Nigerian Journal of Mathematics and Applications
Volume **31**, (2021), 97 – 107. *Printed by Unilorin press*
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On Character Amenability of Fréchet Algebras

S. A. Ayinde[1*] and S. O. Agboola[2]

Abstract

In this paper we examine the notion of character amenability for Fréchet algebras A (:= Arens-Michael algebras A) with emphasis on the continuous point derivation on A. We show that a Fréchet algebra with a locally bounded approximate identity (*lbai*) and locally bounded approximate diagonal (*lbad*) is character amenable.

1. Introduction

The concept of amenability was introduced by [7], where he connected ammenability of Banach algebras to bounded approximate identity (*bai*) and bounded approximate diagonal (*bad*). [4] and [5] introduced generalized notion of amenability with the hope of achieving Banach algebras without *bai*. On the other hand [11] developed the theory of amenable Fréchet algebras with the introduction of *lbai* and *lbad* for ammenable Frechet algebras. [8] introduced the notion of approximate amenability of Fréchet algerbas in line with the work of [11]. In all, this shows that ammenable algebras admit (locally) *bai* and (locally) *bad*. [10], developed and introduced the concept of character amenability for Banach algebra where he emphasized that amenability is a stronger concept than character amenability. In this paper, we employ the ideas of [11] to consider the concept of character amenability of Fréchet algerbas where the derivation considered is a

Received: 21/09/2021, Accepted: 15/11/2021, Revised: 29/11/2021. * Corresponding author.
2015 *Mathematics Subject Classification.* 46H99 & 43A10.
Keywords and phrases. Fréchet algebras, Amenability, locally bounded approximate identity, locally bounded approximate diagonal
[1]Department of Basic Sciences, Babcock University, Ilishan-Remo, Ogun State, Nigeria
[2]Department of Mathematical Sciences, Nigeria Army University Biu, Borno State, Nigeria
E-mail of the corresponding author: ayindes@babcock.edu.ng
ORCID of the corresponding author: 0000-0003-4917-4709

continuous point derivation. Given the set of non-zero characters Φ_A of A and $\varphi \in \Phi \cup \{0\}$, we prove that a Fréchet algebra with a *lbad* is character amenable if the kernel of φ contains the *lbai* of A. The Main results are proved in section 2 of this paper with some examples in section 3, whereas subsections 1.1 and 1.2 are devoted to preliminaries and further preliminaries needed for the work.

1.1 Preliminaries:

Here, we give some definitions and established results that are used in the sequel. For further details see ([2], [3], [6], [12] & [13]).

A topological linear space E is a Fréchet space if it is a complete metrizable locally convex space (*lcs*) whose topology is defined by a countable system of continuous seminorms. Let E be a Fréchet space. E' and E'' denote the dual and second dual of E . i_E denotes the canonical embedding of E in E''. The image of E in E'' under i_E is denoted by $\widehat{E}$. For each $x \in E''$ there is a net (x_α) in E such that $x_\alpha \longrightarrow x$ in $(E'', \sigma(E'', E'))$. Here and hereafter, $\sigma(E'', E')$ represents the weak $*$-topology on E''.

When a vector space A is equipped with an associative multiplication $A \times A \to A$. $(a, b) \mapsto ab$, whereby the map $(a, b) \longmapsto ab$ is a continuous bilinear map, then A is called a topological algebra. If continuity is in each one of the two variables, it is said to be separately continuous. On the other hand, if continuity is in both variables, it is said to be jointly continuous.

If the underlying linear space of a topological algebra A is locally convex, then A is said to be a locally convex algebra (*lca*). A complete *lca* with jointly continuous multiplication is called a $\hat{\otimes}$ -algebra. A *lca* A is a locally multiplicatively convex algebra (*lmca*) if and only if its topology is determined by a system of continuous seminorms $p_i \forall i \in I$ satisfying sub multiplicative condition $p_i(ab) \leq p_i(a)p_i(b) \quad \forall \quad a, b \in A$. A Fréchet algebra is a *lmca* which is a Fréchet space. In this case, the topology is determined by a countable system of continuous seminorms $(p_j)_{j\in\mathbb{N}}$ satisfying the submultiplicative condition.

Suppose A be a topological algebra and E a topological vector space. E is a left (right) A-module, if there exists a bilinear map $A \times E \to E$ $(E \times A \to E)$, such that $(a, x) \mapsto ax$, $((x, a) \mapsto xa)$ are continuous for $a \in A$ and $x \in E$. If E is both a left A-module and a right A-module, then E is an A-bimodule.

If E is an A-bimodule, a linear map $D : A \to E$ so that $D(st) = D(s)t + sD(t)$ for $s, t \in A$ is a derivation from A to E. For $r \in E$, the map $\delta_r : A \to E$ defined by $\delta_r(s) = s \cdot r - r \cdot s \quad (s \in A)$ is called an inner derivation. A character φ on a Fréchet algebra A is a complex-valued homomorphism on A. The characteristic space of A is denoted by $\Phi \cup \{0\}$, where Φ_A is the set of non-zero characters of A. Let $\varphi \in \Phi \cup \{0\}$. The induced relative topology on Φ_A by the weak topological dual A' of A is the weak topology, see ([9], 139). A complex linear functional d

on A is a point derivation at φ if

$$d(st) = \varphi(s)d(t) + \varphi(t)d(s) \quad s, t \in A.$$

The space of point derivations at φ is denoted by $Z^1(A, \mathbb{C})$.
Dual space E' of a Fréchet A-bimodule E is a Fréchet A-bimodule with respect to the module operation:

$$\langle r, s \cdot \phi \rangle = \langle r \cdot s, \phi \rangle,\ \langle r, \phi \cdot s \rangle = \langle s \cdot r, \phi \rangle \quad (r \in E)$$

$s \in A$ and $\phi \in E'$. Hence, E' is referred to as the dual module of E.
With the foregoing preparations, we can now define the notion of amenable Fréchet algbera. If A be a Frechet algebra and E a Frechet A-bimodule, then A is called amenable if every continuous derivation $D : A \to E'$ is inner. ([11], Theorem. 9.8).
Given a topological algebra A, a net $(e_\alpha)_{\alpha \in I}$ in A is a right (left) approximate identity (ai) if $s = \lim_\alpha se_\alpha$ $(s = \lim_\alpha e_\alpha s)$ for $s \in A$. A net $(e_\alpha)_{\alpha \in I}$ is an ai if it is both a left and a right ai. An ai is bounded if the set $(e_\alpha)_{\alpha \in I}$ is bounded. A has a right (left) *lbai* if for each 0-neighborhood $U \subset A$ there exists $c > 0$ such that for each finite subset $F \subset A$ there exists $t \in cU$ with $s - st \in U$ $(s - ts \in U)$ for all $s \in F$. A has a *lbai* if it has both right and left *lbai*.

$A\hat{\otimes}A$ denotes the complete projective tensor product of A and A. Suppose $\mathbf{B}$ is a 0-neighbourhood base in A. Given $U \in \mathbf{B}$ let $\overline{\Gamma(U \otimes U)}$ denote the closure of the absolutely convex hull of the set

$$U \otimes U = \{s \otimes t : s, t \in U\} \subset A\hat{\otimes}A.$$

Then $\overline{\Gamma(U \otimes U)}$ is a neighbourhood base at 0 in $A\hat{\otimes}A$ (see [11], 96). The relation $r(u) = \inf \sum_i p(s_i)q(t_i)$ with the infimum taken over the finite sums $\sum_i s_i \otimes t_i$ in $A\hat{\otimes}A$ where $u = \sum_i s_i \otimes t_i$ defines r as a seminorm on $A\hat{\otimes}A$. If $u = \{s \in A : p(s) \leq 1\}$ and $v = \{t \in A : q(t) \leq 1\}$, then r is the seminorms corresponding to the convex hull $\Gamma(U \otimes V)$. Hence, $r_{\Gamma(u \otimes v)} = p_u \otimes q_v$. (see [9], 365-368).
Given a product map $\pi : A\hat{\otimes}A \to A$, a net $(m_i)_{i \in I}$ in $A\hat{\otimes}A$ is an approximate diagonal (ad) for A if $s \cdot m_i - m_i \cdot s \to 0$ and $\pi(m_i)s \to s$ for each $s \in A$. An ad is bounded if the set $(m_i)_{i \in I}$ is bounded. A has a *lbad* if for each 0-neighbourhood $U \subset A$ there exists $c > 0$ so that for each finite subset $F \subset A$ there exists $m \in c\overline{\Gamma(U \otimes U)}$ with $sm - ms \in \overline{\Gamma(U \otimes U)}$ and $\pi(m)s - s \in U$ for all $s \in F$.
Lemmas 1.1 and Proposition 1.2 show that an amenable Frechet algebra admits both the *lbai* and *lbad*. The proofs of these results can be found in [11].

Lemma 1.1. [11]. *Suppose A is an Arens-Michael algebra (Frechet algebra) and let $A = \lim_{\leftarrow} A_\lambda$. Then A is amenable if it has a lbai.*

Proof. For the proof see [11], Theorem 9.5,(i) and (viii). ∎

Proposition 1.2. *([11], Theorem 9.7) Suppose A is a Fréchet algebra and let $A = \lim_{\leftarrow} A_\lambda$. Then A is amenable, if and only if it has a lbad.*

1.2 Further Preliminaries

1.2.1 Arens Product:

Here we give a brief description of Arens multiplication on the second dual of Frechet algebras A.
Given $m : A'' \times A'' \longrightarrow A''$, we define $m(h, g) = h\Box g$ for $h, g \in A''$ and $m(h, g) = h \diamond g$ for $h, g \in A''$
$\Box$ and $\diamond$ are the two Arens operations defined below. The definition are in stages.

Definition 1.3. Let A be a lca. Let $s, t \in A$, $\phi \in A'$ and $h, g \in A''$. Then

$$(\phi \cdot s)t = \phi(st), \text{ then, } \quad \phi \cdot s \text{ is an element of } A'$$

$$(h \cdot \phi)s = h(\phi s), \text{ then, } \quad h \cdot \phi \text{ is an element of } A'$$

$$(h\Box g)\phi = h(g \cdot \phi), \text{ then, } \quad h\Box g \text{ is an element of } A''.$$

We call $h\Box g$ the first Arens multiplication. Hence, the multiplication makes A'' a lca endowed with the weak $*$ topology.

Definition 1.4. Let A be a lca. Let $s, t \in A, \phi \in A'$ and $h, g \in A''$. Then

$$(s \cdot \phi)t = \phi(ts), \text{ then, } \quad s \cdot \phi \text{ is an element of } A'$$

$$(\phi \cdot h)s = h(s \cdot \phi), \text{ then, } \quad \phi \cdot h \text{ is an element of } A'$$

$$(h \diamond g)\phi = g(\phi \cdot h), \text{ then, } \quad h \diamond g \text{ is an element of } A''.$$

We call $h \diamond g$ the second Arens multiplication, on A''.

The weak $*$ topology makes $(A'', \Box)$ a lca.

Theorem 1.5. [14] *Let A be a topological algebra, A has a right bai if, and only if, $(A'', \Box)$ the second dual of A endowed with the weak * topology has a right identity.*

1.3 Projective Tensor Product of Frechet Algebras

In this subsection, we examine the second dual of tensor product of Fréchet algebras A and B.

Definition 1.6 ([9], 369)**.** Let A, B be lca and $A \otimes B$ the respective tensor product of A, B. A lcs topology, say, τ on $A \otimes B$ is said to be a compactible topology (with the tensor product linear space structure of $A \otimes B$, we denote the lcs thus defined by $A \otimes_\tau B$ and its completion by $A\hat{\otimes}_\tau B$), if the following two conditions are satisfied:

(i) *Canonical linear map*

$$R : A \times B \longrightarrow A\hat{\otimes}_\tau B$$

is separately continuous.

(ii) *For any equicontinuous subsets* $S \subseteq A'$ *and* $T \subseteq B'$*, the set* $S \otimes T (= \{s' \otimes t' : s' \in S,\ t' \in T\})$ *is a subset of* $(A \otimes_\tau B)'$ *which is equicontinuous.*

Definition 1.7 ([9], 376)**.** Let A, B be *lca* and $A \otimes B$ the respective tensor product algebra of A, B. A *lcs* topology τ on $A \otimes B$ is said to be a compatible topology with tensor product algebra structure of $A \otimes B$ if the following conditions are satisfied:

(i) *The lcs* $A \otimes_\tau B$ *is a lca.*

(ii) *The topology* τ *is a compatible topology with the tensor product space structure of* $A \otimes B$*.*

Lemma 1.8. *([9], 375, Lemma 3.1) Let* A, B *be lca and* $A \otimes_\pi B$ *the respective projective tensor product of lcs* A, B*. Then* $A \otimes_\pi B$ *is a lca in such a way that* π *is a compatible topology with the tensor product algebra structure of* $A \otimes B$*. In particular, if the given lca* A, B *have continuous multiplication then the same is true for the lca* $A \otimes_\pi B$*.*

Lemma 1.9. *([9], 337) Let* A *and* B *be two given algebras and* p, q *sub multiplicative seminorms on* A, B*, respectively. Then, "the tensor product seminorm"* $r = p \otimes q$ *defined on (the vector space)* $A \otimes B$ *by the relation* $r(u) = \inf \sum_{i=1}^{n} p(s_i)q(t_i)$*, for which "inf" is taken over all expressions of the form* $u = \sum_{i=1}^{n} s_i \otimes t_i \in A \otimes B$*, gives a sub multiplicative seminorm on the tensor product algebra* $A \otimes B$*.*

Proposition 1.10. *([1], Proposition 3.2) Let* A *and* B *be Fréchet algebras. Then there is a continuous injection* $T : A\widehat{\otimes}B \longrightarrow A''\widehat{\otimes}B''$ *so that*

$$T(s \cdot e_i \otimes t \cdot e_j) = s\Box E \otimes t\Box F = s \otimes t,$$

where E *and* F *are units in* A'' *and* B'' *respectively.*

Lemma 1.11. *([1], Lemma 3.3) Let* A''*,* B'' *be lca and* $A'' \otimes_\pi B''$ *the respective projective tensor product of lcs* A'', B''*. Then* $(A \otimes_\pi B)''$ *the projective tensor product of second dual of* $(A \otimes_\pi B)$ *with Arens multiplication and endowed with weak * topology is a lca in such a way that* π *is a compatible topology with the tensor product algebra structure of* $A'' \otimes B''$*. In particular if the given lca* A'', B'' *have continuous multiplication, then the same is true for the lca* $(A \otimes_\pi B)''$*.*

Remark. (1.1): The second dual of a Fréchet space is a Fréchet space when given the strong topology, by ([11], Proposition 16). When the second dual is given an Arens multiplication, it becomes a Fréchet algebra. Moreover, if A and B are Frechet algebra, its tensor product $A\widehat{\otimes}B$ is also a Frechet algebra by Lemma (1.9). Hence its second dual $(A\widehat{\otimes}B)''$ is also a Frechet algebra by Lemma (1.11).

2. Main Results

Our results in this section show that a Frechet algebra A with *lbai* and *lbad* is character amenable with continuous point derivation d on A.

The following definition and notations will be convenient.

Definition 2.1. Let A be a Fréchet algebra. A is said to be left character amenable if $\forall\ \varphi \in \Phi_A \cup \{0\}$ and all Fréchet A-bimodules X and the right module action $r \cdot s = \varphi(s) \cdot r \quad (s \in A,\ r \in X)$, every continuous derivation $D : A \longrightarrow X'$ is inner analogous definition of right character amenability suffices by considering Fréchet A-bimodules X with the left module action defined by $s \cdot r = \varphi(s) \cdot r$.

If A is both left and right character amenable, then A is called character amenable.

Remark. (2.1) (i) The definition of character amenability is based on the condition that the continuous derivation from A into its dual A-bimodules is inner with modules considered in such a way that the left or right module action is defined by character of A (including the zero character).
(ii) The point derivation d is a derivation into the A-bimodule $\mathbb{C}$.

Theorem 2.2. *Let A be a Fréchet algebra with $\varphi \in \Phi_A \cup \{0\}$. Then A is character amenable if A has a lbai which belongs to* $\ker\varphi$.

Proof. Let $\{e_i\}_i \subset A$ be a *lbai*. We define a continuous inner derivation $\delta \in Z^1(A, \mathbb{C})$. *w.l.o.g*, we may assume that $\phi = w - \lim_i(\delta e_i)$. It follows that $\delta(s) = \lim_i \delta(se_i) = \lim_i s(\delta e_i) = s \cdot \phi$, where $s \in A$ and $\phi \in \mathbb{C}$. This is a left module action. Let $d \in Z^1(A, \mathbb{C})$ and defined by $d(st)\varphi(s)d(t) + \varphi(t)d(s)$ for $s, t \in A$. We need to show that $d \in Z^1(A, \mathbb{C})$ is inner and that $s \cdot \phi = \varphi(s) \cdot \phi$ where $\varphi(s) \in \Phi_A$. Then,

$$\begin{aligned} d(s) &= \lim_i d(se_i) = \lim_i [\varphi(s)(e_i) + \varphi(e_i)d(s)] \\ &= \varphi(s) \lim_i d(e_i) + \lim_i \varphi(e_i)d(s) \end{aligned}$$

Since $e_i \subset \ker\phi, \Longrightarrow \varphi(e_i) = 0$.

$$\begin{aligned} \therefore\ \ d(s) &= \varphi(s) \lim_i d(e_i) = \varphi(s) \lim_i (de_i) \\ d(s) &= \varphi(s) \cdot \phi \end{aligned}$$

Hence, $d(s)$ is inner. Therefore

$$s \cdot \phi = \varphi(s) \cdot \phi \qquad (i)$$

Similarly, we now consider the right module action. For $\delta \in Z^1(A, \mathbb{C})$. So also *w.l.o.g.* we may assume that $\phi = w - \lim_i(\delta e_i)$. It follows that $\delta(s) = \lim_i \delta(e_i s) = \lim_i(\delta e_i)s = \phi \cdot s$, where $s \in A$ and $\phi \in \mathbb{C}$.

Let $d \in Z^1(A, \mathbb{C})$ and defined by

$$d(ts) = \varphi(t)d(s) + \varphi(s)d(t)$$

where $s, t \in A$. We need to show that $d \in Z^1(A, \mathbb{C})$ is inner and that $s \cdot \phi = \varphi(s) \cdot \phi$ where $\varphi(s) \in \Phi_A$.

$$\begin{aligned} d(s) &= \lim_i d(e_i s) = \lim_i [\varphi(e_i)d(s) + \varphi(s)d(e_i)] \\ &= \lim_i \varphi(e_i)d(s) + \lim_i \varphi(s)d(e_i) \end{aligned}$$

Since $e_i \subset \ker \varphi \Longrightarrow \varphi(e_i) = 0$

$$d(s) = \lim_i \varphi(s)(de_i) = \varphi(s) \cdot \phi.$$

Therefore this shows that $d(s)$ inner.
Hence,

$$\phi \cdot s = \varphi(s) \cdot \phi \qquad (ii)$$

□

Thus, (i) and (ii) satisfied both left and right character amenability, hence, A is character amenable. □

Theorem 2.3. *Let A be a Fréchet algebra, $\varphi \in \Phi_A \cup \{0\}$ and let A have a lbad such that the lbai of A is in $\ker \varphi$. Then A is character amenable.*

Proof. Let $(m_\alpha)_\alpha$ be a *lbad* for A and $\pi : A\hat{\otimes}A \longrightarrow A$ be a product map, then $\pi(m_\alpha)$ is a *lbai* for A. Consider the natural injection $A\hat{\otimes}A \longrightarrow (A\hat{\otimes}A)''$ (from Lemma 1.11 & Remark 1.1) $A\hat{\otimes}A$ and $(A\hat{\otimes}A)''$ are Fréchet algebras) then, the net $(m_\alpha)_\alpha \in A\hat{\otimes}A$ converges to an element in $(A\otimes A)''$. m_α has a representation $m_\alpha = \sum_{n=1}^\infty s_n^\alpha \otimes t_n^\alpha$ and is a locally bounded net in $A\hat{\otimes}A$. Also $\pi\left(\sum_{n=1}^\infty s_n^\alpha \otimes t_n^\alpha\right) = \sum_{n=1}^\infty s_n^\alpha t_n^\alpha$. We define a continuous inner derivation $\delta \in Z^1(A, \mathbb{C})$ and without loss of generality we may have $\phi = w - \lim_\alpha \left(\delta\left(\sum_{n=1}^\infty s_n^\alpha t_n^\alpha\right)\right)$. Hence, $\delta(s) = \lim_\alpha \left(\delta\left(\sum_{n=1}^\infty s_n^\alpha t_n^\alpha\right) s\right) = \phi \cdot s$ for $s \in A$ and $\phi \in \mathbb{C}$.

Let $d \in Z^1(A,\mathbb{C})$. We need to show that $d \in Z^1(A,\mathbb{C})$ is an inner derivation and that $\phi \cdot s = \varphi(s) \cdot \phi$ where $\varphi(s) \in \Phi_A$. Therefore, for $s \in A$, we have

$$\begin{aligned}
\phi \cdot s &= \lim_\alpha \sum_{n=1}^{\infty} d(s_n^\alpha t_n^\alpha)s \\
&= \lim_\alpha \sum_{n=1}^{\infty} (\varphi(s_n^\alpha)d(t_n^\alpha) + \varphi(t_n^\alpha)d(s_n^\alpha))s \\
&= \lim_\alpha \sum_{n=1}^{\infty} (\varphi(s_n^\alpha)d(t_n^\alpha)s + \varphi(t_n^\alpha)d(s_n^\alpha)s) \\
&= \lim_\alpha \sum_{n=1}^{\infty} (\varphi(s_n^\alpha)d(t_n^\alpha s) + \varphi(t_n^\alpha)d(s_n^\alpha s)) \\
&= \lim_\alpha \sum_{n=1}^{\infty} (\varphi(s_n^\alpha)\varphi(t_n^\alpha)d(s) + \varphi(s_n^\alpha)\varphi(s)d(t_n^\alpha) + \varphi(t_n^\alpha)\varphi(s_n^\alpha)d(s) + \varphi(t_n^\alpha)\varphi(s)d(s_n^\alpha)) \\
&= \lim_\alpha \sum_{n=1}^{\infty} ((\varphi(s_n^\alpha)\varphi(t_n^\alpha) + \varphi(t_n^\alpha)\varphi(s_n^\alpha))d(s) + \varphi(s)(\varphi(s_n^\alpha)d(t_n^\alpha) + \varphi(t_n^\alpha)d(s_n^\alpha))) \\
&= \lim_\alpha \sum_{n=1}^{\infty} (\varphi(s_n^\alpha t_n^\alpha) + \varphi(t_n^\alpha s_n^\alpha))d(s) + \varphi(s)(\varphi(s_n^\alpha)d(t_n^\alpha) + \varphi(t_n^\alpha)d(s_n^\alpha)).
\end{aligned}$$

Since $\sum_{n=1}^{\infty} s_n^\alpha t_n^\alpha \subset \ker\varphi \Longrightarrow \varphi\left(\sum_{n=1}^{\infty}(s_n^\alpha t_n^\alpha)\right) = 0$ and $\varphi\left(\sum_{n=1}^{\infty}(t_n^\alpha s_n^\alpha)\right) = 0$. We must have $\phi \cdot s = \lim_\alpha \sum_{n=1}^{\infty}(\varphi(s)(\varphi(s_n^\alpha)d(t_n^\alpha) + \varphi(t_n^\alpha)d(s_n^\alpha)))$
$\phi \cdot s = \varphi(s) \cdot \phi,\ d \in Z^1(A,\mathbb{C})$ is inner.

Hence,

$$\phi \cdot s = \varphi(s) \cdot \phi \qquad (ii)$$

Similarly, for an inner derivation $\delta \in Z^1(A,\mathbb{C})$ and $w.l.o.g$, we may have $\phi = w - \lim_\alpha (\delta \sum_{n=1}^{\infty} s_n^\alpha t_n^\alpha)$. Hence $\delta(s) = \lim_\alpha \delta\,(s \sum_{n=1}^{\infty} s_n^\alpha t_n^\alpha) = s \cdot \phi$ for $s \in A$ and $\phi \in \mathbb{C}$.
Let $d \in Z^1(A,\mathbb{C})$ be a continuous point derivation. We need to show that $d \in Z^1(A,\mathbb{C})$ is an inner derivation and that $\phi \cdot s = \varphi(s) \cdot \phi$ where $\varphi(s) \in \Phi_A$.

Therefore, for $s \in A$, we have

$$\begin{aligned} s \cdot \phi &= \lim_{\alpha} s\left(\sum_{n=1}^{\infty} d(s_n^{\alpha} t_n^{\alpha})\right) \\ &= \lim_{\alpha} \sum_{n=1}^{\infty} s(\varphi(s_n^{\alpha}) d(t_n^{\alpha}) + \varphi(t_n^{\alpha})) d(s_n^{\alpha})) \\ &= \lim_{\alpha} \sum_{n=1}^{\infty} (s\varphi(s_n^{\alpha}) d(t_n^{\alpha}) + s\varphi(t_n^{\alpha}) d(s_n^{\alpha})) \\ &= \lim_{\alpha} \sum_{n=1}^{\infty} (\varphi(s_n^{\alpha}) d(st_n^{\alpha}) + \varphi(t_n^{\alpha}) d(ss_n^{\alpha})) \\ &= \lim_{\alpha} \sum_{n=1}^{\infty} (\varphi(s_n^{\alpha})\varphi(s) d(t_n^{\alpha}) + \varphi(s_n^{\alpha})\varphi(t_n^{\alpha}) d(s) + \varphi(t_n^{\alpha})\varphi(s) d(s_n^{\alpha}) + \varphi(s_n^{\alpha}) d(s)) \\ &= \lim_{\alpha} \sum_{n=1}^{\infty} (\varphi(s)(\varphi(s_n^{\alpha}) d(t_n^{\alpha}) + \varphi(t_n^{\alpha}) d(s_n^{\alpha})) + (\varphi(s_n^{\alpha})\varphi(t_n^{\alpha}) + \varphi(t_n^{\alpha})\varphi(s_n^{\alpha})) d(s)) \\ &= \lim_{\alpha} \sum_{n=1}^{\infty} (\varphi(s)(\varphi(s_n^{\alpha}) d(t_n^{\alpha}) + \varphi(t_n^{\alpha}) d(s_n^{\alpha})) + (\varphi(s_n^{\alpha} t_n^{\alpha}) + \varphi(t_n^{\alpha} s_n^{\alpha})) d(s)) \end{aligned}$$

Since $\sum_{n=1}^{\infty} s_n^{\alpha} t_n^{\alpha} \subset \ker \varphi \implies \varphi\left(\sum_{n=1}^{\infty}(s_n^{\alpha} t_n^{\alpha})\right) = 0$, $\varphi\left(\sum_{n=1}^{\infty}(t_n^{\alpha} s_n^{\alpha})\right) = 0$
$\therefore \quad s \cdot \phi = \lim_{\alpha} \sum_{n=1}^{\infty}(\varphi(s)(\varphi(s_n^{\alpha}) d(t_n^{\alpha}) + \varphi(t_n^{\alpha}) d(s_n^{\alpha})))$
Hence, $\phi \cdot s = \varphi(s) \cdot \phi$, $d \in Z^1(A, \mathbb{C})$ is inner.
Hence, $s \cdot \phi = \varphi(s) \cdot \phi$ *(ii)*
Combining (i) and (ii), A is character amenable. □

Remark. (2.2): Lemma 1.1, Proposition 1.2, Theorem 2.2 and 2.3 further helped to establish the fact that character amenability is weaker that amenability.

3. Examples

(1) Let S be the algebra consisting of infinitely differentiable functions whose derivatives approach zero faster than any polynomial. That is, S consists of all functions $h : \mathbb{R} \to \mathbb{C}$ for which $|s|^r |h^{(m)}(s)| \to 0$ as $|s| \to \infty$ $(r, m \in \mathbb{N})$. Its seminorm ρ_n is defined by

$$\rho_n(h) = \sup_{s \in \mathbb{R}} \left(|s|^n \sum_{s=0}^{n} \frac{|h^{(k)}(s)|}{k!} \right)$$

on S with respect to ρ_n, S is a Frechet algebra. But it has a continuous point derivation $d : h \to h'(s)$ for all $s \in \mathbb{R}$ which is non-zero with respect to the character $\varphi : h \mapsto h(s)$. While its inner derivation into the bimodule $\mathbb{C}$ are all zero. This implies that the point derivation is not equal to the inner derivation. Hence it is not character amenable.

(2) We consider an open $D \subset \mathbb{C}$ and $A = O(D)$ an algebra of analytic functions $g : D \to \mathbb{C}$. Defining a system of seminorms ρ_n, given by

$$\rho_n(g) = \sup_{|s| \leq 1 - \frac{1}{n}} |g(s)|$$

on A makes A a Frechet algebra. A is not character amenable since the continuous point derivation $d : A \to \mathbb{C}$ given by $g \mapsto g'(0)$ is non-zero whereas its inner derivation $\delta : A \to \mathbb{C}$ are all zero. (See [8]).

(3) It follows from the above results in Theorems 2.2 and 2.3 that every projective limit of sequence of C*-algebra is character amenable, since $\ker\varphi$ has a left [right] locally bounded approximate identity for every $\varphi \in \Phi \cup \{0\}$.

Acknowledgement: The authors are very grateful to Babcock University and Nigeria Army University for the supports they received during the compilation of this work and also to anonymous referees for their careful reading of the paper and helpful suggestions.

Competing interests: The manuscript was read and approved by all the authors. They therefore declare that there is no conflicts of interests regarding this paper.

Funding: The Authors received no financial support for the research, authorship, and/or publication of this article.

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