $\Delta_{n+1}(t) \leq \varphi\left(\int_0^t \Delta_n'(s)ds\right)$ in (49), then

$$\int_a^b \Delta_n'(t)\Delta_{n+1}(t)dt \leq \int_a^b \nabla_{n+1}(t)\nabla_n'(t)dt. \tag{50}$$

Adding both sides of inequality (47) and (50) yields

$$\begin{aligned}&\int_a^b \Delta_{n+1}'(t)\Delta_n(t)dt + \Delta_n'(t)\Delta_{n+1}(t)dt \\ &\leq \int_a^b \nabla(t)_n\nabla_{n+1}'(t)dt + \nabla_{n+1}(t)\nabla_n'(t)dt = \int_a^b \left(\nabla_n(t)\nabla_{n+1}(t)\right)' dt\end{aligned}$$

(51) $$\int_a^b \Delta_n(t)\Delta'_{n+1}(t)dt + \Delta'_n(t)\Delta_{n+1}(t)dt \leq \nabla_n(b)\nabla_{n+1}(b).$$

Since $\nabla_n(a) = 0$, then

$$\nabla_n(b) = \int_a^b \Delta'_n(t)dt = \int_a^b \Delta'_n(t)P_1(t)^{-1}P_1(t)dt \leq \left(\int_a^b P_1(t)^{-2}dt\right)^{\frac{1}{2}} \left(\int_a^b P_1(t)^2\Delta'_n(t)^2dt\right)^{\frac{1}{2}}.$$

Suppose $\nabla_{n+1}(a) = 0$, then

$$\nabla_{n+1}(b) = \int_a^b \Delta'_{n+1}(t)dt$$
$$= \int_a^b \Delta'_{n+1}(t)P_1(t)^{-2}P_1^2(t)dt \leq \left(\int_a^b P_2(t)^{-2}dt\right)^{\frac{1}{2}} \left(\int_a^b P_2(t)^{-2}\Delta'_{n+1}(t)^2dt\right)^{\frac{1}{2}}.$$

Thus

(52)

$$\int_a^b \Delta_n(t)\Delta'_{n+1}(t) + \Delta'_n(t)\Delta_{n+1}(t)dt \leq \left(\int_a^b P_1(t)^{-2}dt\right)^{\frac{1}{2}} \left(\int_a^b P_1(t)^2\Delta'_n(t)^2dt\right)^{\frac{1}{2}}$$
$$\left(\int_a^b P_2(t)^{-2}dt\right)^{\frac{1}{2}} \left(\int_a^b P_2(t)^2\Delta'_{n+1}(t)^2dt\right)^{\frac{1}{2}},$$

which can also be written as:

$$\int_a^b \left[(\Delta_1(t)\Delta_2(t))' + (\Delta_3(t)\Delta_4(t))' + \cdots + (\Delta_n(t)\Delta_{n+1}(t))' + (\Delta_{n+1}(t)\Delta_{n+2}(t))'\right] dt$$
$$\leq \int_a^b \left[(\nabla_1(t)\nabla_2(t))' + (\nabla_2(t)\nabla_3(t))' + \cdots + (\nabla_n(t)\nabla_{n+1}(t))'\right] dt,$$

that is

(53) $$\int_a^b \left(\sum_{i=1}^n \Delta_i(t)\Delta_{i+1}(t)\right)' dt \leq \int_a^b \left(\sum_{i=1}^n \nabla_i(t)\nabla_{i+1}(t)\right)' dt,$$

which indeed is a new class of Opial-type inequalities. □

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