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A New Numerical Algorithm for the Solution of General Second Order Ordinary Differential Equations

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Abstract

This paper contains the formulation of an algorithm for solving general second order initial value problems of ordinary differential equations. This method is basically a continuous linear multistep (clmm) block method obtained from the collocation and interpolation of a functional as basis function. Its implementation was on the evaluation of the main method and two additional methods of the block matrix. It is proved that the algorithm is consistent, zero-stable and convergent. Errors obtained using uniform step lengths were also investigated and presented Numerical examples are provided to show the efficiency and suitability of the algorithm.

1. Introduction

The general second order ordinary differential equation has the form:

$$y'' = f\left(x, y, y'\right),\ y\left(a\right) = b_o, y'\left(a\right) = b_1 \tag{1}$$

The equation (1) has numerous applications in physics, in electrical and mechanical engineering. For instance the motion of the mass on spring and the electric circuit problems are all formulated using second order differential equations.
The numerical integration of (1) has become of interest in recent time due to its importance, as such various numerical integral formulas have been proposed

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for its solution. It has been reported that the direct method rather than the method of reduction is more efficient. Direct methods have been proposed by many authors [22], [15], [4], [7], Other direct methods that have been developed are the hybrid methods in which some off-grid points are included; [19], [26], [23], [25] & [27]. Direct methods proposed included the predictor-corrector methods, [22] & [15], and [4], [25] & [27]. Although the predictor-corrector yielded good results; but the implementation is very costly due to the requirement of special techniques for the supply of starting values, [24], [25] & [2]. To provide for this set back, block methods, a more efficient method capable of simultaneous generation of numerical results at all selected grid points was introduced [3], [21], [12], [17], [19], [24]. [20] developed a one-step hybrid method with a single off-step point for the direct solution of general second order ordinary differential equations.
In this work, we propose a two-step continuous linear multistep method with a single off-step point for the direct solution of general second order initial valued problems. This paper is organized as follows: Section 2 discusses the derivation of the method, section 3 discusses the analysis of the stability properties, sections 4 and 5 discusses the numerical experiments and discussion of results respectively.

2. Derivation of the Method

In this section we shall derived a $k-$step multistep method of the form:

$$y_{n+k} + \alpha_v y_{n+v} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \beta_v f_{n+v} \tag{2}$$

Let the exact solution of (1) be $Y(x)$ in the range $[x_n, x_{n+h}]$. We shall represent the exact solution by the approximation solution of the form

$$Y(x) = \sum_{j=0}^{k+4} a_j x^j \tag{3}$$

where the $a_j' s$ are the coefficient to be determined. We shall construct our method by imposing the following conditions on (4), we have

$$Y(x_n) = y_n$$

$$Y'(x) = \sum_{j=1}^{k+4} j a_j x^{j-1} = f_{n+j},\ j = 0(v)k \tag{4}$$

$$f(x, y, y') = \sum_{j=2}^{k+4} j(j-1) a_j x^{j-2} \tag{5}$$

Equations (4) & (5) leads to a system of non-linear equations of the form:

$$AX = U \tag{6}$$

where $X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_{n+\frac{1}{2}} & x_{n+\frac{1}{2}}^2 & x_{n+\frac{1}{2}}^3 & x_{n+\frac{1}{2}}^4 & x_{n+\frac{1}{2}}^5 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x_{n+\frac{1}{2}}^2 & 20x_{n+\frac{1}{2}}^3 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^2 \end{bmatrix}$

$$A = [a_0, a_1, a_2, a_3, a_4, a_5]^T, \; B = \left[y_n, y_{n+\frac{1}{2}}, f_n, f_{n+\frac{1}{2}}, f_{n+1}, f_{n+2}\right]^T$$

Solving equation (6) to obtain the $a_i's, i = 0, 1, 2, ..., 5$ and substituting in equation (3) we have the continuous linear multistep method (clmm) given as:

$$Y(x) = \sum_{j=0}^{k-2} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(x) f_{n+j} + h^2 \beta_v(x) f_{n+v} \tag{7}$$

where α_j and β_j are the continuous coefficients given as:

$$\begin{aligned}
&\alpha_0(t) = 1 - 2t \\
&\alpha_{1/2}(t) = 2t \\
&\beta_0(t) = h^2\left(-\tfrac{1}{20}t^5 + \tfrac{7}{24}t^4 - \tfrac{7}{12}t^3 + \tfrac{1}{2}t^2 - \tfrac{11}{80}t\right) \\
&\beta_{1/2}(t) = h^2\left(\tfrac{2}{15}t^5 - \tfrac{2}{3}t^4 + \tfrac{8}{9}t^3 - \tfrac{53}{360}t\right) \\
&\beta_1(t) = h^2\left(-\tfrac{1}{10}t^5 + \tfrac{5}{12}t^4 - \tfrac{1}{3}t^3 - \tfrac{3}{80}t\right) \\
&\beta_2(t) = h^2\left(\tfrac{1}{60}t^5 - \tfrac{1}{24}t^4 + \tfrac{1}{36}t^3 - \tfrac{1}{360}t\right)
\end{aligned}$$

The derivative of (7) is given by

$$Y'(x) = \sum_{j=0}^{k-2} \alpha_j'(x) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j'(x) f_{n+j} + h^2 \beta_v'(x) f_{n+v} \tag{8}$$

with the coefficients given as

$$\begin{aligned}
&\alpha_0' = \tfrac{-2y_n}{h}, \; \alpha_{1/2}'(t) = \frac{2y_{n+1/2}}{h} \\
&\beta_0'(t) = h\left(-\tfrac{1}{4}t^4 + \tfrac{7}{6}t^3 - \tfrac{7}{4}t^2 + t - \tfrac{11}{80}\right) \\
&\beta_{1/2}'(t) = h\left(\tfrac{2}{3}t^4 - \tfrac{8}{3}t^3 + \tfrac{8}{3}t^2 - \tfrac{53}{360}\right) \\
&\beta_1'(t) = h\left(-\tfrac{1}{2}t^4 + \tfrac{5}{3}t^3 - t^2 - \tfrac{3}{80}\right) \\
&\beta_2'(t) = h\left(\tfrac{1}{12}t^4 - \tfrac{1}{6}t^3 - \tfrac{1}{12}t^2 - \tfrac{1}{360}\right)
\end{aligned}$$

Evaluating (7) and (8) at $t = 1$ and 2 yield the following discrete methods

$$(9)\quad \left.\begin{aligned} y_{n+1} &= -y_n + 2y_{n+\frac{1}{2}} + h^2\left(\tfrac{1}{48}f_n + \tfrac{5}{24}f_{n+\frac{1}{2}} + \tfrac{1}{48}f_{n+1}\right) \\ y_{n+2} &= -3y_n + 4y_{n+\frac{1}{2}} + h^2\left(\tfrac{1}{8}f_n + \tfrac{5}{12}f_{n+\frac{1}{2}} + \tfrac{7}{8}f_{n+1} + \tfrac{1}{12}f_{n+2}\right) \end{aligned}\right\}$$

The derivatives are given as:

(10)

$$\left.\begin{aligned} 720y'_{n+1}h &= -1440y_n + 1440y_{n+\frac{1}{2}} + h^2\left(21f_n + 374f_{n+\frac{1}{2}} + 147f_{n+1} - 2f_{n+2}\right) \\ 720y'_{n+2}h &= -1440y_n + 1440y_{n+\frac{1}{2}} + h^2\left(141f_n - 106f_{n+\frac{1}{2}} + 987f_{n+1} + 238f_{n+2}\right) \\ 2880y'_{n+1}h &= -5760y_n + 5760y_{n+\frac{1}{2}} + h^2\left(159f_n + 656f_{n+\frac{1}{2}} - 102f_{n+1} + 7f_{n+2}\right) \end{aligned}\right\}$$

Considering the methods (9) and (10) we shall define the general k block , r-point block method as follows:

$$(11)\quad Y_\alpha = \sum_{i=1}^{k} A_i Y_{\alpha-i} + h\sum_{i=1}^{k} B_i F_{\alpha-i}$$

where, $Y_\alpha = (y_n, y_{n+1}, y_{n+2}, \cdots, y_{n+r-1})$, $\mathrm{F}_\alpha = (f_n, f_{n+1}, f_{n+2}, \cdots, f_{n+r-1})$. The $A_i's$ and are $r \times r$ matrices of coefficient.

Using (11), we formulate our block form of the new method to be

$$(12)\quad h^\lambda \bar{a} Y_\alpha = h^\lambda \bar{e} y_\alpha + h^{\lambda-u}\left[\bar{d}f\left(y_m\right) + \bar{b}F\left(Y_\alpha\right)\right]$$

The normalization of the above vector equation (12) yield

$$(13)\quad AY_\alpha = h^\lambda E y_\alpha + h^{\mu-\lambda}\left[Df(y_\alpha) + BF\left(Y_\alpha\right)\right]$$

Using (13) the block solution is as given below:

$$(14)\quad \begin{bmatrix} 1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1 \end{bmatrix}\begin{bmatrix} y_{n+\frac{1}{2}}\\ y_{n+1}\\ y_{n+2}\\ y'_{n+\frac{1}{2}}\\ y'_{n+1}\\ y'_{n+2}\end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2}h\\ 1 & h\\ 1 & 2h\\ 1 & h\\ 1 & h\\ 1 & h\end{bmatrix}\begin{pmatrix} y_n\\ y'_n\end{pmatrix} +$$

$$\begin{bmatrix} \frac{11}{160}h^2\\ \frac{19}{120}h^2\\ \frac{2}{5}h^2\\ \frac{37}{192}h^2\\ \frac{1}{6}h^2\\ \frac{1}{3}h^2\end{bmatrix}(f_n) + \begin{bmatrix} \frac{53}{720}h^2 & -\frac{3}{160}h^2 & \frac{1}{720}h^2\\ \frac{16}{45}h^2 & -\frac{1}{60}h^2 & \frac{1}{360}h^2\\ \frac{32}{45}h^2 & \frac{4}{5}h^2 & \frac{4}{45}h^2\end{bmatrix}\begin{pmatrix} f_{n+\frac{1}{2}}\\ f_{n+1}\\ f_{n+2}\end{pmatrix}$$

The above can be written as:

$$
(15)\quad
\begin{aligned}
y_{n+\frac{1}{2}} &= \tfrac{11}{160}h^2 f_n - \tfrac{3}{160}h^2 f_{n+1} + \tfrac{53}{720}h^2 f_{n+\frac{1}{2}} + \tfrac{1}{720}h^2 f_{n+2} + y_n + \tfrac{1}{2}hy_n^{'}\\
y_{n+1} &= \tfrac{19}{120}h^2 f_n - \tfrac{1}{60}h^2 f_{n+1} + \tfrac{16}{45}h^2 f_{n+\frac{1}{2}} + \tfrac{1}{360}h^2 f_{n+2} + y_n + hy_n^{'}\\
y_{n+2} &= \tfrac{2}{5}h^2 f_n + \tfrac{4}{5}h^2 f_{n+1} + \tfrac{32}{45}h^2 f_{n+\frac{1}{2}} + \tfrac{4}{45}h^2 f_{n+2} + y_n + 2hy_n^{'}\\
y_{n+\frac{1}{2}}^{'} &= \tfrac{37}{192}h^2 f_n - \tfrac{7}{96}h^2 f_{n+1} + \tfrac{3}{8}h^2 f_{n+\frac{1}{2}} + \tfrac{1}{192}h^2 f_{n+2} + hy_n^{'}\\
y_{n+1}^{'} &= \tfrac{1}{6}h^2 f_n + \tfrac{1}{6}h^2 f_{n+1} + \tfrac{2}{3}h^2 f_{n+\frac{1}{2}} + hy_n^{'}\\
y_{n+2}^{'} &= \tfrac{1}{3}h^2 f_n + \tfrac{4}{3}h^2 f_{n+1} + \tfrac{1}{3}h^2 f_{n+2} + hy_n^{'}
\end{aligned}
$$

3. Analysis of the new method

The new method in respect of order and error constant, consistency and zero stability are analyzed in this section. The region of absolute stability of the method will also be discussed.

3.1. **Order and Error constant.** The general k-step linear multistep method is given by

$$(16)\qquad \sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j}$$

where $f_{n+j} = f\left(x_{n+j}, y_{n+j}, y_{n+j}^{'}\right)$
The linear operator associated with (16) is defined as:

$$(17)\qquad L\left\{y\left(x\right);h\right\} = \sum \left[\alpha_j y\left(x+jh\right) - h^2 \beta_j y''\left(x+jh\right)\right]$$

By expanding (16) in Taylor's series and comparing the coefficients in terms of the powers of h, we have

(18)
$$L\left\{y\left(x\right);h\right\} = c_0 y\left(x\right) + c_1 hy\left(x\right) + ... + c_p h^p y^p\left(x\right) + c_{p+1}h^{p+1}y^{p+1}\left(x\right) + c_{p+2}h^{p+2}y^{p+2}\left(x\right) + ...$$

Definition 3.1. ([18]): The linear k-step method (2) is said to be order p if in (18), we have

$$(19)\qquad c_0 = c_1 = c_2 = ... = c_{p+1} = 0, c_{p+2} \neq 0$$

The truncation error of the k-step method (2) of order p is therefore given as:

$$(20)\qquad t_{n+1} = c_{p+2}h^{p+2}y^{p+2}\left(x_n\right)$$

Thus the order of the method is the largest integer p for which

$$(21)\qquad \left|\frac{1}{h^2}t_{n+1}\right| = O\left|h^p\right|$$

Expanding equation (14) in Taylor series and consider the block method in (11) as follows:

(22)
$$\left(\begin{array}{l}
\sum_{j=0}^{\infty} \frac{\left(1/2\right)^j}{j!} y_n^j - y_n - \frac{1}{2}hy_n' - \frac{11}{160}h^2 y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \frac{53}{720}\left(\frac{1}{2}\right)^j - \frac{3}{160}\left(1\right)^j + \frac{1}{720}\left(2\right)^j \right\} = 0 \\
\sum_{j=0}^{\infty} \frac{(1)^j}{j!} y_n^j - y_n - hy_n' - \frac{19}{120}h^2 y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \frac{16}{45}\left(\frac{1}{2}\right)^j - \frac{1}{160}\left(1\right)^j + \frac{1}{360}\left(2\right)^j \right\} = 0 \\
\sum_{j=0}^{\infty} \frac{(2)^j}{j!} y_n^j - y_n - 2hy_n' - \frac{2}{5}h^2 y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \frac{32}{45}\left(\frac{1}{2}\right)^j + \frac{4}{45}\left(1\right)^j + \frac{4}{45}\left(2\right)^j \right\} = 0 \\
\sum_{j=0}^{\infty} \frac{\left(1/2\right)^j}{j!} y_n^{(j+1)} - y_n' - \frac{37}{192}h^2 y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \frac{3}{8}\left(\frac{1}{2}\right)^j - \frac{7}{96}\left(1\right)^j + \frac{1}{192}\left(2\right)^j \right\} = 0 \\
\sum_{j=0}^{\infty} \frac{(1)^j}{j!} y_n^{(j+1)} - y_n' - \frac{1}{6}h^2 y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ \frac{2}{3}\left(\frac{1}{2}\right)^j - \frac{1}{6}\left(1\right)^j + 0\left(2\right)^j \right\} = 0 \\
\sum_{j=0}^{\infty} \frac{(2)^j}{j!} y_n^{(j+1)} - y_n' - \frac{1}{3}h^2 y_n'' - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \left\{ 0\left(\frac{1}{2}\right)^j - \frac{4}{3}\left(1\right)^j + \frac{1}{3}\left(2\right)^j \right\} = 0
\end{array}\right)$$

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = 0,$$

Therefore, $c_6 = \left[-\frac{29}{92160}, -\frac{1}{1440}, -\frac{1}{180}, -\frac{31}{5760}, -\frac{19}{720}, -\frac{38}{45}\right]^T$. The new method has accurate order 4.

3.2. **Zero stability of Method.** In the spirit of [3], the block method (14) is said to be zero stable if as $h \to 0$ the roots $\lambda_i, i = 1, 2, 3$ of the first characteristics polynomial $\rho(\lambda) = 0$ satisfies $|\lambda_i| \leq 1,\ i = 1, 2, 3$. The characteristic polynomial of the above is given by

(23)
$$\rho(\lambda) = \det\left[\sum A^0 \lambda^{k-1}\right] = 0$$

Thus $\rho(\lambda) = \det\left(\lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right)$

$$= \begin{bmatrix} \lambda & 0 & -1 & 0 & 0 & 0 \\ 0 & \lambda & -1 & 0 & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix} = \lambda^5 (\lambda - 1) = 0 \lambda = 0, 0, 0, 0, 0, 1$$

The block method (14) is zero stable since from (1) ; if $\rho(\lambda) = 0$ satisfies $|\lambda_i| \leq 1,\ i = 1, 2, 3$ and for $|\lambda_i| = 1$, the multiplicity does not exceed 1.

3.3. **Consistentency of Method.** [16], stated that method (14) is consistent if it has order $p \geq 1$. [13] asserted that the hybrid linear multistep method (HLMM) is consistent if the following additional conditions hold:

(1) $\sum_{j=0}^{k} \alpha_j = 0$, where $\alpha_j{}'s$ are coefficients to be determined
(2) $\rho(r) = \rho'(r) = 0$ for $r = 1$
(3) $\rho''(r) = 2!\sigma(r)$ for $r = 1$

It should be noted that $\rho(r)$ and $\sigma(r)$ are the first and second characteristic polynomials the method respectively.

3.4. **Convergence of the new method.** [24] stated that the two sufficient conditions for the convergence of a linear multistep method are that:

(1) they are consistent; and
(2) they are zero stable.

Since our new method satisfies the above two conditions, it is therefore convergent.

3.5. **Region of Absolute Stabilityof Method.** The block method of equation (14) can be express in the form:

$$\left[\frac{Y}{Y_{\alpha+1}}\right] = \left[\frac{A \qquad \mathrm{U}}{B \qquad \mathrm{V}}\right]\left[\frac{h^2 f(y)}{Y_{\alpha-1}}\right] \tag{24}$$

where $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{11}{160} & \frac{53}{720} & -\frac{3}{160} & \frac{1}{720} \\ \frac{19}{120} & \frac{16}{45} & -\frac{1}{60} & \frac{1}{360} \\ \frac{2}{5} & \frac{32}{45} & \frac{4}{5} & \frac{4}{45} \end{bmatrix}$, $\mathrm{B} = \begin{bmatrix} \frac{11}{160} & \frac{53}{720} & -\frac{3}{160} & \frac{1}{720} \\ \frac{2}{5} & \frac{32}{45} & \frac{4}{5} & \frac{4}{45} \end{bmatrix}$,

$\mathrm{U} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathrm{V} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $Y = \begin{bmatrix} y_n \\ y_{n+1/2} \\ y_{n+1} \\ y_{n+2} \end{bmatrix}$,

$f(y) = \begin{bmatrix} f_n \\ f_{n+1/2} \\ f_{n+1} \\ f_{n+2} \end{bmatrix}$, $\mathrm{Y}_{m-1} = \begin{bmatrix} y_{n+1/2} \\ y_n \end{bmatrix}$, $\mathrm{Y}_{m+1} = \begin{bmatrix} y_{n+1/2} \\ y_{n+2} \end{bmatrix}$

From (24) the stability matrix as given in [10] and [9] is

$$M(z) = V + zB(I - zA)^{-1} U \tag{25}$$

Therefore the stability function is given as:

$$\rho\left(n,z\right)=\det\left(nI-M(z)\right) \tag{26}$$

If the matrices A, B, U, V, M and I are substituted in (25) and (26) we obtained the stability polynomial.

$$R\left(z\right)=-\frac{32z^3+1101z^2+5340z+2880}{4z^3-42z^2+840z-5760} \tag{27}$$

The region of absolute stability curve is as shown below:

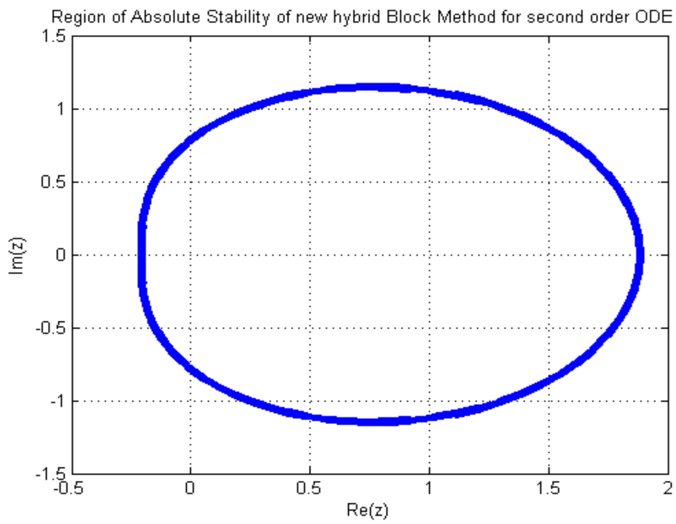

Figure 9: Region of Absolute Stability of the Method

4. Numerical Experiments

In this section, the efficiency and accuracy of the new method are tested on six test problems. We present some numerical experiments that are widely solved.

Problem 1. $y''=y',\ y\left(0\right)=0,\ y\left(0\right)=-1;$
Exact solution $y\left(x\right)=1-e^{x}$.

Table 1: Results of Problem 1 with errors in new method compared with [2]

X	Exact solution	Computed solution	Error in the New method	Error in [2]	Error in [8]
0.1	-0.10517091807565	-0.10517091880000	7.24×10^{-10}	7.24×10^{-9}	-
0.2	-0.22140275816017	-0.221402760202954	2.04×10^{-9}	2.03×10^{-9}	4.25×10^{-8}
0.3	-0.34985880757600	-0.349858811648246	4.07×10^{-9}	4.43×10^{-8}	7.47×10^{-8}
0.4	-0.49182469764127	-0.491824704588975	6.95×10^{-9}	7.706×10^{-8}	1.52×10^{-7}
0.5	-0.64872127070013	-0.648721281524800	1.08×10^{-8}	1.25×10^{-7}	2.45×10^{-7}
0.6	-0.82211880039051	-0.822118816272506	1.59×10^{-8}	1.86×10^{-7}	3.54×10^{-7}
0.7	-1.01375270747048	-1.013752729795550	2.23×10^{-8}	2.67×10^{-7}	5.31×10^{-7}
0.8	-1.22554092849247	-1.225540958881920	3.04×10^{-8}	3.66×10^{-7}	7.37×10^{-7}
0.9	-1.45960311115695	-1.459603151501840	4.05×10^{-8}	4.93×10^{-7}	9.73×10^{-7}
1.0	-1.71828182845905	-1.718281880959000	5.25×10^{-8}	6.45×10^{-7}	1.31×10^{-7}

Problem 2: $y'' + \lambda^2 y = 0,\ y(0) = 1, y'(0) = 2;$
Exact solution: $y(x) = \cos 2x + \sin 2x$.

Table 2: Results of Problem 2 with errors compared with [1]

X	Exact solution	Computed solution	Error in the New method	[23]
0.01	1.01979867335991	1.01979867339400	3.409×10^{-11}	2.6577×10^{-11}
0.02	1.0391894408476	1.03918944091706	6.945×10^{-11}	8.4761×10^{-10}
0.03	1.05816454641465	1.05816454652070	1.061×10^{-10}	6.4146×10^{-9}
0.04	1.07671640027179	1.07671640041564	1.439×10^{-10}	6.7071×10^{-9}
0.05	1.09483758192485	1.09483758210764	1.827×10^{-10}	7.1209×10^{-9}
0.06	1.11252084314278	1.11252084336561	2.228×10^{-10}	7.6530×10^{-9}
0.07	1.12975911085687	1.12975911112080	2.639×10^{-10}	8.3601×10^{-9}
0.08	1.14654548998987	1.14654549029590	3.060×10^{-10}	9.0592×10^{-9}
0.09	1.16287326621394	1.16287326652377	3.491×10^{-10}	9.9268×10^{-9}
0.10	1.17873590863630	1.17873590902939	3.930×10^{-10}	1.0899×10^{-8}

Problem 3: $y'' + 1001y' + 1000y = 0,\ y(0) = 1, y'(0) = -1;$
Exact solution: $y(x) = e^{-x}$.

Table 3: Results of Problem 3 with errors in new method compared with [20] and [8].

X	Exact solution	Computed sol	Error in the New method	Error in [20]	Error in [8]
0.1	0.904837418035960	0.90483741809000	5.40×10^{-11}	2.90×10^{-9}	2.00×10^{-10}
0.2	0.818730753077982	0.818730753249490	1.72×10^{-10}	1.87×10^{-8}	3.15×10^{-10}
0.3	0.740818220681718	0.740818220981503	2.99×10^{-10}	9.97×10^{-8}	2.74×10^{-10}
0.4	0.670320046035639	0.670320046453101	4.18×10^{-10}	5.25×10^{-7}	5.44×10^{-10}
0.5	0.606530659712633	0.606530660229675	5.17×10^{-10}	2.75×10^{-7}	7.53×10^{-10}
0.6	0.548811636094026	0.548811636691118	5.97×10^{-10}	1.44×10^{-6}	2.76×10^{-10}
0.7	0.496585303791410	0.496585304450096	7.04×10^{-10}	7.50×10^{-6}	1.18×10^{-10}
0.8	0.449328964117222	0.449328964821040	6.59×10^{-10}	3.92×10^{-5}	1.76×10^{-10}
0.9	0.406569659740599	0.406569660475300	7.35×10^{-10}	2.04×10^{-4}	-
1.0	0.367879441171442	0.367879441924913	7.54×10^{-10}	1.07×10^{-3}	-

Table 4: Results of Problem 3 with errors in new method compared with [11].

X	Exact solution	Computed sol H=0.05	Error in the New method	Error in [11]
0.1	0.951229424500714	0.95122942455000	4.928×10^{-11}	$2.00e^{-10}$
0.2	0.904837418035960	0.904837418122679	8.672×10^{-11}	$3.15e^{-10}$
0.3	0.860707976425058	0.860707976542597	1.1754×10^{-10}	$2.74e^{-10}$
0.4	0.818730753077982	0.818730753221981	1.4399×10^{-10}	$5.44e^{-10}$
0.5	0.778800783071405	0.778800783238540	1.6714×10^{-10}	$7.53e^{-10}$
0.6	0.740818220681718	0.740818220869203	1.8749×10^{-10}	$2.76e^{-10}$
0.7	0.704688089718713	0.704688089924091	2.0538×10^{-10}	$1.18e^{-10}$
0.8	0.670320046035639	0.670320046256687	2.2105×10^{-10}	$1.76e^{-10}$
0.9	0.637628151621773	0.637628151856462	2.3469×10^{-10}	–
1.0	0.606530659712633	0.606530659959102	2.4647×10^{-10}	–

Problem 4: $y'' = 3y' + 8e^{2x},\ y(0) = 1, y'(0) = 1,\ h = 0.005;$
Exact solution: $y(x) = -4e^{2x} + 3e^{3x} + 2$.

Table 5: Result of problem 4 with errors compared with errors in [14].

X	Exact solution	Computed sol h=0.005	Error in the New method	Error in [14]
0.0050	1.00513852551048	1.00513852551027	2.100×10^{-13}	1.2349×10^{-9}
0.0100	1.01055824175352	1.01055824175503	2.390×10^{-12}	2.6905×10^{-9}
0.0150	1.01626544391208	1.01626544391738	1.510×10^{-12}	4.3738×10^{-9}
0.0200	1.02226654286652	1.02226654287879	5.300×10^{-12}	6.2921×10^{-9}
0.0250	1.02856806714981	1.02856806717048	1.131×10^{-11}	8.9697×10^{-9}
0.0300	1.03517666493419	1.03517666496580	3.056×10^{-11}	1.0863×10^{-8}
0.0350	1.04209910605025	1.04209910609545	4.411×10^{-11}	1.6463×10^{-8}
0.0400	1.04934228403830	1.04934228409993	6.049×10^{-11}	
0.0450	1.05691321823309	1.05691321831416	7.989×10^{-11}	
0.0500	1.06481905588224	1.06481905598591	1.024×10^{-10}	

Problem 5 : $y'' = -100y + 99\sin x,\ y(0) = 1, y'(0) = 11;$
Exact solution: $y(x) = \cos(10x) + \sin(10x) + \sin(x)$.

Table 6: Result of problem 5 with errors compared with errors in [20]

X	Exact solution	Computed sol h=0.005	Error in the New method	Error in [20]
$\frac{1}{320}$	1.03388166738420	1.03388166740420	2.000×10^{-11}	7.9800×10^{-11}
$\frac{3}{320}$	1.09859628036501	1.09859628038310	1.809×10^{-11}	8.3780×10^{-10}
$\frac{6}{320}$	1.18762551125002	1.18762551133405	8.403×10^{-11}	3.3600×10^{-10}
$\frac{9}{320}$	1.26638728692280	1.26638728707680	1.5400×10^{-10}	7.3481×10^{-9}
$\frac{12}{320}$	1.33427136255382	1.33427136282755	2.7373×10^{-10}	1.2557×10^{-8}
$\frac{15}{320}$	1.39076300660681	1.39076300695380	3.4699×10^{-10}	1.8721×10^{-8}
$\frac{18}{320}$	1.43544751437359	1.43544751474454	3.7095×10^{-10}	2.5555×10^{-8}
$\frac{21}{320}$	1.46801384301307	1.46801384345270	4.8682×10^{-10}	3.2762×10^{-8}
$\frac{24}{320}$	1.48825733616990	1.48825733671978	5.4988×10^{-10}	4.0036×10^{-8}
$\frac{27}{320}$	1.49608151424405	1.49608151485014	6.0609×10^{-10}	4.7066×10^{-8}

Problem 6: $y'' = -w^2 y,\ y(0) = 1, y(0) = 0, h = \frac{\pi}{800}, \frac{\pi}{1600};$
when $h = \frac{\pi}{1600}$.

Point	Exact solution	Computed solution H=	Error in the new method	Error in [6]
5π	0.951056516270117	0.951056515914432	3.55685×10^{-10}	1.6532×10^{-7}
10π	0.233445363521082	0.233445362480218	1.06439×10^{-9}	3.1128×10^{-7}
15π	-0.271440450323200	-0.271440450831872	5.08672×10^{-10}	6.8875×10^{-7}
20π	-0.785316931275750	-0.785316930235050	1.040700×10^{-9}	9.5563×10^{-7}

when $h = \frac{\pi}{1600}$.

Point	Exact solution	Computed solution H=	Error in the new method	Error in [6]
5π	0.951056516270117	0.951056516478219	2.08102×10^{-10}	2.5567×10^{-09}
10π	0.785316930774156	0.785316931220417	4.46261×10^{-10}	5.2398×10^{-09}
15π	0.603555941763811	0.603555942314076	5.50265×10^{-10}	6.9971×10^{-08}
20π	0.327630179260283	0.327630179694381	4.34098×10^{-10}	9.20138×10^{-8}

Problem 7:

$y'' = y + xe^{3x},\ y(0) = -\frac{3}{32}, y'(0) = -\frac{5}{32},\ h = 0.0025;$

Exact: Solution**:** $y(x) = \frac{4x-3}{32e^{-3x}}$.

Table 7: Results of problem 6 with errors in new method compared with [5].

X	Exact solution	Computed sol 0.0025	Error in the New method	Error in [5]
0.0025	-0.0941409157618499	-0.0941409157616699	1.782×10^{-13}	1.341510×10^{-12}
0.0050	-0.0945324041423391	-0.0945324041419803	$3{,}588\times10^{-12}$	3.21479×10^{-12}
0.0075	-0.0949244516083881	-0.0949244516078481	5.400×10^{-13}	4.745607×10^{-12}
0.0100	-0.0953170443907006	-0.0953170443899786	7.220×10^{-13}	5.983755×10^{-12}
0.0125	-0.0957101684809812	-0.0957101684800758	9.054×10^{-13}	8.162346×10^{-12}
0.0150	-0.0961038096291138	-0.0961038096280240	1.0898×10^{-12}	1.089376×10^{-11}
0.0175	-0.0964979533403162	-0.0964979533390411	1.2751×10^{-12}	1.326500×10^{-11}
0.0200	-0.0968925848722641	-0.0968925848708023	1.4618×10^{-12}	1.531951×10^{-11}
0.0225	-0.0972876892321841	-0.0972876892305345	1.6496×10^{-12}	1.834202×10^{-11}
0.0250	-0.0976832511739197	-0.0976832511720809	1.8388×10^{-12}	2.193921×10^{-11}

Discussion of the results. We have developed computer programs to implement our methods using MAPLE 18 software. The methods were tested on some initial value problems in ordinary differential equations. The results are then compared with the results obtained from the use of some existing methods in the literature. The performance of the new method as compared with those of the existing is displayed in Tables 1-7 above. As observed in the tables our method performed better in terms of accuracy and error.

Conclusion: We have developed and implemented a new algorithm for solving general second order initial value problems in ordinary differential equations. The analysis of the properties was carried out and it was found that the new method is consistent, zero stable and convergent. Numerical experiment shows that our new method performed better and compared favourably with some of the existing methods in the literature.

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