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Mathematical Modelling of an Insurer's Portfolio and Reinsurance Strategy under the CEV Model and CRRA Utility

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Abstract

One of the major problems encountered by most insurance companies is portfolio management and payment of claims. Hence the study of optimal portfolio strategy (OPS) and optimal reinsurance strategy (ORS) becomes necessary. In this paper, the insurer is allowed to invest in a risk-free-asset and a risky-asset, where the risky-asset price follows the constant elasticity variance model and can buy proportional reinsurance policy as a backup. By optimal control approach, the Hamilton-Jacobi-Bellman (HJB) equation is obtained. Using Legendre transformation method, the HJB-equation is transformed to a linear partial differential equation and solved for OPS and ORS for an insurer with logarithm utility. Finally, numerical and theoretical analyses were presented to study the impact of model parameters on ORS and OPS.

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1. Introduction

In general, insurance can be viewed as a risk transfer tool but can also be used to prevent risk. In view of the importance and rapid increase in the study of portfolio optimization and reinsurance problems, many authors have carried out a number of researches in this area of study. Also, considering the fact that the insurer and the reinsurer are allowed to invest their surplus wealth in a financial market comprising of both risk free and risky assets, these has help to build their portfolios not only by receiving premiums but also by investing in different assets available in the financial market.

Unlike other financial institutions such as pension system, banks etc, the insurer and the reinsurer are faced with dual risks which exist both in the insurance market and the stock market. In order to ease the risk of claims, the insurer can afford to buy reinsurance contracts from the reinsurer thereby transferring a certain agreed proportion of the risk of claims to the reinsurer since the reinsurer is more risk-seeking than the insurer. Consequently, the insurer and the reinsurer are faced with how to design a suitable reinsurance contract; this has led to the study of optimal portfolio strategies which basically involves the determination of the best possible way of investing in the risky assets for optimal profit with minimal risk.

In order to take a near perfect decision while investing in risky assets such as stock, the stochastic volatility models become necessary to understand the random nature of the stock market price and one of such model is the CEV model which is used in describing the behaviour of the stock market price. This model, was first used in [6] and in their work, they showed that the CEV model degenerate into a constant volatility model called the geometric Brownian motion (GBM) when its elasticity parameter is zero. Also, they pointed out that one outstanding characteristic of this model is in the ability to capture implied volatility skew. A lot of researches have been carried on utility maximization and optimal control strategies by different authors, some of which include [10,20 ,24], who studied reinsurance problem under the CEV model and optimal control strategies. [17], studied the optimal investment problem with taxes, dividend and transaction cost using different utility functions under the CEV process; they showed that the optimal investment strategy increases with dividends and decreases with transaction cost for logarithm utility function. [13] & [23] solved the optimal investment problem for a defined contribution (DC) pension plan with return of premiums clauses under different assumptions and assumed that the stock market price follows the CEV process; they used the game theoretic approach and solved the resultant extended Hamilton Jacobi Bellman equation for the optimal investment plan. The optimal investment plan with the CEV process under several assumptions has been studied by different authors when the risk free interest rate are constant as seen in the above literatures except for

[1], [2] & [11], who investigated the optimal control strategies under the CEV model with stochastic interest rate. In each case, they assumed the interest is either modelled by the Cox- Ingersoll-Ross (CIR) process, Vasicek model, or the O-U process.
Generally, for insurance company, they are concerned not only on investment problems but also risk management problems associated with insurance/reinsurance. In [4], the attention of many researchers was drawn to risk management with insurance. [8], were the first to propose how insurance can be used as a risk prevention tool. [7] & [19] give early contributions to insurance and reinsurance. Subsequently, a lot of researchers such as [18] & [27] studied an insurer optimal investment strategy under exponential utility function with the jump-diffusion process. [21] used an investment model that followed the Hull and White stochastic volatility model and obtained a reinsurance and investment strategy to maximize expected utility function. [3] Studied portfolio strategy for an investor with stochastic premium under exponential utility using Legendre transform and dual theory. They assumed the premiums paid to the insurance companies are stochastic. [14] Studied the optimal reinsurance and investment problem of the maximum expected two exponential utility function whose claim process are modelled by Brownian motion with drift. [9] Focused on the optimal Excess-of-Loss reinsurance and investment with maximizing exponential utility. [25] Considered the optimal reinsurance and investment problem of maximizing the expected power utility function. In [31], the time-consistent investment and reinsurance strategies for mean-variance insurers under stochastic interest rate and stochastic volatility were studied; in their work, they assumed the stock market price was modelled by Heston stochastic volatility and the interest rate follows the vasicek model and used the mean variance utility function as their objective function to determine the optimal investment strategies. [14] studied time consistent reinsurance investment strategy for an insurer and a reinsurer under mean-variance criterion. [28] Studied time-consistent investment-reinsurance strategies for the insurer and the reinsurer under the generalized mean-variance criteria in a discrete time setting. Their results indicate that the intertemporal restrictions will urge the insurer and the reinsurer to shrink the position invested in the risky asset; however, for the time-consistent reinsurance strategy, the impact of the intertemporal restrictions depends on who is the leader in the proposed model. [15] & [17] discussed the problem of ruin probability minimization, the ruin probability for the insurer, and maximizing the exponential utility function. [26] focused on the optimal investment problem with consumption.
Very recently, the optimal reinsurance and investment strategy under the CEV model with fractional power utility function was studied in [20]; in their work, they obtained the optimal reinsurance policy and optimal investment strategy for an insurer with power utility function by solving the resultant non-linear partial

differential equation using power transformation method. They observed that the greater the value of the reinsurer's safety loading, the smaller the optimal reinsurance policy and to maintain a stable income, the insurer would prefer buying less reinsurance.

In this paper, we study the optimal portfolio strategy and reinsurance strategy for an insurance company exhibiting the CRRA and whose risky asset is modelled by the CEV process. Furthermore, the Legendre transformation method is used to solve for the solutions of the optimal portfolio and reinsurance strategy by reducing the resultant non-linear partial differential equation to a linear partial differential equation, we also give some numerical simulations to explain our results. The difference between our work and [20] is that their utility function is power utility while ours is logarithm utility. Also they used power transformation method to solve the resultant Hamilton Jacobi Bellman equation while we used the Legendre transformation method and dual theory.

The rest of the paper is outlined as follows; section 2 gives a preliminary notion on the surplus process of an insurance company and describes the financial models of the insurance company. Section 3 describes the optimization problem and the method used in obtaining the optimization problem. Section 4 provides explicit solutions of the optimal portfolio strategy and the reinsurance strategy under logarithm utility function. Sections 5 and 6 present numerical simulations of the impact of some sensitive parameters on the optimal portfolio and reinsurance strategies and discussions. Section 7 concludes the work.

2. Preliminaries

2.1. **The Surplus Process.** In this subsection, we formulate the surplus process of the insurer. In insurance, the surplus process is the process of accumulation of wealth. To derive the surplus process, we need the claim process. Following the framework of [5] & [15], we model the claim process $\mathcal{C}(t)$ which follows the Brownian motion with drift as

$$d\mathcal{C}\left(t\right)=adt-bd\mathcal{B}_{0}\left(t\right), \tag{1}$$

where a and b are positive constant, and $\mathcal{B}_0\left(t\right)$is a standard Brownian motion defined on the complete probability space(?, $\mathcal{F}$, $\mathcal{P}$). According to the expected value principle [22], the premium rate of insurer is $c=\left(1+\vartheta\right)a$ and $\theta>0$ is the safety loading of insurer. In this paper, we assume a classical Cramer-Lundberg model for surplus process [22] as

$$d\mathcal{R}\left(t\right)=x_0+ct-\mathcal{C}\left(t\right)t \ \geq \ 0, \tag{2}$$

where $\mathcal{R}\left(t\right)$ and x_0 are the insurers capital at time t and initial capital $\mathcal{R}\left(0\right)=x_0$, respectively. According to (8), the surplus process for the insurer is given as

$$d\mathcal{R}\left(t\right)=cdt-d\mathcal{C}\left(t\right)=a\vartheta dt+bd\mathcal{B}_{0}\left(t\right). \tag{3}$$

Furthermore, the insurer can buy reinsurance contract to reduce risk [16]. Suppose the insurer pays reinsurance premium continuously at rate $c_1 = (1+\eta)\,a$ where $\eta > \vartheta > 0$ is the safety loading of the reinsurer. So, the surplus process $R_1(t)$ associated with reinsurance of the insurer follows

$$d\mathcal{R}_1(t) = cdt - (1-p(t))\,d\mathcal{C}(t) - c_1 p(t)\,dt, \tag{4}$$

$$d\mathcal{R}_1(t) = (1+\vartheta)\,adt - (1-p(t))\,(adt - bd\mathcal{B}_0(t)) - (1+\eta)\,ap(t)\,dt,$$

$$d\mathcal{R}_1(t) = (\vartheta - \eta p(t))\,adt + b\,(1-p(t))\,d\mathcal{B}_0(t)\,. \tag{5}$$

where $p(t)$ is proportion reinsurance at time t.

2.2. **The Financial Market Model.** Consider a portfolio comprising of one risk free asset and two risky assets in a financial market which is open continuously for an interval $t \in [0,\ T]$ where T the expiration date of the investment is. Let $\{\mathcal{B}_0(t)\,,\ \mathcal{B}_1(t) : t =\ 0\}$ be standard Brownian motion defined on a complete probability space $(?,\ \mathcal{F},\ \mathcal{P})$ where ? is a real space and $\mathcal{P}$ is a probability measure and F is the filtration which represents the information generated by the three Brownian motions.
Let $\mathcal{H}_0(t)$ denote the price of the risk free asset at time t and from [17], [23] & [30], the model is given as:

$$\frac{d\mathcal{H}_0(t)}{\mathcal{H}_0(t)} = r(t)\,dt\mathcal{H}_0(0) = h_0 > 0\ . \tag{6}$$

where r is the predetermined interest rate process.
Let $\mathcal{H}_1(t)$ denote the price of the stocks described by the CEV model is given by the stochastic differential equations at $t \geq 0$ as follows

$$\frac{d\mathcal{H}_1(t)}{\mathcal{H}_1(t)}\ = \mu dt +\ \sigma \mathcal{H}_1^{\beta}(t)\,d\mathcal{B}_1(t)\,. \tag{7}$$

where μ is appreciation rate of the stock, σ is instantaneous volatility and $\beta < 0$ represent elasticity parameter. see [1], [3], [13], [17] & [30] for details. Note if $\beta = 0$, the stock market price is modelled by GBM.

3. The Optimization Problem and Methodology

3.1. **Optimization Problem.** Let π be the optimal portfolio strategy and we define the utility attained by the investor from a given state k at time t as:

$$\mathcal{N}_\pi(t, h_1, k) = E_\pi\, U(\mathcal{K}(T)) \mid \mathcal{H}_1(t) = h_1,\ \mathcal{K}(t) = k]\,,$$

where t is the time, r is the risk free interest rate and k is the wealth. The objective here is to determine the optimal portfolio strategy and the optimal value function of the investor given as
π^* and $\mathcal{N}(t, h_1, k) = \sup_\pi \mathcal{N}_\pi(t, h_1,\ k)$,

Respectively such that

$$\mathcal{N}_{\pi^*}(t, h_1,\ k) = \mathcal{N}(t, h_1,\ k).$$

Let $\mathcal{K}(t)$ be the insurer's wealth at time t and the amount of wealth of insurer invested on risk asset at time t denoted by $\pi(t)$ and $\mathcal{K}(t)$ represents the wealth of the insurer. So, the remainder $\mathcal{K}(t) - \pi(t)$ invested in risk-free asset, and then the differential form associated with the fund size is given as:

$$\text{d}\mathcal{K}(t) = (\mathcal{K}(t) - \pi)\frac{d\mathcal{H}_0(t)}{\mathcal{H}_0(t)} + \pi\frac{d\mathcal{H}_1(t)}{\mathcal{H}_1(t)} + d\mathcal{R}_1(t). \tag{8}$$

Substituting (5), (6) and (7) into (8), we have

$$\text{d}\mathcal{K}(t) = \begin{pmatrix} (\mathcal{K}(t) - \pi) r(t) dt + \pi\left(\mu dt + \ \sigma\mathcal{H}_1^{\beta}(t) d\mathcal{B}_1(t)\right) \\ +(\vartheta - \eta p(t)) a dt + b(1 - p(t)) d\mathcal{B}_0(t) \end{pmatrix},$$

$$\text{d}\mathcal{K}(t) = \begin{pmatrix} (r\mathcal{K}(t) + \pi(\mu - r) + (\vartheta - \eta p(t)) a) dt \\ +b(1 - p(t)) d\mathcal{B}_0(t) + \pi\sigma\mathcal{H}_1^{\beta}(t) d\mathcal{B}_1(t) \end{pmatrix}, \tag{9}$$

Applying the Ito's lemma and maximum principle, the Hamilton Jacobi Bellman (HJB) equation which is a nonlinear PDE associated with (9) is obtained by maximizing $\mathcal{N}_{\pi^*}(t, h_1,\ k)$ subject to the insurer's wealth in (8) as follows

$$\left\{\begin{matrix} \mathcal{N}_t + \mu h_1\mathcal{N}_{h_1} + [rk + \pi(\mu - r) + (\vartheta - \eta p(t)) a]\mathcal{N}_k \\ +\frac{1}{2}\sigma^2 h_1^{2\beta+2}\mathcal{N}_{h_1h_1} + \frac{1}{2}\left(b^2(1 - p(t))^2 + \pi^2\sigma^2 h_1^{2\beta}\right)\mathcal{N}_{kk} \\ -(\mu - r) h_1\frac{\mathcal{N}_k\mathcal{N}_{ks_1}}{\mathcal{N}_{kk}} + \pi\sigma^2 h_1^{2\beta+1}\mathcal{N}_{h_1k} \end{matrix}\right\} = 0. \tag{10}$$

Differentiating (10) with respect to π *and* p, we obtain the first order maximizing condition for (10) as follows

$$(\mu - r)\mathcal{N}_k + \pi\sigma^2 h_1^{2\beta}\mathcal{N}_{kk} + \sigma^2 h_1^{2\beta+1}\mathcal{N}_{h_1k} = 0, \tag{11}$$

$$-\eta a\mathcal{N}_k - b^2(1 - p(t))\mathcal{N}_{kk} = 0, \tag{12}$$

by solving for π and p in (11) and (12) respectively, we have

$$\pi^* = -\frac{(\mu - r)\mathcal{N}_k + \sigma^2 h_1^{2\beta+1}\mathcal{N}_{h_1k}}{\sigma^2 h_1^{2\beta}\mathcal{N}_{kk}}, \tag{13}$$

$$p^* = \frac{\eta a}{b^2}\frac{\mathcal{N}_k}{\mathcal{N}_{kk}} + 1. \tag{14}$$

Substituting (13) and (14) into (10), we have

$$\left\{\begin{matrix} \mathcal{N}_t + \mu h_1\mathcal{N}_{h_1} + [rk + (\vartheta - \eta) a]\mathcal{N}_k + \frac{1}{2}\sigma^2 h_1^{2\beta+2}\left[\mathcal{N}_{h_1h_1} - \frac{\mathcal{N}_{kh_1}^2}{\mathcal{N}_{kk}}\right] \\ -\frac{1}{2}\left[\frac{(\mu-r)^2}{\sigma^2 h_1^{2\beta}} - \frac{\eta^2 a^2}{k^2}\right]\frac{\mathcal{N}_k^2}{\mathcal{N}_{kk}} - (\mu - r) h_1\frac{\mathcal{N}_k\mathcal{N}_{kh_1}}{\mathcal{N}_{kk}} \end{matrix}\right\} = 0 \tag{15}$$

3.2. **Legendre Transformation and Dual theory.** The differential equation obtained in (15) is a non linear PDE and is somehow complex to solve. In this section, we will introduce the Legendre transformation and dual theory and use it to transform the non linear PDE to a linear PDE.

Theorem 3.1: Let $f:\ R^n \to R$ be a convex function for $z > 0$, define the Legendre transform

$$\mathcal{M}(z) = \max_k \{f(k) - zk\},$$

where $\mathcal{M}(z)$ is the Legendre dual of $f(k)$.

Since $f(k)$ is convex, from Theorem 3.1 and [12] & [29], the Legendre transform for the value function $\mathcal{N}(t, h_1, k)$ can be defined as follows

$$\widehat{\mathcal{N}}(t, h_1, z) = \sup \mathcal{N}(t, h_1, k) - zk \mid 0 < k < \infty\} \quad 0 < t < T$$

where $\widehat{\mathcal{N}}$ is the dual of $\mathcal{N}$ and $z > 0$ is the dual variable of k.

The value of k where this optimum is achieved is represented by $g(t, h_1, z)$, such that

$$g(t, h_1,\ z) = \inf k \,\Big|\, \mathcal{N}(t, h_1,\ k) \geq zk + \widehat{\mathcal{N}}(t, h_1,\ z)\Big\} \quad 0 < t < T. \tag{16}$$

From (16), the function g and $\widehat{\mathcal{N}}$ are very much related and can be refers to as the dual of $\mathcal{N}$ and are related thus

$$\mathcal{N}(t, h_1,\ k) = \widehat{\mathcal{N}}(t, h_1,\ z) + zg, \tag{17}$$

where

$$g(t, h_1,\ z) = k,\ \mathcal{N}_k = \mathcal{N}_g = z, \quad g = -\widehat{\mathcal{N}}_z. \tag{18}$$

differentiating (17) with respect to $t,\ h_1$ *and* k,

$$\begin{cases} \mathcal{N}_t = \widehat{\mathcal{N}}_t, \quad \mathcal{N}_{h_1} = \widehat{\mathcal{N}}_{h_1}\ ,\ \mathcal{N}_k = z,\ \mathcal{N}_{kk} = \frac{-1}{\widehat{\mathcal{N}}_{zz}}\ , \\ \mathcal{N}_{h_1 k} = \frac{-\widehat{\mathcal{N}}_{s_1 z}}{\widehat{\mathcal{N}}_{zz}}, \mathcal{N}_{h_1 h_1} = \widehat{\mathcal{N}}_{h_1 h_1} - \frac{\widehat{\mathcal{N}}^2_{h_1 z}}{\widehat{\mathcal{N}}_{zz}} \end{cases}, \tag{19}$$

At terminal time T, we define the dual utility in terms of the original utility function $U(k)$ as

$$\hat{U}(z) = sup\, U(k) - zk \mid 0 < k < \infty\}\ ,$$

and

$$G(z) = \sup\{k \mid U(k) \geq zk + \hat{U}(z)\,.$$

As a result $\widehat{\mathcal{N}}(t, h_1,\ z) = \mathcal{N}(t, h_1,\ g) - zg$.

$$G(z) = (U')^{-1}(z), \tag{20}$$

where G is the inverse of the marginal utility U and note that $\mathcal{N}(T, h_1, k) = U(k)$

At terminal time T, we can define

$$g\left(T,h_1,\ z\right)=\underset{k>0}{inf}\ k\ \middle|\ U\left(k\right)\geq zk+\widehat{\mathcal{N}}\left(t,h_1,\ z\right)\Big\}\ and\ \widehat{\mathcal{N}}\left(t,h_1,\ z\right)=\underset{k>0}{sup}\{\ U\left(k\right)-zk\}$$

so that

$$g(T,h_1,\ z)=\left(U'\right)^{-1}(z). \tag{21}$$

Substituting (19) into (13), (14) and (15), we have

$$\left\{\begin{array}{c}\widehat{\mathcal{N}}_t+\mu h_1\widehat{\mathcal{N}}_{h_1}+\left[rg+\left(\vartheta-\eta\right)a\right]z+\frac{1}{2}\sigma^2h_1^{2\beta+2}\widehat{\mathcal{N}}_{h_1h_1}\\ +\frac{1}{2}\left[\frac{(\mu-r)^2}{\sigma^2h_1^{2\beta}}-\frac{\eta^2a^2}{b^2}\right]z^2\widehat{\mathcal{N}}_{zz}-\left(\mu-r\right)h_1z\widehat{\mathcal{N}}_{h_1z}\end{array}\right\}=0, \tag{22}$$

$$p^*=1-\frac{\eta a}{b^2}z\widehat{\mathcal{L}}_{zz}, \tag{23}$$

$$\pi^*=\frac{-\left(\mu-r\right)z\widehat{\mathcal{N}}_{zz}-\sigma^2h_1^{2\beta+1}\widehat{\mathcal{N}}_{h_1z}}{\sigma^2h_1^{2\beta}}. \tag{24}$$

From equation (18), differentiating (22), (23) and (24) with respect to z, we have

$$\left\{\begin{array}{c}g_t+rh_1g_{h_1}-\left[rg+\left(\vartheta-\eta\right)a\right]+\frac{1}{2}\sigma^2h_1^{2\beta+2}g_{h_1h_1}\\ +\left[\frac{(\mu-r)^2}{\sigma^2h_1^{2\beta}}-\frac{\eta^2a^2}{b^2}-r\right]z\ g_z+\frac{1}{2}\left[\frac{(\mu-r)^2}{\sigma^2h_1^{2\beta}}-\frac{\eta^2a^2}{b^2}\right]z^2g_{zz}-\left(\mu-r\right)h_1zg_{h_1z}\end{array}\right\}=0, \tag{25}$$

$$p^*=\left(1-\frac{\eta a}{b^2}\right)\left(g_z+z\ g_{zz}\right), \tag{26}$$

$$\pi^*=\frac{-\left(\mu-r\right)z\ g_z-\sigma^2h_1^{2\beta+1}g_{h_1}}{\sigma^2h_1^{2\beta}}, \tag{27}$$

where, $\mathcal{N}\left(T,h_1,z\right)=U\left(z\right)$ and $U\left(z\right)$ is the marginal utility of the investor. Next, we proceed to solve (25) for g considering an insurer with logarithm utility, then substitute the solution into (26) and (27) for the optimal investment plan using change of variable method.

4. RESULT AND DISCUSSION

4.1. **Insurer's Investment and Reinsurance Strategy under Logarithm Utility.** Here, we consider an insurer with utility function exhibiting constant relative risk aversion (CRRA) different from the one in [11] where the insurer exhibited constant absolute risk aversion (CARA). Since our interest here is to determine the optimal investment plan for the insurer with CRRA utility, we choose the logarithm utility function similar to the one in [29] & [30].

From [17] & [29], The logarithm utility function is given as:

$$U(k) = lnkk > 0, \tag{28}$$

From (21),

$$g(T, h_1,\ z) = (U')^{-1}(z) = \frac{1}{z}. \tag{29}$$

Next, we conjecture a solution to (25) similar to the one in [29] with the form:

$$\begin{cases} g(t, h_1, z) = \frac{1}{z} e(t, h_1) + f(t) \\ e(T, h_1) = 1, \quad f(T) = 0, \end{cases} \tag{30}$$

$$\begin{cases} g_t = \frac{1}{z} e_t + f_t, \quad g_{h_1} = \frac{1}{z} e_{h_1} g_{h_1 h_1} = \frac{1}{z} e_{h_1 h_1}, \\ g_z = -\frac{e}{z^2}, \ g_{zz} = \frac{2e}{z^3}\ , \ g_{h_1 z} = -\frac{1}{z^2} e_{h_1}, \end{cases} \tag{31}$$

Substituting (31) into (**??**), we have

$$\left\{ [f_t - rf - (\vartheta - \eta) a] + \frac{1}{z} \left[e_t + \mu h_1 e_{h_1} + \frac{1}{2} \sigma^2 h_1^{2\beta+2} e_{h_1 h_1} \right] = 0. \right. \tag{32}$$

Splitting (32) we have

$$\begin{cases} [f_t - rf - (\vartheta - \eta) a] = 0 \\ f(T) = 0 \end{cases}, \tag{33}$$

$$\begin{cases} \left[e_t + \mu h_1 e_{h_1} + \frac{1}{2} \sigma^2 h_1^{2\beta+2} e_{h_1 h_1} \right] = 0 \\ e(T, h_1) = 1 \end{cases}, \tag{34}$$

Solving equation (33) for f, we have

$$f(t) = \frac{(\vartheta - \eta) a}{r} \left[e^{r(t-T\}} - 1 \right]. \tag{35}$$

Lemma 4.1
The solution of equation (34) is given as

$$e(t, h_1) = 1,$$

Proof. Assume

$$\begin{cases} e(t, h_1) = u(t, m), \qquad m = h_1^{-2\beta} \\ e(T, h_1) = 1 \end{cases}, \tag{36}$$

then

$$\left. \begin{array}{c} e_t = u_t, \quad e_{h_1} = -2\beta h_1^{-2\beta-1} u_m\ , \\ e_{h_1 h_1} = 2\beta(2\beta+1) h_1^{-2\beta-2} u_m + 4\beta^2 h_1^{-4\beta-2} u_{mm} \end{array} \right\}. \tag{37}$$

Substituting (37) into (34), we have

$$\begin{cases} u_t - 2\mu\beta m u_m + \sigma^2 \beta (2\beta+1) u_m + 2\beta^2 \sigma^2 m u_{mm} = 0 \\ u(T, m) = 1 \end{cases} \tag{38}$$

Next, we assume a solution for (38) as follows

$$u(t,m) = \mathcal{A}_0(t) + m\mathcal{A}_1(t), \tag{39}$$

and

$$u_t = \mathcal{A}_{0t} + m\mathcal{A}_{1t}, \quad u_m = \mathcal{A}_1, \; u_{mm} = 0, \tag{40}$$

Substituting (40) in (39), we have

$$\begin{cases} \mathcal{A}_{0t} + \beta\sigma^2(2\beta+1)\mathcal{A}_1 = 0 \\ \mathcal{A}_0(T) = 1 \end{cases}, \tag{41}$$

$$\begin{cases} \mathcal{A}_{1t} - 2\mu m\mathcal{A}_1 = 0 \\ \mathcal{A}_1(T) = 0 \end{cases}, \tag{42}$$

Solving (42), we have

$$\mathcal{A}_1(t,r) = 0, \tag{43}$$

Putting (43) into (41) solving it, we have

$$\mathcal{A}_0(t,r) = 1, \tag{44}$$

Hence from (36)

$$e(t,h_1) = u(t,m) = \mathcal{A}_0(t) + m\mathcal{A}_1(t) = 1. \tag{45}$$

Hence Lemma 4.1 is proved. □

Proposition 4.1.

$$g(t,h_1,z) = \frac{1}{z} + \frac{(\vartheta-\eta)a}{r}\left[e^{r(t-T\}} - 1\right] \tag{46}$$

Proof. Recall from (30),

$$g(t,h_1,z) = \frac{1}{z}e(t,h_1) + f(t)$$

substituting (35) and (45) into (30), Proposition 4.1 is proved. □

Proposition 4.2. *The optimal portfolio strategy and reinsurance strategy are given as*

$$\pi^* = \frac{(\mu-r)}{\sigma^2 h_1^{2\beta}}\left[k + \frac{(\vartheta-\eta)a}{r}\left[1-e^{r(t-T)}\right]\right], \tag{47}$$

and

$$p^* = \left(1-\frac{\eta a}{b^2}\right)\left[k + \frac{(\vartheta-\eta)a}{r}\left[1-e^{r(t-T)}\right]\right]^2. \tag{48}$$

Proof. Recall that equation (26) and (27) are given as

$$p^* = \left(1 - \frac{\eta a}{b^2}\right)\left(g_z + z\ g_{zz}\right),$$

$$\pi^* = \frac{-\left(\mu - r\right) z\ g_z - \sigma^2 s_1^{2\beta+1} g_{s_1}}{\sigma^2 s_1^{2\beta}},$$

From proposition 4.1 and equation (18), we have
$g\left(t, h_1, z\right) = \frac{1}{z} + \frac{(\vartheta-\eta)a}{r}\left[1 - e^{rr(t-T)}\right]$ and $g = k$
this implies that $g = \frac{1}{z} + \frac{(\eta-\vartheta)a}{r}\left[1 - e^{r(t-T)}\right] = k$
hence

$$\text{(49)} \qquad \frac{1}{z} = k - \frac{\left(\vartheta - \eta\right) a}{r}\left[1 - e^{r(t-T)}\right] = k + \frac{\left(\eta - \vartheta\right) a}{r}\left[1 - e^{r(t-T)}\right].$$

differentiating g with respect to s_1, z and substitute into equation (26) and (27), we have

$$\pi^* = \frac{-\left(\mu - r\right) z\ g_z}{\sigma^2 h_1^{2\beta}} = \frac{-\left(\mu - r\right) z}{\sigma^2 h_1^{2\beta}} \times \frac{-1}{z^2} = \frac{\left(\mu - r\right) z}{\sigma^2 h_1^{2\beta}} \times \frac{1}{z},$$

Substituting (49) into the above equation, we have

$$\pi^* = \frac{\left(\mu - r\right)}{\sigma^2 h_1^{2\beta}}\left[k + \frac{\left(\eta - \vartheta\right) a}{r}\left[1 - e^{r(t-T)}\right]\right].$$

Similarly,

$$p^* = \left(1 - \frac{\eta a}{b^2}\right)\left(g_z + z\ g_{zz}\right) = \left(1 - \frac{\eta a}{b^2}\right)\left(\frac{-1}{z^2} + \frac{2z}{z^3}\right) = \left(1 - \frac{\eta a}{b^2}\right)\left(\frac{-1}{z^2} + \frac{2}{z^2}\right),$$

$$p^* = \left(1 - \frac{\eta a}{b^2}\right)\left(\frac{1}{z^2}\right) = \left(1 - \frac{\eta a}{b^2}\right)\left(\frac{1}{z}\right)^2$$

Substituting (**??**) into the above equation, we have

$$p^* = \left(1 - \frac{\eta a}{b^2}\right)\left[k + \frac{\left(\eta - \vartheta\right) a}{r}\left[1 - e^{r(t-T)}\right]\right]^2.$$

□

Proposition 4.3. *Suppose* $\eta > 0,\ a > 0,\ b > 0, k > 0, r > 0$ *and* $b^2 - \eta a > 0$, *then* $\frac{dp^*}{d\eta} < 0$

Proof. recall from (49),

$$p^* = \left(1 - \frac{\eta a}{b^2}\right)\left[k + \frac{\left(\eta - \vartheta\right) a}{r}\left[1 - e^{r(t-T)}\right]\right]^2,$$

Differentiating the above equation with respect to η, we have

$$\frac{dp^*}{d\eta} = \begin{bmatrix} -\frac{a}{b^2}\left[k + \frac{(\eta-\vartheta)a}{r}\left[1-e^{r(t-T)}\right]\right]^2 \\ -\left(1-\frac{\eta a}{b^2}\right)\frac{a}{r}\left[1-e^{r(t-T\}}\right]\left[k+\frac{(\eta-\vartheta)a}{r}\left[1-e^{r(t-T)}\right]\right] \end{bmatrix},$$

Since $(\eta-\vartheta)>0, \left[1-e^{r(t-T)}\right]>0$ and $0<\frac{\eta a}{b^2}<1$, then $\left(1-\frac{\eta a}{b^2}\right)>0$,

$$\left[k + \frac{(\eta-\vartheta)\,a}{r}\left[1-e^{r(t-T)}\right]\right] > 0.$$

Hence

$$\frac{dp^*}{d\eta} = \begin{bmatrix} -\frac{a}{b^2}\left[k + \frac{(\eta-\vartheta)a}{r}\left[1-e^{r(t-T)}\right]\right]^2 \\ -\left(1-\frac{\eta a}{b^2}\right)\frac{a}{r}\left[1-e^{r(t-T)}\right]\left[k+\frac{(\eta-\vartheta)a}{r}\left[1-e^{r(t-T)}\right]\right] \end{bmatrix},$$

$$\frac{dp^*}{d\eta} = -\begin{bmatrix} \frac{a}{b^2}\left[k + \frac{(\eta-\vartheta)a}{r}\left[1-e^{r(t-T)}\right]\right]^2 \\ +\left(1-\frac{\eta a}{b^2}\right)\frac{a}{r}\left[1-e^{r(t-T)}\right]\left[k+\frac{(\eta-\vartheta)a}{r}\left[1-e^{r(t-T)}\right]\right] \end{bmatrix} < 0,$$

Therefore

$$\frac{dp^*}{d\eta} < 0.$$

□

Proposition 4.4. *Suppose* $\sigma>0, h_1^{2\beta}>0,\ a>0,\ b>0, k>0, r>0, \mu>0,\ \eta>\vartheta>0,$ *then*
(i) $\frac{d\pi^*}{d\mu}>0$ *(ii)* $\frac{d\pi^*}{d\sigma}<0$ (iii) $\frac{d\pi^*}{d\sigma}<0$

Proof. (1) Recall from (49),

$$\pi^* = \frac{(\mu-r)}{\sigma^2 h_1^{2\beta}}\left[k + \frac{(\eta-\vartheta)\,a}{r}\left[1-e^{r(t-T)}\right]\right],$$

Differentiating the above equation with respect to μ, we have

$$\pi^* = \frac{1}{\sigma^2 h_1^{2\beta}}\left[k + \frac{(\eta-\vartheta)\,a}{r}\left[1-e^{r(t-T)}\right]\right],$$

Since $(\eta-\vartheta)>0, \left[1-e^{r(t-T)}\right]>0$, then $\left[k+\frac{(\eta-\vartheta)a}{r}\left[1-e^{r(t-T)}\right]\right]>0$.
Hence,

$$\frac{d\pi^*}{d\mu} = \frac{1}{\sigma^2 h_1^{2\beta}}\left[k + \frac{(\eta-\vartheta)\,a}{r}\left[1-e^{r(t-T)}\right]\right]>0,$$

Therefore,

$$\frac{d\pi^*}{d\mu} > 0.$$

(1) Recall from (49),

$$\pi^* = \frac{(\mu - r)}{\sigma^2 h_1^{2\beta}} \left[k + \frac{(\eta - \vartheta)\, a}{r} \left[1 - e^{r(t-T)}\right]\right],$$

Differentiating the above equation with respect to σ, we have

$$\frac{d\pi^*}{d\sigma} = -\frac{2\,(\mu - r)}{\sigma^3 h_1^{2\beta}} \left[k + \frac{(\eta - \vartheta)\, a}{r} \left[1 - e^{r(t-T)}\right]\right],$$

Since $\left[k + \frac{(\eta-\vartheta)a}{r}\left[1 - e^{r(t-T)}\right]\right] > 0$ and $\frac{2(\mu-r)}{\sigma^3 h_1^{2\beta}} > 0$
then,

$$\frac{d\pi^*}{d\sigma} = -\frac{2\,(\mu - r)}{\sigma^3 h_1^{2\beta}} \left[k + \frac{(\eta - \vartheta)\, a}{r} \left[1 - e^{r(t-T)}\right]\right] < 0,$$

Therefore,

$$\frac{d\pi^*}{d\sigma} < 0.$$

(1) Recall from (49),

$$\pi^* = \frac{(\mu - r)}{\sigma^2 h_1^{2\beta}} \left[k + \frac{(\eta - \vartheta)\, a}{r} \left[1 - e^{r(t-T)}\right]\right],$$

Differentiating the above equation with respect to β, we have

$$\frac{d\pi^*}{d\beta} = -\frac{2\,(\mu - r)}{\sigma^2 h_1^{2\beta+1}} \left[k + \frac{(\eta - \vartheta)\, a}{r} \left[1 - e^{r(t-T)}\right]\right],$$

Since $\left[k + \frac{(\eta-\vartheta)a}{r}\left[1 - e^{r(t-T)}\right]\right] > 0$ and $\frac{2(\mu-r)}{\sigma^2 h_1^{2\beta+1}} > 0$
then,

$$\frac{d\pi^*}{d\beta} = -\frac{2\,(\mu - r)}{\sigma^2 h_1^{2\beta+1}} \left[k + \frac{(\eta - \vartheta)\, a}{r} \left[1 - e^{r(t-T)}\right]\right] < 0,$$

Therefore,

$$\frac{d\pi^*}{d\beta} < 0.$$

which complete the proof □. □

4.2. **Numerical Simulations.** In this section, some numerical simulations are presented to study the impact of some sensitive parameters on the optimal investment plans under logarithm utility. To achieve this, the following data extracted from [20], will be use unless otherwise stated $\eta = 2$; µ $= 0.5$; $b = 1$; $T = 10 years$; $\sigma = 1$; $a = 1.5$; $r = 0.3$; $h_1 = 10$; $\vartheta = 0.5$; $k = 0.1$:

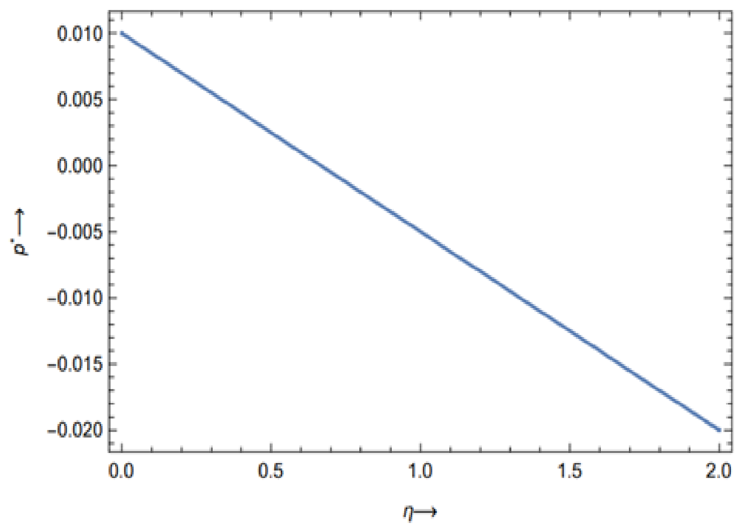

Figure 4.1: The impact of $\boldsymbol{\eta}$ **on** p^*

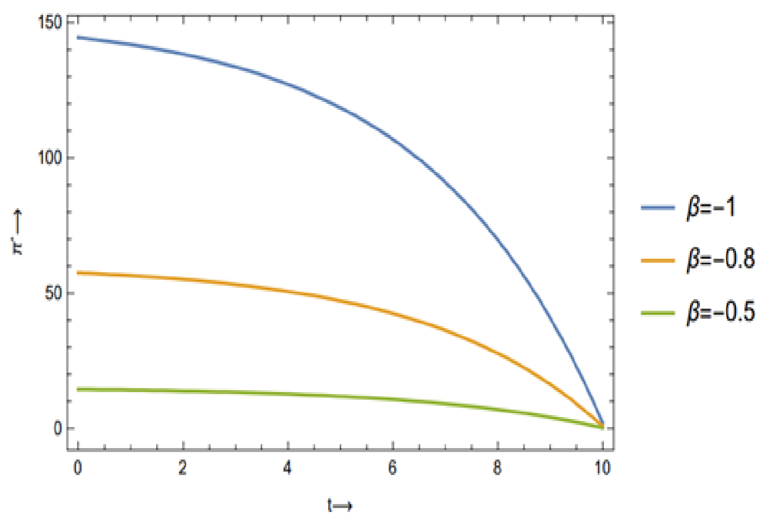

Figure 4.2: The impact of $\boldsymbol{\beta}$ **on** $\boldsymbol{\pi}^*$

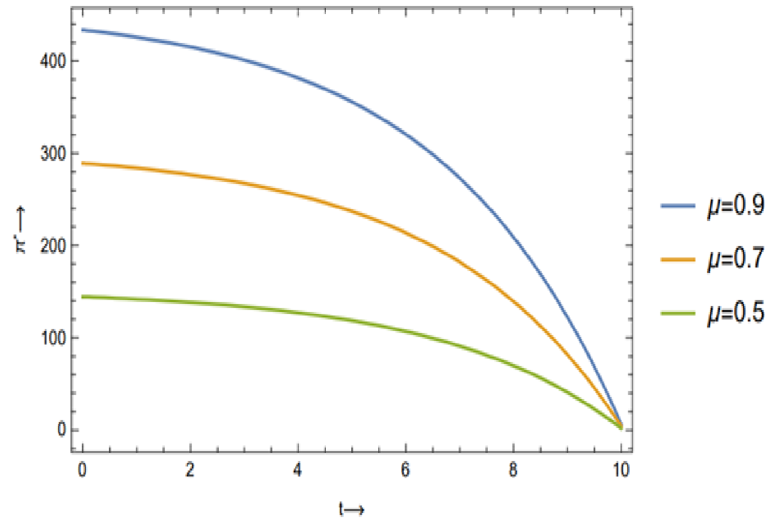

Figure 4.3: The impact of $\boldsymbol{\mu}$ **on** $\boldsymbol{\pi^*}$

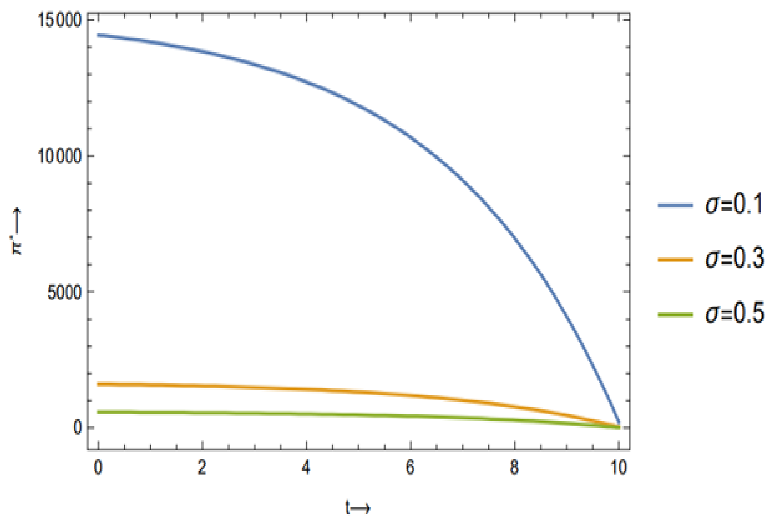

Figure 4.4: The impact of instantaneous volatility $\boldsymbol{\sigma}$ **on** $\boldsymbol{\pi^*}$

4.3. **Discussion.** Figure 4.1 presents a relationship between the optimal reinsurance strategy and the safety loading of the reinsurance policy. From Proposition 4.3 and figure 4.1, the optimal reinsurance policy is a decreasing function of the safety loading of the reinsurance policy. This implies that greater value of the safety loading (η) will lead to a smaller value of p^*. Hence to maintain a stable income, there is need for the insurer to buy fewer insurance policies.

Figure 4.2, presents simulation of π^* against time with different values of the elasticity parameterβ. The graph shows that as β reduces, the optimal portfolio strategy π^* for the risky asset increases which implies that an insurer with high risk averse coefficient will invest more in risk free asset to prevent more loss especially when the market is very volatile, this is consistent with Proposition 4.4 (ii).

Figure 4.3 present graph showing the relationship between the optimal portfolio strategy π^* and the appreciation rate μ of the risky asset. We observed that as the appreciation rate increases, the optimal portfolio strategy of the risky assets increases and vice versa. The implication here is that the insurance company will invest more in a stock where there is high appreciation rate and invest less in a stock with less appreciation rate. This can be confirmed in Proposition 4.4 (i).

Figure 4.4, presents simulation of π^* against time with different values of instantaneous volatility. The graph shows that as σ reduces, the optimal investment strategy π^* for the risky asset increases. The implication here is that highly volatile stock will discourage the insurance company from investing more of its capital in such asset, hence a reduction in the fraction to be invested in such asset and vice versa. This again, is consistent with Proposition 4.4 (iii).

Finally, in figures 4.2, 4.3, 4.4, it is observed that the optimal portfolio strategy is a decreasing function of the investment time. This implies that, as the expiration time of the investment in risky asset draw closer, the insurance company will reduce the proportion of its wealth to be invested in risky asset in order not to lose what has been gained already.

Conclusions: In conclusion, this paper focused on the determination of the optimal portfolio strategy and the optimal reinsurance strategy for insurance companies with reinsurance policy. It is required that the insurance company invest the premiums paid by its members and at the same time be willing and ready to pay claims to its members whenever there is need for claims to be paid. The basic claim process was assumed to follow the geometric Brownian motion with drift, and the insurance company can buy proportional reinsurance to enable them cope with the pressure of claims payment whenever it occurs. The insurance company were allowed to invest in a risk-free asset and a risky asset, where the price process of the risky was modelled by the constant elasticity variance (CEV) model which is a natural extension of the geometric Brownian motion. By the optimal control approach and Ito's lemma a non linear partial differential equation (PDE) known as the Hamilton Jacobi Bellman (HJB) equation was obtained. Also, we applied the Legendre transformation and dual theory method to transform the resultant non-linear partial differential equation to a linear partial differential equation. More so, the transformed equation is then solved to obtain the optimal portfolio strategy and the optimal reinsurance strategy for an insurance company with logarithm utility function exhibiting constant relative risk averse (CRRA). Finally, some numerical simulations and theoretical analysis were presented to explain the impact of model parameters on the insurance company optimal reinsurance and investment strategies. It was observed that the optimal reinsurance strategy (p^*) is a decreasing function of the safety loading of the reinsurance policy. Also, the optimal portfolio strategy for the risky asset was observed to be a decreasing function of the elasticity parameter (β), instantaneous volatility (σ) and the investment time (t) while the optimal investment strategy for the risky asset was observed to be an increasing function of the appreciation rate of the of the risky asset (μ).

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