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On the Convergence and Consistence of Rational Integrator for the Solution of Ordinary Differential Equation

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Abstract

This research work is concerned with the determination of solution to different classes of problems in Ordinary Differential Equations (ODEs). We derived an A_0-Stable rational integrators for the solution of ordinary differential equations. We establish the convergence, consistency and the stability of our scheme in the interpolants of order $m = 3$, through the rational integrator. The stability analysis of the method was carried with the use of MAPLE-18 and MATLAB software's. We compared our new and solve real-life problems which ascertain the convergence and consistency of scheme. Our result shows that our integrator is stable analytically and computationally.

1. Introduction

This research work is concerned with the determination of solution to different classes of problems in Ordinary Differential Equations (ODEs). There have been three major directions in which researches have been channelled, namely: the modelling and simulation group, the abstract and classical analysis group and the computerized group, [4]. The modelling and simulation group looks at real-life problems, [18] observed the rate at which a given real-life problem is changing

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with respect to an observable single variable and then goes ahead to model and simulate an ODE based on the observations. [12], [19] compiled a collection of modelled ODE from various industries which today are used in testing the power and direction of computerized group.
The second group which is the abstract and classical analysis group, considered an ODE with specific unknown parameters and coefficients of the ODE terms, subjects the ODE to certain constraints, then obtain conditions under which the coefficients or parameters must exist in order for the given ODE to be able to have solutions or classes of solutions. A typical example is the work due to [14], where he considered the fourth order ODE in the form:

$$a_0x^{(4)} + a_1x^{(3)} + a_2x^{(2)} + a_3x^{(1)} + a_4x = p(t, x, \cdots) \tag{1}$$

He considered several approaches, one of which involves Routh-Hurwitz conditions and arrived at $a_i > 0$ $for\ all$ $1 \leq i \leq 4$.
The researchers in the computerized group concern themselves with the development, analysis, testing and implementation of methods to solve problems in ODE. This group handles effectively problems whose solutions cannot be established with ease through abstract and classical methods but to evaluate them without computational methods becomes extremely difficult. These are the main reasons why in today's world of ODE, emphasis is drifting to computerized mathematics whenever one is handling most crucial problems in this area, [2].
This research work is concern with finding a suitable numerical solution to the initial value problem:

$$y^{'} = f(x, y), \qquad y(x_0) = y_0, \quad a \leq x \leq b \tag{2}$$

where $f(x, y)$ must satisfy a Lipschitz condition with respect to y and is defined and continuous in a region D $\subset$ [a,b] that are either Stiff or Singular. Also to determine its stability from both analytical and computational method.
In the course of this work, we shall make use of some terms, which are carefully defined for more understanding.
Local Truncation Error according to Lambert (1978), at x_{n+1} of the general explicit one-step method is defined to be T_{n+1} where

$$T_{n+1} {=} y\left(x_{n+1}\right) - h\emptyset\left(x_{n,} y\left(x_n\right), h\right) \tag{3}$$

and $y(x)$ is the theoretical solution of the initial value problems.
Then the local truncation error at x_{n+k} of an explicit linear $k-$Step method satisfies $T_{n+k}{=}y\left(x_{n+k}\right) - y_{n+k}$. The relationship between global and local truncation error is $e_{n+1} \leq KT_{n+1}$. where K is a constant. The local truncation error is directly proportional to the global error introduce at each step mostly when the derivation and computation of local truncation error is rigorous on the exact solution.

Lipschitz constant according to [10], from (2) above,$f(x,y)$ is said to satisfy a Lipschitz condition in y, over the region D, if there exist a constant L such that

$$||f(x,y_1) - f(x,y_2)|| \leq L||y_1 - y_2|| \tag{4}$$

where L is called the Lipschitz constant and $f(x,y)$ is said to be Lipschitzian. By virtue of the relation. Hence,

$$\frac{\partial f(x,y)}{\partial y} = \lim_{(\alpha \to 0}[\frac{||f(x,y_1) - f(x,y_2)||}{\alpha}] \tag{5}$$

where $\alpha = ||y_1 - y_2||$. Consequently, we have

$$L = \frac{\partial f}{\partial y} \tag{6}$$

which relates to the test equation for systems of ivp (2), given by,

$$\frac{\partial f}{\partial y} = \lambda \equiv y' = \lambda y \tag{7}$$

The function $\frac{\partial f}{\partial y} = J$ where J is called the Jacobian matrix.
For treatment of stiff problems, heavy reliance is placed on the eigenvalue of this Jacobian matrix. Then the Lipschitz condition here guarantees the existence and uniqueness of solution (2) to a unique one as this convergence of solution is needed in analyzing and comparing our solutions with existing ones.
According to [16], a system is said to be stiff over the finite interval [a,b] if for every $x \in [a,b]$, the eigenvalues $\{\lambda_s$(x), s = 1,2,. . . ,m$\}$ of the Jacobian matrix:

$$J = \frac{\partial f}{\partial y} \tag{8}$$

satisfy the following conditions:

(1) Re λ_s (x) < 0, s = 1, 2,. . . ,m

where Re λ_s (x) represents the real component of the complex number λ_s(x).

(1) Stiffness Ratio = max$\frac{\text{Re } \lambda_r \text{ (x)}}{\text{Re } \lambda_s \text{ (x)}}$>> 1, r,s = 1,. . . ,m

For the linear initial value problem:
y' = Ay + g, y(x_0) = y_0

$$\frac{\partial f}{\partial y} = \frac{\partial y'}{\partial y} = A$$

is m x m matrix and $g \epsilon R^m$.
[15] analyzed the rational integrator using Taylor's series method for derivation of order and error constant of the method. Associating the linear difference operator

L defined by:
(9)

$$L\left[y\left(x\right);h\right]=\sum_{i=0}^{k}\alpha_{i}y\left(x+ih\right)-\sum_{j=1}^{k}h^{j}(\beta_{ik}y^{(j)}\left(x+ih\right)-\beta_{jk}^{*}y^{(j)}\left(x+(i+1)h\right))$$

where $y(x)$ is an arbitrary function, continuously differentiable on [a,b], expanding $y\left(x+ih\right)$ and its derivatives $y^{(j)}\left(x+ih\right)$ as Taylor's series about x, and collecting terms gives:

$$L\left[y\left(x\right);h\right]=c_{0}y\left(x\right)+c_{1}hy^{(1)}y\left(x\right)+\cdots+c_{q}h^{(q)}y^{(q)}\left(x\right)+\ldots \tag{10}$$

where c_q are constant. The class of the method is order p if:

$$L\left[y\left(x\right);h\right]=c_{p+1}h^{(p+1)}y^{(p+1)}\left(x\right)+o(h^{(p+2)});c_{p+1}\neq 0 \tag{11}$$

where $c_0=\alpha_0+\alpha_1+\alpha_2+\cdots+\alpha_k, \quad \alpha_k=1$

$$c_1=\alpha_1+2\alpha_2+\cdots+k\alpha_k-(\beta_{ik}-\beta_{jk}^{*})$$

$$c_q=\frac{1}{q}\left(\alpha_1+2^{q}\alpha_2+\cdots+k^{q}\alpha_k\right)-\sum_{j=1}^{k}\frac{(k)^{q-j}}{(q-j)}\beta_{ik}+\sum_{j=1}^{k}\frac{(k+)^{q-j}}{(q-j)}\beta_{jk}^{*}$$

[11] constructed a quartic based denominator of order six rational integrator which is defined by:

$$U\left(x\right)=\frac{\sum_{i=0}^{2}p_{1}x^{i}}{1+\sum_{i=0}^{4}q_{1}x^{i}} \tag{12}$$

where p_0, p_1, p_2 and q_0, q_1, q_2, q_3, q_4 are called the integrator parameters

$$U\left(x\right)=\sum_{r=0}^{\infty}c_{r}x^{r} \tag{13}$$

Hence equation (12) becomes

$$\sum_{r=0}^{\infty}c_{r}x^{r}\left(1+\sum_{i=0}^{4}q_{1}x^{i}\right)=\sum_{i=0}^{2}p_{1}x^{i} \tag{14}$$

Which leads to the integrator as

$$y_{n+1}=\frac{\sum_{r=0}^{2}\frac{h^{r}y_{n}^{(r)}}{r!}+A\sum_{r=0}^{1}\frac{h^{r}y_{n}^{(r)}}{r!}+By_{n}}{1+A+B+C+D}$$

[7] considered a case of the stability of order 7 rational interpolation scheme for solving initial value problems in ordinary differential equations. The general rational integrator formula for k = 4, we have

$$U\left(x\right)=\frac{\sum_{i=0}^{3}p_{1}x^{i}}{1+\sum_{i=0}^{4}q_{1}x^{i}} \tag{15}$$

$$\sum_{r=0}^{\infty} c_r x^r \left(1+\sum_{i=0}^{4} q_1 x^i\right) = \sum_{i=0}^{3} p_1 x^i \tag{16}$$

With the stability function

$$\frac{y_{n+1}}{y_n} = \frac{\left((1+A+B+C)+h\,(1+A+B)+\frac{\overline{h}^2}{2}(1+A)\frac{\overline{h}}{6}\right)}{1+A+B+C+D} \tag{17}$$

2. Derivation of the Method

The rational interpolant U : R → R is defined by

$$y_{n+1} = r + \frac{\sum_{i=0}^{m-1} p_i x_{n+1}^i}{1+\sum_{i=1}^{m} q_i x_{n+1}^i} \tag{18}$$

where p_i and q_i are called the integrator parameters, and r is a constant parameter.
From (18) when $m = 3$, we have:

$$y_{n+1} = r + \frac{\sum_{i=0}^{2} p_i x_{n+1}^i}{1+\sum_{i=1}^{3} q_i x_{n+1}^i} \tag{19}$$

$$\begin{aligned} y_{n+1} &= r + \left[p_0 + p_1 x + p_2 x^2\right]\left[1+q_1 x + q_2 x^2 + q_3 x^3\right]^{-1} \\ &= r + \left[p_0 + p_1 x + p_2 x^2\right]\{\sum_{i=0}^{\infty}(-1)^i [q_1 x + q_2 x^2 + q_3 x^3]^i\} \end{aligned}$$

By the use of binomial expansion for rational functions and Maple 18 software complier we obtain the matrix form:

$$\begin{bmatrix} \frac{y_n^{(5)}}{5!(n+1)^5} & \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} \\ \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} & \frac{y_n^{(1)}}{(n+1)} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = - \begin{bmatrix} \frac{y_n^{(6)}}{6!(n+1)^6} \\ \frac{y_n^{(5)}}{5!(n+1)^5} \\ \frac{y_n^{(4)}}{4!(n+1)^4} \end{bmatrix} \tag{20}$$

Applying the Crammer's rule to determine q_1, q_2, and q_3 from (19) where $q_i = \frac{x_i}{|A|}$, for all $i = 1, 2, 3$. Let

$$A = \begin{bmatrix} \frac{y_n^{(5)}}{5!(n+1)^5} & \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} \\ \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} & \frac{y_n^{(1)}}{(n+1)} \end{bmatrix}, \tag{21}$$

then

$$|A| = (n+1)\, 18{y_n}^{(2)^2} y_n^{(5)} - 60y_n^{(2)} y_n^{(4)} y_n^{(3)} - 12y_n^{(1)} y_n^{(5)} y_n^{(3)} + 15{y_n^{(4)}}^2 y_n^{(1)} + 40{y_n^{(3)}}^3$$

If x_1, x_2 and x_3 are respectively:

$$x_1 = \begin{bmatrix} -\frac{y_n^{(6)}}{6!(n+1)^6} & \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} \\ -\frac{y_n^{(5)}}{5!(n+1)^5} & \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ -\frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(2)}}{2!(n+1)^2} & \frac{y_n^{(1)}}{(n+1)} \end{bmatrix}, \tag{22}$$

$$x_2 = \begin{bmatrix} \frac{y_n^{(5)}}{5!(n+1)^5} & -\frac{y_n^{(6)}}{6!(n+1)^6} & \frac{y_n^{(3)}}{3!(n+1)^3} \\ \frac{y_n^{(4)}}{4!(n+1)^4} & -\frac{y_n^{(5)}}{5!(n+1)^5} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ \frac{y_n^{(3)}}{3!(n+1)^3} & -\frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(1)}}{(n+1)} \end{bmatrix} \tag{23}$$

and

$$x_3 = \begin{bmatrix} \frac{y_n^{(5)}}{5!(n+1)^5} & \frac{y_n^{(4)}}{4!(n+1)^4} & -\frac{y_n^{(6)}}{6!(n+1)^6} \\ \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} & -\frac{y_n^{(5)}}{5!(n+1)^5} \\ \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} & -\frac{y_n^{(4)}}{4!(n+1)^4} \end{bmatrix}, \tag{24}$$

then

$$|x_1| = 6{y_n}^{(2)^2} - 12y_n^{(1)} y_n^{(5)} y_n^{(3)} - 15{y_n^{(4)}}^2 y_n^{(2)} + 6y_n^{(1)} y_n^{(5)} y_n^{(4)} - 4y_n^{(1)} y_n^{(6)} y_n^{(3)} + 120{y_n^{(3)}}^2 y_n^{(4)}$$

$$|x_2| = 15y_n^{(2)} y_n^{(5)} y_n^{(4)} - 10y_n^{(2)} y_n^{(6)} y_n^{(3)} - 16{y_n^{(5)}}^2 y_n^{(1)} + 5y_n^{(1)} y_n^{(6)} y_n^{(4)} + 20{y_n^{(3)}}^2 y_n^{(5)} - 25{y_n^{(4)}}^2 y_n^{(3)}$$

and

$$|x_3| = 36{y_n}^{(5)^2} y_n^{(2)} - 30y_n^{(2)} y_n^{(4)} y_n^{(6)} - 120y_n^{(3)} y_n^{(5)} y_n^{(4)} + 75{y_n}^{(4)^3} + 40{y_n^{(3)}}^2 y_n^{(6)}$$

At the integration point $x = x_{n+1}$.

$$U(x_{n+1}) = \sum_{i=0}^{\infty} c_r x_{n+1}^r. \tag{25}$$

Since

$$U(x_{n+1}) = y_{n+1}, \tag{26}$$

then $y_{n+1} = \sum_{r=0}^{\infty} c_r$. By the Taylor's Series expansion of y_{n+1} we have:

(27)
$$y_{n+1} = \sum_{r=0}^{\infty} \frac{h^r y_n^{(r)}}{r!}$$

(28)
$$\sum_{r=0}^{\infty} c_r x_{n+1}^r = \sum_{r=0}^{\infty} \frac{h^r y_n^{(r)}}{r!}.$$

Which gives

(29)
$$c_r x_{n+1}^r = \frac{h^r y_n^{(r)}}{r!} \ \forall \ r = 0, 1, 2 \ldots$$

With these definitions for $c_r s$, the determinant |A| can be written as follows:
(30)
$$|A| = \frac{h^9}{x_{n+1}^9}\left[(n+1)\, 18 {y_n}^{(2)^2} y_n^{(5)} - 60 y_n^{(2)} y_n^{(4)} y_n^{(3)} - 12 y_n^{(1)} y_n^{(5)} y_n^{(3)} + 15 y^{(4)^2} y_n^{(1)} + 40 y_n^{(3)^2}\right]$$

$$|x_1| = \frac{h^{10}}{-2x_{n+1}^{10}}\left[6 {y_n}^{(2)^2} - 12 y_n^{(1)} y_n^{(5)} y_n^{(3)} - 15 y_n^{(4)^2} y_n^{(2)} + 6 y_n^{(1)} y_n^{(5)} y_n^{(4)} - 4 y_n^{(1)} y_n^{(6)} y_n^{(3)} + 120 y^{(3)^2} y_n^{(4)}\right]$$

$$|x_2| = \frac{h^{11}}{-10x_{n+1}^{11}}\left[15 y_n^{(2)} y_n^{(5)} y_n^{(4)} - 10 y_n^{(2)} y_n^{(6)} y_n^{(3)} - 16 y_n^{(5)^2} y_n^{(1)} + 5 y_n^{(1)} y_n^{(6)} y_n^{(4)} + 20 y_n^{(3)^2} y_n^{(5)} - 25 y_n^{(4)^2} y_n^{(3)}\right]$$

$$|x_3| = \frac{h^{12}}{-10x_{n+1}^{12}}\left[36 {y_n}^{(5)^2} y_n^{(2)} - 30 y_n^{(2)} y_n^{(4)} y_n^{(6)} - 120 y_n^{(3)} y_n^{(5)} y_n^{(4)} + 75 {y_n}^{(4)^3} + 40 y_n^{(3)^2} \overset{(6)}{y_n}\right]$$

Since $q_i's$ are defined by (**??**), we have
(31)
$$q_1 x_{n+1}^1 = \frac{h^1}{-2}\frac{\left[6 {y_n}^{(2)^2} - 12 y_n^{(1)} y_n^{(5)} y_n^{(3)} - 15 y_n^{(4)^2} y_n^{(2)} + 6 y_n^{(1)} y_n^{(5)} y_n^{(4)} - 4 y_n^{(1)} y_n^{(6)} y_n^{(3)} + 120 y_n^{(3)^2} y_n^{(4)}\right]}{\left[(n+1)\, 18 {y_n}^{(2)^2} y_n^{(5)} - 60 y_n^{(2)} y_n^{(4)} y_n^{(3)} - 12 y_n^{(1)} y_n^{(5)} y_n^{(3)} + 15 y^{(4)^2} y_n^{(1)} + 40 y_n^{(3)^2}\right]}$$

(32)
$$q_2 x_{n+1}^2 = \frac{h^2}{-10}\frac{\left[15 y_n^{(2)} y_n^{(5)} y_n^{(4)} - 10 y_n^{(2)} y_n^{(6)} y_n^{(3)} - 16 y_n^{(5)^2} y_n^{(1)} + 5 y_n^{(1)} y_n^{(6)} y_n^{(4)} + 20 y_n^{(3)^2} y_n^{(5)} - 25 y_n^{(4)^2} y_n^{(3)}\right]}{\left[(n+1)\, 18 {y_n}^{(2)^2} y_n^{(5)} - 60 y_n^{(2)} y_n^{(4)} y_n^{(3)} - 12 y_n^{(1)} y_n^{(5)} y_n^{(3)} + 15 y^{(4)^2} y_n^{(1)} + 40 y^{(3)^3}\right]}$$

(33)
$$q_3 x_{n+1}^3 = \frac{h^3}{-120}\frac{\left[36 {y_n}^{(5)^2} y_n^{(2)} - 30 y_n^{(2)} y_n^{(4)} y_n^{(6)} - 120 y_n^{(3)} y_n^{(5)} y_n^{(4)} + 75 {y_n}^{(4)^3} + 40 y_n^{(3)^2} \overset{(6)}{y_n}\right]}{\left[(n+1)\, 18 {y_n}^{(2)^2} y_n^{(5)} - 60 y_n^{(2)} y_n^{(4)} y_n^{(3)} - 12 y_n^{(1)} y_n^{(5)} y_n^{(3)} + 15 y_n^{(4)^2} y_n^{(1)} + 40 y_n^{(3)^2}\right]}$$

From (21), the rational integrator is expanded to give

$$y_{n+1} = \frac{p_0 + p_1x_{n+1} + p_2x^2_{n+1} + r + rq_1x_{n+1} + rq_2x^2_{n+1} + rq_3x^3_{n+1}}{1 + q_1x_{n+1} + q_2x^2_{n+1} + q_3x^3_{n+1}} \tag{34}$$

$$y_n = r + p_0 \tag{35}$$

$$r = y_n - p_0 \tag{36}$$

$$p_1 = \frac{hy_n^{(1)}}{1!x_{n+1}} + p_0q_1$$

$$p_1x_{n+1} = hy_n^{(1)} + p_0q_1x_{n+1} \tag{37}$$

$$p_2 = \frac{h^2y_n^{(2)}}{2!x^2_{n+1}} + \frac{hy_n^{(1)}}{1!x_{n+1}} + p_0q_2$$

$$p_2x^2_{n+1} = \frac{h^2y_n^{(2)}}{2!} + \ hy_n^{(1)}q_1x_{n+1} + p_0q_2x^2_{n+1} \tag{38}$$

$$p_0q_3 = \frac{h^3y_n^{(3)}}{3!x^3_{n+1}} + \frac{h^2y_n^{(2)}}{2!x^2_{n+1}}q_1 + \frac{h^1y_n^{(1)}}{1!x_{n+1}}q_2$$

$$p_0 = \frac{-1}{q_3}\left[\frac{h^3y_n^{(3)}}{3!} + \frac{h^2y_n^{(2)}q_1x_{n+1}}{2!} + hy_n^{(1)}q_2x^2_{n+1}\right] \tag{39}$$

$$r = y_n - \frac{1}{q_3}\left[\frac{h^3y_n^{(3)}}{3!} + \frac{h^2y_n^{(2)}q_1x_{n+1}}{2!} + hy_n^{(1)}q_2x^2_{n+1}\right] \tag{40}$$

Then

$$p_0 + r = r + p_0 = y_n \tag{41}$$

$$p_1x_{n+1} + rq_1x_{n+1} = hy_n^{(1)} - \frac{q_1x_{n+1}}{q_3x^3_{n+1}}\left[\frac{h^3y_n^{(3)}}{3!} + \frac{h^2y_n^{(2)}q_1x_{n+1}}{2!} + hy_n^{(1)}q_2x^2_{n+1}\right]$$
$$+ q_1x_{n+1}y_n + \frac{q_1x_{n+1}}{q_3x^3_{n+1}}\left[\frac{h^3y_n^{(3)}}{3!} + \frac{h^2y_n^{(2)}q_1x_{n+1}}{2!} + hy_n^{(1)}q_2x^2_{n+1}\right]$$

$$p_1x_{n+1} + rq_1x_{n+1} = hy_n^{(1)} + q_1x_{n+1}y_n \tag{42}$$

$$p_2x_{n+1}^2+rq_2x_{n+1}^2=\frac{h^2y_n^{(2)}}{2!}+hy_n^{(1)}q_1x_{n+1}-\frac{q_2x_{n+1}^2}{q_3x_{n+1}^3}\left[\frac{h^3y_n^{(3)}}{3!}+\frac{h^2y_n^{(2)}q_1x_{n+1}}{2!}+hy_n^{(1)}q_2x_{n+1}^2\right]$$

$$+q_2x_{n+1}^2y_n+\frac{q_2x_{n+1}^2}{q_3x_{n+1}^3}\left[\frac{h^3y_n^{(3)}}{3!}+\frac{h^2y_n^{(2)}q_1x_{n+1}}{2!}+hy_n^{(1)}q_2x_{n+1}^2\right]$$

$$p_2x_{n+1}^2+rq_2x_{n+1}^2=\frac{h^2y_n^{(2)}}{2!}+\ hy_n^{(1)}q_1x_{n+1}+q_2x_{n+1}^2y_n \tag{43}$$

$$rq_3x_{n+1}^3=q_3x_{n+1}^3y_n+\frac{q_3x_{n+1}^3}{q_3x_{n+1}^3}\left[\frac{h^3y_n^{(3)}}{3!}+\frac{h^2y_n^{(2)}q_1x_{n+1}}{2!}+hy_n^{(1)}q_2x_{n+1}^2\right] \tag{44}$$

Substituting (39)-(42) into (34), we obtain our general integrator

$$y_{n+1}=\frac{\begin{array}{c}y_n+hy_n^{(1)}+q_1x_{n+1}y_n+\frac{h^2y_n^{(2)}}{2!}+\ hy_n^{(1)}q_1x_{n+1}+q_2x_{n+1}^2y_n+\\ \frac{h^3y_n^{(3)}}{3!}+\frac{h^2y_n^{(2)}q_1x_{n+1}}{2!}+hy_n^{(1)}q_2x_{n+1}^2+q_3x_{n+1}^3y_n\end{array}}{1+q_1x_{n+1}+q_2x_{n+1}^2+q_3x_{n+1}^3}$$

Now, we let
A= $q_1x_{\text{n}+1}$, B $=q_2x_{n+1}^2$, and C $=q_3x_{n+1}^3$
Therefore, our general integrator formula for m = 3 becomes

$$y_{n+1}=\frac{\sum_{i=0}^{3}\frac{h^iy_n^{(i)}}{i!}+A\sum_{i=0}^{2}\frac{h^iy_n^{(i)}}{i!}+B\sum_{i=0}^{1}\frac{h^iy_n^{(i)}}{i!}+Cy_n}{1+A+B+C} \tag{45}$$

3. Convergence, Consistency

[1] asserted that convergence is a minimal property of any given numerical integrator and that convergence must satisfies for all initial value problems. Again, Lambert (1995) reported that one-step method is convergent if initial value problems satisfy the Lipschitz conditions:

$$\lim_{h\to 0}\max_{0\le n\le N}||y(x_n)-y_n||=0 \tag{46}$$

Theorem 3.1. *The rational integrator for the solution of ordinary differential equations is given as follows:*

$$y_{n+1}=\frac{\sum_{i=0}^{3}\frac{h^iy_n^{(i)}}{i!}+A\sum_{i=0}^{2}\frac{h^iy_n^{(i)}}{i!}+B\sum_{i=0}^{1}\frac{h^iy_n^{(i)}}{i!}+Cy_n}{1+A+B+C} \tag{47}$$

where the functions A, B and C are defined by equation and state in equation (31), (32), and (33) respectively is convergent and consistent.

Proof: Subtracting y_n from both sides of (47), we have

$$y_{n+1} - y_n = \frac{\sum_{i=0}^{3} \frac{h^i y_n^{(i)}}{i!} + A\sum_{i=0}^{2} \frac{h^i y_n^{(i)}}{i!} + B\sum_{i=0}^{1} \frac{h^i y_n^{(i)}}{i!} + Cy_n}{1 + A + B + C} - y_n$$

$$y_{n+1} - y_n = \frac{\sum_{i=1}^{3} \frac{h^i y_n^{(i)}}{i!} + A\sum_{i=1}^{2} \frac{h^i y_n^{(i)}}{i!} + Bh^i y_n^{(i)}}{1 + A + B + C}$$

$$\frac{y_{n+1} - y_n}{h} = \frac{y_n^{(i)} + \sum_{i=2}^{3} \frac{h^{i-1} y_n^{(i)}}{i!} + A\sum_{i=1}^{2} \frac{h^{i-1} y_n^{(i)}}{i!} + Bhy_n^{(i)}}{1 + A + B + C}$$

$$\underset{h\to 0}{\text{Limit}} \left(\frac{y_{n+1} - y_n}{h}\right) : \quad \underset{x\to 0}{\text{Limit}} \quad A = \underset{x\to 0}{\text{Limit}} \quad B = 0$$

$$\underset{h\to 0}{\text{Limit}} \left(\frac{y_{n+1} - y_n}{h}\right) = y_n^{(1)}$$

$\text{Limit}_{h\to 0} \left(\frac{y_{n+1} - y_n}{h}\right) = y_n^{(1)} = f(x_n,\ y_n)$ □.

It implies that the rational integrator for the solution of ordinary differential equations is consistent with the initial value. Hence, the integrator is convergent. But the IVP was chosen arbitrarily, therefore our integrator is convergent (Lambert, 1996).

4. Numerical Experiments

We shall be concerned with the application of our new integrator in solving a number of problems in this section. This integrator have been designed for and also determine the convergence and consistency structure of the method.

Problem 1: [13].

$$y' = -y; \qquad y(0) = 0.5; \quad 0 \le x \le 1, \qquad h = 0.1$$

The exact solution is $y(x_n) = \frac{1}{e^{x_n}}$.

Table 1. Error in Numerical Integration

XN	TSOL	YN (4^{th}-stage)	New Method
.1D+00	0.90D+00	-.81D-07	1.864e-08
.2D+00	0.81D+00	-.14D-06	3.374e-08
.3D+00	0.74D+00	-.20D-06	4.579e-08
.4D+00	0.67D+00	-.24D-06	5.525e-08
.5D+00	0.60D+00	-.27D-06	6.249e-08
.6D+00	0.54D+00	-.29D-06	6.785e-08
.7D+00	0.49D+00	-.31D-06	7.163e-08
.8D+00	0.44D+00	-.32D-06	7.407e-08
.9D+00	0.40D+00	-.33D-06	7.540e-08
1D+00	0.36D+00	-.33D-06	7.581e-08

The table above shows the performance of our new numerical integrator. Our new integrator compete favorably well with the explicit fourth-stage, fourth-order Runge-Kutta methods of [13] with a very rate of convergence.

Problem 2: (Real Life Problem).
A new cereal product is introduced through an advertising campaign to a population of 1 million potential customers. The rate at which the population hears about the product is assumed to be proportional to the number of people who are not yet aware of the product. By the end of 1 year, half of the population has heard of the product. How many will have heard of it by the end of 2 years? The differential equation for this problem is given by $\frac{dy}{dt} = k\,(1-y)\,; \quad y = 1 - e^{-0.693t}$ where y represents the number (in millions) of people at time t who have heard of the product. This means that $(1-y)$ is the number of people who have not heard and dy/dt is the rate at which the population hears about the product. Exact solution of this problem can easily be obtained as $y = 1 - e^{-kt}$, for $k = 0.693$.

Table 2: Error in Numerical Integration

XN (Time)	TSOL	[11]	New method
1.0e+00	4.9992e-01	-7.3595e-05	2.2665e-06
2.0e+00	7.4992e-01	-5.2051e-05	7.4408e-06
3.0e+00	8.7494e-01	-3.6816e-05	2.0639e-06
4.0e+00	9.3746e-01	-2.6044e-05	2.0664e-06
5.0e+00	9.6872e-01	-1.8447e-05	1.5976e-06
6.0e+00	9.8436e-01	-1.3038e-05	1.0928e-06
7.0e+00	9.9217e-01	-9.2181e-06	6.9643e-07
8.0e+00	9.9608e-01	-6.5177e-06	4.2395e-07
9.0e+00	9.9804e-01	-4.6085e-06	2.5002e-07
10.e+00	9.9902e-01	-3.2586-e06	1.4407e-07

The result which is shown in the numerical solution in Table 2 agree excellently with the result at 2 years. □.

Conclusions: This article established the convergence, consistency and the stability of the scheme in the interpolants of order $m = 3$, through the rational integrator. The stability analysis of the method was carried out with the help of mathematical software. The Results showed that our integrator is stable analytically and computationally.

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References

[1] Aashikpelokhai U. S. U. (1991). *A class of non-linear one-step rational integrator*. Ph.D Thesis, University of Benin, Benin city, Nigeria.

[2] Aashikpelokhai U. S. U. (1997). *A class of non-linear one-step rational integrator*. Pon Publishers Limited Agbor, Delta State, Nigeria.

[3] Aashikpelokhai U. S. U. (2010). A general [L, M] One- step integrator for initial value problems. *International Journal of Computer Mathematics*. **1**, 1-12.

[4] Aashikpelokhai U. S. U. (2014). Renaissance of Mathematics in our time. *Pub. FNS AAU Ekpoma*

[5] Abid A. Bras M. & Hojjati G. (2014). On the construction of second derivative diagonally implicit multi-stage integration methods for ODEs. *Appl. Numer. Math.* **76**, 1-18.

[6] Ebhohimen F. & Anetor O. (2017). The stability of the rational interpolation method in Ordinary Differential Equations at $k = 6$. *Transactions of the Nigerian Association of Mathematical Physics* **5**, (September and November), 33-38.

[7] Ehiemua M. & Agbeboh G. U. (2020). On the stability of order 7 rational interpolation scheme for solving initial value problems in ODEs. *WJIR* ISSN: 2454-8236, **9** (3), 110-114.

[8] Elakhe O. A. & Aashikpelokhai U. S. U. (2010). On a singulo-stiff rational integrator. *International Journal of Natural and Applied Science*. **2**, 43-53.

[9] Elakhe O. A. (2011). *Singulo-Oscillatory Stiff Rational Integrator*. Ph.D. Thesis, Ambrose Alli University, Ekpoma, 173pp.

[10] Elakhe O. A. & Aashikpelokhai U. S. U. (2013). On a singulo oscillatory-stiff rational integrator. *International Journal of Natural and Applied Science*. **8**, 1703-1715.

[11] Elakhe O. A., Ehika E. & Ehika S. (2020). A quartic based denominator of order six rational integrator. *J. Physical & Applied Sciences*. **2** (1), Ambrose Alli University, Ekpoma.

[12] Enright W. H. & Pryce J. D. (1987). Two Fortran packages for assessing initial value methods. *ACM Transaction son Mathematical Software*. **13**, 1-27.

[13] Esekhaigbe C. A. (2017). Transformation and implementation of a highly efficient fully implicit forth-order Runge-Kutta method. *International Journal of Innovative Research*. **5**(1), 171-183.

[14] Ezeilo . (2000). Periodic boundary value problems for some fourth order differentials. *Proceedings of the National Mathematical Centre. Abuja*. **1** (1), 2000

[15] Ezzeddine A. K., Hajjati G. & Abdi A. (2014). Sequential second derivative general linear method for stiff systems. *Bulletin of the Iranian Mathematical Society*. **40** (1), 83-100.

[16] Fatunla S. O. (1978). An implicit two-part numerical integration formula for linear and non-linear stiff system of ODE. *Mathematics of Computation*. **32**, 1-11.

[17] Hojjati G. (2015). A Class of Parallel Mrthods with Super-future Points Technique for the Numerical Solution of Shiff Systems. *Journal of order Methods in Numerical Mathematics*. **6** (2), 57-63.

[18] Obodi. (2011). Application of anseir epidemic model with delay, *ABACUS- Maths Series*. **28** (2), Sept, 2011.

[19] Vyawahare V. & Nataraj P. S. V. (2014). Proceedings of the 4^{th} international conference on simulation and modeling methodologies, technologies and applications. *SIMULTECH*. **1**, 352-360.