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Development of a New One step Scheme for the Solution of Initial Value Problems in Ordinary Differential Equations

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Abstract

This paper presents the development and implementation of an explicit rational one step method of order four for solving initial value problems of ordinary differential equations. The scheme has been derived through the combination of two interpolants namely, polynomial and rational transcendental form of exponential and trigonometric functions. The method was used for the solution of four initial values problems in which two of them are nonlinear IVPs. The numerical results showed that the new scheme is consistent with the initial value, robust and efficient.

1. Introduction

Differential equation is one of the most important tools in the field of mathematics because it has useful and great applications in forming models in almost all areas of human endeavors, from physical sciences, biological sciences, social sciences and engineering just to mention but a few ([3] - [5], [10], [14] - [16]).
Such mathematical formulation of physical phenomena in these fields often lead to initial value problems of the form:

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$$y' = f(x, y),\ \ y(a) = a \tag{1}$$

It is also a known fact that most of the models or relationships that emanates from these areas of life situations are mostly not amenable to classical methods of solution which makes the exact solutions difficult to obtain or not available, thereby we turn to numerical approximate method of solutions. Numerical method is a complete and unambiguous set of procedures for the solution of a problem together with computable error estimates [6]. This has attracted the attention of many researchers into development many numerical methods which have performed excellently in providing accurate solutions to initial value problems in ordinary differential equations ([6], [7], [11] - [15], [16]).

2. Development of the new scheme

The solution to equation (1) is found at a set of discretized points in the continuous interval $[a, b$, let y_n represent the numerical estimate of the theoretical value $y(x_n)$ and f_n represent $f(x_n, y_n)$.
We assume that the theoretical solution $y(x)$ to the initial value problem (1) can be locally represented in the interval $[x_n, x_{n+1}]$ by the interpolating function of the form

$$F(x) = \sum_{j=0}^{2} a_j x^j + a_3 \left(\frac{e^x}{\cos x} \right) \tag{2}$$

where a_0, a_1, a_2, a_3, are undetermined constants. The integration interval of $[a, b]$ is defined as $a = x_0 \leq x \leq x_n = b$
The step length is defined as

$$h = \frac{b-a}{N} \tag{3}$$

We define the mesh points as

$$\begin{cases} x_n = a + nh,\ \ n = 0, 1, 2, 3, ..., N \\ \\ x_{n+1} = a + (n+1)h,\ \ n = 0, 1, 2, 3, ..., N \end{cases} \tag{4}$$

We shall make the following assumptions:

(1) That the interpolating function coincides with the theoretical solution using equation (3) so that we have

$$F(x_n) = \sum_{j=0}^{3} a_j x_n^j + a_3 \left(\frac{e^{x_n}}{\cos x_n} \right) \tag{5}$$

and

$$F(x_{n+1}) = \sum_{j=0}^{3} a_j x_{n+1}^{j} + a_3 \left(\frac{e^{x_{n+1}}}{\cos x_{n+1}} \right) \tag{6}$$

(1) That the first, second and third derivatives of the interpolating function with respect to x coincide with the differential equation respectively, as well as its first and second derivatives with respect to x at $x = x_n$. That is,

$$F^{'}(x_n) = f_n;\; F^{''}(x_n) = f_n^{(1)};\; F^{'''}(x_n) = f_n^{(2)} \tag{7}$$

This yields

$$f_n = \sum_{j=1}^{3} j a_j x_n^{j-1} \tag{8}$$

$$f_n^1 = \sum_{j=2}^{3} j(j-1) a_j x_n^{j-2} \tag{9}$$

$$f_n^2 = \sum_{j=3}^{3} j(j-2) a_j x_n^{j-3} \tag{10}$$

Representing equations (8) - (10) as a system of linear equations $AX = b$ we have a matrix of the form:

(11)

$$\begin{pmatrix} 1 & 2x_n & \frac{e^{x_n}}{\cos(x_n)} + \frac{\sin(x_n)e^{x_n}}{\cos^2(x_n)} \\ & \frac{e^{x_n}}{\cos(x_n)} & \frac{2\sin(x_n)e^{x_n}}{\cos^2(x_n)} + \frac{2\sin^2(x_n)e^{x_n}}{\cos^3(x_n)} \\ & & \frac{4e^{x_n}}{\cos(x_n)} + \frac{8\sin(x_n)e^{x_n}}{\cos^2(x_n)} + \frac{6\sin^3(x_n)e^{x_n}}{\cos^3(x_n)} + \frac{6\sin^3(x_n)e^{x_n}}{\cos^2(x_n)} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} f_n \\ f_n^1 \\ f_n^2 \end{pmatrix}$$

where

$$A = \begin{bmatrix} 1 & 2x_n & \frac{e^{x_n}}{\cos(x_n)} + \frac{\sin(x_n)e^{x_n}}{\cos^2(x_n)} \\ & \frac{e^{x_n}}{\cos(x_n)} & \frac{2\sin(x_n)e^{x_n}}{\cos^2(x_n)} + \frac{2\sin^2(x_n)e^{x_n}}{\cos^3(x_n)} \\ & & \frac{4e^{x_n}}{\cos(x_n)} + \frac{8\sin(x_n)e^{x_n}}{\cos^2(x_n)} + \frac{6\sin^3(x_n)e^{x_n}}{\cos^3(x_n)} + \frac{6\sin^3(x_n)e^{x_n}}{\cos^2(x_n)} \end{bmatrix}$$

$$X = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$b = \begin{pmatrix} f_n \\ f_n^1 \\ f_n^2 \end{pmatrix}$$

Solving (11) using Gaussian Elimination method we obtain

$$a_1 = \frac{\begin{pmatrix} -4\cos^3(x_n) f_n^{(1)} x_n + 2\cos^3(x_n) f_n^{(2)} x_n - \\ 8\cos^2(x_n)\sin(x_n) f_n^{(1)} x_n + 2\cos^2(x_n)\sin(x_n) f_n^{(2)} x_n \\ -6\cos(x_n)\sin^2(x_n) f_n^{(1)} x_n + 2\cos(x_n)\sin^2(x_n) f_n^{(2)} x_n \\ -6\sin^2(x_n) f_n^{(1)} x_n + 4\cos^2(x_n) f_n - \cos^2(x_n) f_n^{(2)} \\ +8\cos^2(x_n)\sin(x_n) f_n - \cos^2(x_n)\sin(x_n) f_n^{(2)} + \\ 6\cos(x_n)\sin^2(x_n) f_n + 6\sin^2(x_n) f_n \end{pmatrix}}{2\begin{pmatrix} 2\cos^2(x_n) + 4\cos^2(x_n)\sin x_n + \\ 3\cos(x_n)\sin^2(x_n) + 3\sin^3(x_n) \end{pmatrix}} \tag{12}$$

$$a_2 = \frac{\begin{pmatrix} 2\cos^3(x_n) f_n^{(1)} - \cos^3(x_n) f_n^{(2)} + 4\cos^2(x_n)\sin(x_n) f_n^{(1)} \\ -\cos^2(x_n)\sin(x_n) f_n^{(2)} + 3\cos(x_n)\sin^2(x_n) f_n^{(1)} \\ -\cos(x_n)\sin^2(x_n) f_n^{(2)} + 3\sin^3(x_n) f_n^{(1)} \end{pmatrix}}{2\begin{pmatrix} 2\cos^3(x_n) + 4\cos^2(x_n)\sin x_n + \\ 3\cos(x_n)\sin^2(x_n) + 3\sin^3(x_n) \end{pmatrix}} \tag{13}$$

$$a_3 = \frac{\left(f_n^{(2)}\cos^4(x_n)\right)}{2e^{x_n}\begin{pmatrix} 2\cos^3(x_n) + 4\cos^2(x_n)\sin x_n + \\ 3\cos(x_n)\sin^2(x_n) + 3\sin^3(x_n) \end{pmatrix}} \tag{14}$$

Subtracting equation (5) from (6) yields

$$y_{n+1} - y_n = F(x_{n+1}) + F(x_n) = \sum_{j=1}^{3} a_j (x_{n+1}^j - x_n^j) \tag{15}$$

Since it is a one step method equation (15) yields

$$y_{n+1} - y_n = F(x_{n+1}) + F(x_n)$$

$$= a_1 h + a_2(h^2 + 2x_n h) + a_3 e^{x_n}\left(\frac{e^h + \tan(x_n)\sin(h) - \cos(h)}{\cos(x_n)\cos(h) - \sin(x_n)\sin(h)}\right) \tag{16}$$

Substituting the values in equations (12) - (14) into (16) we have our one step scheme as follows:

$$y_{n+1} - y_n = \frac{h\left(\begin{array}{c}\begin{pmatrix}4\cos^3(x_n) + 8\cos^2(x_n)\sin x_n + \\ 6\cos(x_n)\sin^2(x_n) + 6\sin^3(x_n)\end{pmatrix} f_n + \\ \begin{pmatrix}-6\sin^3(x_n)x_n - 6\sin^2(x_n)\cos(x_n)x_n - \\ 8\sin(x_n)\cos^2(x_n)x_n - 4\cos^3(x_n)x_n\end{pmatrix} f_n^{(1)} + \\ \begin{pmatrix}2\sin^2(x_n)\cos(x_n)x_n + 2\sin(x_n)\cos^2(x_n)x_n + \\ 2\cos^3(x_n)x_n - \cos^2(x_n)\sin(x_n) - \cos^3(x_n)\end{pmatrix} f_n^{(2)}\end{array}\right)}{\begin{pmatrix}4\cos^3(x_n) + 8\cos^2(x_n)\sin x_n + \\ 6\cos(x_n)\sin^2(x_n) + 6\sin^3(x_n)\end{pmatrix}} + \tag{17}$$

$$\frac{h^2\left(\begin{array}{c}\begin{pmatrix}3\sin^3(x_n) + 6\sin^3(x_n)\frac{1}{h}x_n + 3\sin^2(x_n)\cos(x_n) + \\ 6\sin^2(x_n)\cos(x_n)\frac{1}{h}x_n + 4\sin(x_n)\cos^2(x_n) + \\ 8\sin(x_n)\cos^2(x_n)\frac{1}{h}x_n + 2\cos^3(x_n) + 4\cos^3(x_n)\frac{1}{h}x_n\end{pmatrix} f_n^{(1)} \\ + \begin{pmatrix}-\sin^2(x_n)\cos(x_n) - 2\sin^2(x_n)\cos(x_n)\frac{1}{h}x_n - \\ \sin(x_n)\cos^2(x_n) - 2\sin(x_n)\cos^2(x_n)\frac{1}{h}x_n - \\ \cos^3(x_n) - 2\cos^3(x_n)\frac{1}{h}x_n\end{pmatrix} f_n^{(2)}\end{array}\right)}{\begin{pmatrix}4\cos^3(x_n) + 8\cos^2(x_n)\sin(x_n) + \\ 6\cos(x_n)\sin^2(x_n) + 6\sin^3(x_n)\end{pmatrix}} +$$

$$\frac{1}{2}\left(\frac{\cos^4(x_n)(e^h + \tan(x_n)\sin(h) - \cos(h))}{\begin{pmatrix}2\cos^3(x_n) + 4\cos^2(x_n)\sin(x_n) + 3\cos(x_n)\sin^2(x_n) + \\ 3\sin^3(x_n)(\cos(x_n)\cos(h) - \sin(x_n)\sin(h))\end{pmatrix}}\right) f_n^{(2)}$$

If we let

$$K_1 = \frac{\left(\begin{array}{c}\begin{pmatrix}4\cos^3(x_n) + 8\cos^2(x_n)\sin x_n + \\ 6\cos(x_n)\sin^2(x_n) + 6\sin^3(x_n)\end{pmatrix} f_n + \\ \begin{pmatrix}-6\sin^3(x_n)x_n - 6\sin^2(x_n)\cos(x_n)x_n - \\ 8\sin(x_n)\cos^2(x_n)x_n - 4\cos^3(x_n)x_n\end{pmatrix} f_n^{(1)} + \\ \begin{pmatrix}2\sin^2(x_n)\cos(x_n)x_n + 2\sin(x_n)\cos^2(x_n)x_n + \\ 2\cos^3(x_n)x_n - \cos^2(x_n)\sin(x_n) - \cos^3(x_n)\end{pmatrix} f_n^{(2)}\end{array}\right)}{\begin{pmatrix}4\cos^3(x_n) + 8\cos^2(x_n)\sin x_n + \\ 6\cos(x_n)\sin^2(x_n) + 6\sin^3(x_n)\end{pmatrix}} + \tag{18}$$

$$K_2 = \frac{\left(\begin{array}{c}\begin{pmatrix}3\sin^3(x_n) + 6\sin^3(x_n)\frac{1}{h}x_n + 3\sin^2(x_n)\cos(x_n) + \\ 6\sin^2(x_n)\cos(x_n)\frac{1}{h}x_n + 4\sin(x_n)\cos^2(x_n) + \\ 8\sin(x_n)\cos^2(x_n)\frac{1}{h}x_n + 2\cos^3(x_n) + 4\cos^3(x_n)\frac{1}{h}x_n\end{pmatrix} f_n^{(1)} \\ + \begin{pmatrix}-\sin^2(x_n)\cos(x_n) - 2\sin^2(x_n)\cos(x_n)\frac{1}{h}x_n - \\ \sin(x_n)\cos^2(x_n) - 2\sin(x_n)\cos^2(x_n)\frac{1}{h}x_n - \\ \cos^3(x_n) - 2\cos^3(x_n)\frac{1}{h}x_n\end{pmatrix} f_n^{(2)}\end{array}\right)}{\begin{pmatrix}4\cos^3(x_n) + 8\cos^2(x_n)\sin(x_n) + \\ 6\cos(x_n)\sin^2(x_n) + 6\sin^3(x_n)\end{pmatrix}} \tag{19}$$

$$K_3 = \left(\frac{\cos^4(x_n)(e^h + \tan(x_n)\sin(h) - \cos(h))}{\begin{pmatrix} 2\cos^3(x_n) + 4\cos^2(x_n)\sin(x_n) + 3\cos(x_n)\sin^2(x_n) + \\ 3\sin^3(x_n)(\cos(x_n)\cos(h) - \sin(x_n)\sin(h)) \end{pmatrix}} \right) f_n^{(2)} \tag{20}$$

$$y_{n+1} = y_n + \frac{1}{2}h[2K_1 + 2hK_2 + \frac{1}{h}K_3] \tag{21}$$

Equation (21) is the required one step scheme.

3. Basic Concepts [7, 9]

In this section, we will state some concepts which are very important in the development of the new scheme.

3.1. **Stability.** A numerical method is said to be stable if the difference between the numerical solution and the theoretical solution can be made as small as possible, that is if there exists two positive numbers e_0 and K such that the following holds:

$$||y_n - y(x_n)|| \leq K||e_0|| \tag{22}$$

3.2. **Consistency.** A numerical method with an increment function $\phi(x_n, y_n; h)$ is said to be consistent with the initial value problem (1) if we have

$$\phi(x_n, y_n; 0) = f(x, y) \tag{23}$$

3.3. **Convergence.** A numerical method is said to be convergent if for all initial value problems satisfying the hypothesis of the Lipschitz condition given by:

$$|f(x, y) - f(x, y^*)| \leq L|y - y^*| \tag{24}$$

L is the Lipschitz constant and it is denoted by $L = \max|f_y(x, y)|$. The necessary and sufficient conditions for convergence is both the stability and consistency.

3.4. **Round off Error.** The round off error in any numerical computation can be written mathematically as

$$R_{n+1} = y_{n+1} - p_{n+1} \tag{25}$$

where y_{n+1} is the approximate solution and p_{n+1} is the computer output. The magnitude of the error depends on the storage capacity of the device being used, for a reliable output a machine with double precision is needed for the approximation.

4. Numerical Experiments and Discussion of Results

Problem 1.
Consider the non-linear initial value problem $y' = \frac{y^3}{2}$, $y(0) = 1$ [14, 15] with theoretical solution $y(x) = 1\frac{1}{\sqrt{1+x}}$, Step size h is 0.05.
Table 1 is the comparative analysis of scheme (17) with the theoretical solution for problem 1.

Table 1. Comparison of percentages.

n	x_n	$y_n(Scheme2.16)$	$y(x_n)$	T_{n+1}
1.0	0.000	1.0000000e+00	1.0000000e+00	0.0000000e+00
2.0	0.050	9.7589693e-01	9.7590007e-01	3.1457838e-06
3.0	0.100	9.5347756e-01	9.5346259e-01	1.4972860e-05
4.0	0.150	9.3255271e-01	9.3250481e-01	4.7903161e-05
5.0	0.200	9.1296189e-01	9.1287093e-01	9.0958996e-05
6.0	0.250	8.9456791e-01	8.9442719e-01	1.4071965e-04
7.0	0.300	8.7725270e-01	8.7705802e-01	1.9468003e-04
8.0	0.350	8.6091397e-01	8.6066297e-01	2.5100339e-04
9.0	0.400	8.4546260e-01	8.4515425e-01	3.0834458e-04
10.0	0.450	8.3082052e-01	8.3045480e-01	3.6572234e-04
11.0	0.500	8.1691901e-01	8.1649658e-01	4.2242622e-04

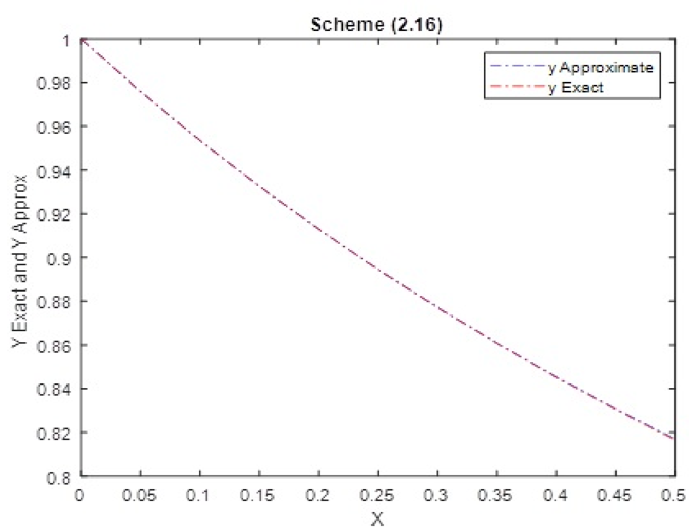

Figure 1: Comparison of the Scheme (17) and the Exact solution for problem 1.

Problem 2

Consider the mildly stiff initial value problem $y^{'} = -30y\ y(0) = 1$, with theoretical solution $y(x) = e^{-30}$ in interval $[0, 1], r$ the step size $h = 0.01$.

Table 2 is the comparative analysis of scheme (17) with the theoretical solution for Problem 2.

TABLE 2. Comparison of percentages.

N	x_n	$y_n(Scheme\ 2.16)$	$y(x_n)$	T_{n+1}
1.0	0.000	1.0000000e+00	1.0000000e+00	0.0000000e+00
2.0	0.010	7.4046625e-01	7.4081822e-01	3.5197068e-04
3.0	0.020	5.4829027e-01	5.4881164e-01	5.2136870e-04
4.0	0.030	4.0599044e-01	4.0656966e-01	5.7922154e-04
5.0	0.040	3.0062222e-01	3.0119421e-01	5.7199460e-04
6.0	0.050	2.2260061e-01	2.2313016e-01	5.2955423e-04
7.0	0.060	1.6482824e-01	1.6529889e-01	4.7065231e-04
8.0	0.070	1.2204975e-01	1.2245643e-01	4.0668251e-04
9.0	0.080	9.0373718e-02	9.0717953e-02	3.4423575e-04
10.0	0.090	6.6918688e-02	6.7205513e-02	2.8682501e-04
11.0	0.100	4.9551030e-02	4.9787068e-02	2.3603861e-04

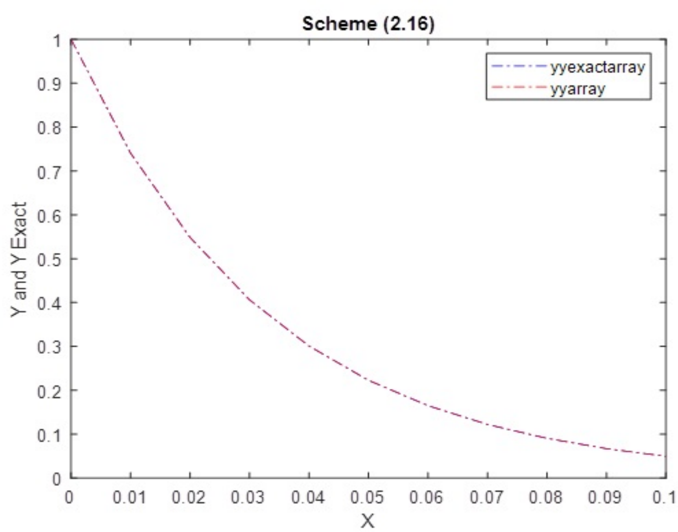

Figure 2: Comparison of the Scheme (17) and the Exact solution for problem 2.

Problem 3

Consider the initial value problem $y' = x + y,\ y(0) = 1$ and $y(x) = 2 * e^x - x - 1$; in interval $[0, 1]$, Step size $h = 0.05$.

Table 3. is the comparative analysis of scheme (17) with the theoretical solution for Problem 3.

TABLE 3. Comparison of percentages.

N	x_n	$y_n(Scheme\ 2.16)$	$y(x_n)$	T_{n+1}
1.0	0.000	1.0000000e+00	1.0000000e+00	0.0000000e+00
2.0	0.050	1.0512710e+00	1.0525422e+00	1.2712045e-03
3.0	0.100	1.1077342e+00	1.1103418e+00	2.6075866e-03
4.0	0.150	1.1696560e+00	1.1736685e+00	4.0124895e-03
5.0	0.200	1.2373161e+00	1.2428055e+00	5.4894285e-03
6.0	0.250	1.3110087e+00	1.3180508e+00	7.0421002e-03
7.0	0.300	1.3910432e+00	1.3997176e+00	8.6743937e-03
8.0	0.350	1.4777447e+00	1.4881351e+00	1.0390403e-02
9.0	0.400	1.5714549e+00	1.5836494e+00	1.2194448e-02
10.0	0.450	1.6725332e+00	1.6866244e+00	1.4091144e-02
11.0	0.500	1.7813556e+00	1.7974425e+00	1.6086987e-02

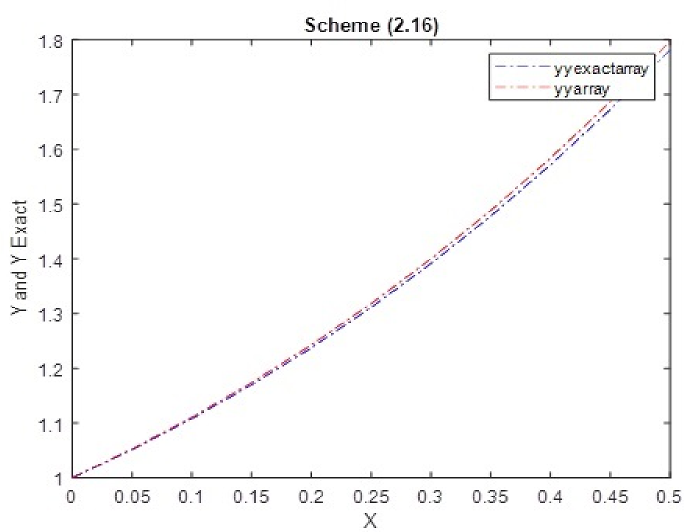

Figure 3: Comparison of the Scheme (17) and the Exact solution for problem 3.

Problem 4
Consider the Riccati initial value problem $y^{'} = 1 + y^2,\ y(0) = 0$, with exact solution $y(x) = \tan(x)$, Step size $h = 0.1$.

Table 4. is the comparative analysis of scheme (17) with [13] and the theoretical solution for Problem 4.

TABLE 4. Comparison of percentages.

N	x_n	$y_n(Scheme\ 2.16)$	[13]	$y(x_n)$	T_{n+1}
1.0	0.000	0.0000000e+00	0.000000000000000	0.0000000e+00	0.0000000e+00
2.0	0.100	1.0035994e-01	0.100334672085451	1.0033467e-01	2.5266803e-05
3.0	0.200	2.0275548e-01	0.202710035508470	2.0271004e-01	4.5446459e-05
4.0	0.300	3.0939959e-01	0.309336249567961	3.0933625e-01	6.3336523e-05
5.0	0.400	4.2287746e-01	0.422790712339784	4.2279322e-01	8.4243633e-05
6.0	0.500	5.4642406e-01	0.546302451536160	5.4630249e-01	1.2156595e-04
7.0	0.600	6.8434715e-01	0.684136340501477	6.8413681e-01	2.1034559e-04
8.0	0.700	8.4273164e-01	0.842284292051472	8.4228838e-01	4.4325944e-04
9.0	0.800	1.0307117e+00	1.029610217610683	1.0296386e+00	1.0731557e-03
10.0	0.900	1.2629892e+00	1.259991097538113	1.2601582e+00	2.8310182e-03
11.0	1.000	1.5654375e+00	1.556523840478488	1.5574077e+00	8.0297304e-03

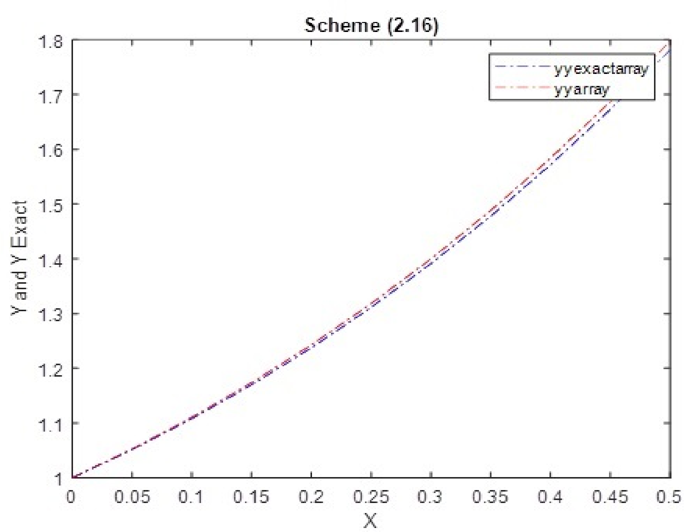

Figure 4: Comparison of the Scheme (17) and the Exact solution for problem 4.

Discussion of Results: The numerical experiment shows that the newly developed scheme is simple to implement and the result obtained for the problems presented compares favorably with their theoretical solutions respectively, only problem 4 is compared with [13] and the result also shows that it compares well with the existing method. [8] was used to perform the numerical computation and to plot the graphs.

Conclusion: In this paper we have developed a new scheme for the solution of initial value problems in ODEs, the scheme was used to solve numerically four problems. Comparative analysis of the results obtained from numerical experiments were also presented in Tables 1-4 and Figures 1-4. The numerical experiment showed that the numerical scheme is efficient and compares favorably with the theoretical solutions, also Problem 4 also compared well with [13]. Hence, the new scheme is efficient computationally and easy to use.

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