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Approximate Analytical Methods for the Solution of Fractional Order Integro-differential Equations

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Abstract

In this paper, two numerical methods for obtaining the solution of fractional order integro-differential equations were disscussed. The modified variational iteration method (MVIM) which is a modified general Lagrange multiplier method and the modified homotopy perturbation method (MHPM) were applied to solve both linear and non-linear cases of fractional order integro-differential equations. Although the modified variational iteration method accelerate fast in convergence to the exact solution, the results obtained as presented on numerical tables and graphs revealed that the two methods were effective and of high accuracy for solving this type of problem.

1. Introduction

In recent years, problem arising from mathematical modeling of physical occurrences such as biological modelling of diseases, modelling of dynamical systems and modelling in mathematical finance usually results to fractional order integro-differential equations. The closed form of the exact solution is tedious to find and most time, not possible. Because of this many numerical approaches such as [3]

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& [4] have been applied by researchers to solve the problem. To begin with is [11] that applied the variational iteration method and homotopy perturbation method to solve Kawahra equation. Also, [9] applied this variational iteration method to obtain the solution of coupled system of non linear partial differential equations. A reliable algorithm for solving boundary value problems for higher-order integro-differential equations was proposed by [7] and [12] applied the variational iteration method to obtain the numerical solution of linear and non linear Fredholm integral equations. A comparison between the variational iteration method and the successive approximation method was carried out by [13] and recently, [14] applied the variational iteration method to obtain the solution of higher-order integro-differential equations. The convergence of variational iteration method for second order delay differential equations was also researched by [10], and this variational iteration method was applied by [1] to obtain the solution of fourth order integro-differential equation. The application of both Variational iteration method and homotopy perturbation method was extended by [6] to obtain the solution of fourth-order fractional integro-differential equations, in the same vein [2] & [8] applied the variational iteration method to obtain the solution of non-linear fractional order integro-diferential equation. The solution of nonlinear Volterra-Fredholm integro-differential equations was obtained by [5] applying hybrid Legendre polynomials and Block-pulse functions approach. In this paper, two approximate analytical methods namely the variational iteration method and homotopy perturbation method will be applied for obtaining the solution of fractional order integro-differential equations of the form:

$$y^{\alpha}(t) = p(t)y(t) + \mu \int_0^t k(x,t)\,[y(x) + F(y(x))]\,dx \qquad \forall \;\; (t) \in [0,1] \tag{1}$$

which may be subjected to the following initial condition:

$$y^n(\Gamma) = \beta_n \qquad n = 0,1,2,3\ldots \tag{2}$$

or boundary condition:

$$y^n(\Gamma_1) = \gamma_1 \; and \; y^m(\Gamma_2) = \gamma_2 \qquad \forall \; \Gamma_n, \; \gamma_1, \gamma_2, \; \beta_n \in R \tag{3}$$

α is the order of the integro-differental equation, the $k(x,t)$ is the kernel of the integral equation, $y(x)$ is a given linear function, $F(y(x))$ is a given non linear function and $y(t)$ is the function to be determined. The problem presented in (1) is assumed to have a unique solution and the results presented shows that the two methods are potent and efficient in handling this type of problem.

2. Preliminaries

In this section, some definitions on fractional calculus applicable in the paper are given.

2.1 Caputo derivative. The caputo derivative of a real function $f(x)$ is defined as:

$$D^{(\alpha)}f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}\frac{d^{(m)}F(t)}{dt^m}dt, \quad m-1<\alpha\leq m \qquad \forall \ m\in N$$

2.2 Riemann-liouville integral. The riemann lioville integral of a function $f(x)$ is defined as:

$$I^{\alpha}F(x) = \frac{1}{\Gamma(\alpha)}\int_0^t (x-t)^{(\alpha-1)}f(t)dt \ \alpha>0$$

if $\alpha = 0$, then

$$I^0F(x) = F(x)$$

For a positive definite real function $f(x), \forall \ \alpha,\beta \geq 0, \ \epsilon \geq -1$, the following properties hold for operator I^{α},

(1) $I^{\alpha}I^{\beta}F(x) = I^{(\alpha+\beta)}F(x)$
(2) $I^{\alpha}x^{\epsilon} = \frac{\Gamma(\epsilon+1)}{\Gamma(\alpha+\epsilon+1)}x^{\alpha+\epsilon}$
(3) $D^{\alpha}[I^{\alpha}F(x)] = F(x)$
(4) $I^{\alpha}[D^{\alpha}F(x)] = F(x) - \sum_{k=0}^{m-1}F^k(0)\frac{x^k}{k!} \ \ m-1<\alpha\leq m$

3. Methods

3.1 Modified Variational Iteration Method. The basic idea of the Variational iteration method can be explained by considering the following equation,

(4) $$Ly(x) + Ny(x) = g(x,t)$$

by variational iterational method,

(5) $$y_{n+1}(x) = y_n(x) + \int_0^x \lambda[Ly_n(s) + Ny(s) - y(s)]ds$$

where

(6) $$\lambda(s) = \frac{(-1)^n}{\Gamma(n)}(s-x)^{n-1}$$

replacing (4) by (1) in (5),

(7)
$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(x)\left(y_n^{\alpha}(x) - p(x)y_n(x) - \int_0^x k(s,t)\left(y_n(s) - F(y_n(s))ds\right)\right)dx$$

Constructing a Riemann-Liouville correctional formula for (7) yields

(8)
$$y_{n+1}(t) = y_n(t) + \frac{1}{\Gamma(\beta)}\int_0^t (t-x)^{\beta-1}\lambda(x)\left([y_n^{\alpha}(x) - p(x)\bar{y}_n(x) - \int_0^x k(s,t)\left(y_n(s) - F(\bar{y}_{n(s)})ds\right)\right)dx$$

$\beta = \alpha+1-m \ \forall \ \ m \in N, and \ \ \lambda(x)$ is the general lagrange multiplier $\bar{y}_n$ denotes the constricted variation.

In this research, the exact solutions of the problem solved are known at $\alpha = 4$. Since the lagrange multiplier

$$\lambda(t) = \frac{(-1)^k}{\Gamma(k)}(t-x)^{k-1}$$

letting the order $k = 4$ and substituting the expression of lagrange multiplier λ at stationary points into (8),

(9)
$$y_{n+1}(t) = y_n(t) + \frac{1}{\Gamma(\alpha-3)} \int_0^t (t-x)^{\alpha-m} \frac{(-1)^{4-1}}{\Gamma(4)}(x-t)^{4-1} \times \left(y_n^{\alpha}(x) - p(x)\bar{y}_n(x) - \int_0^x k(s,t)\left(y_n(s) - F(\bar{y}_n(s)ds \right)\right) dx.$$

By simplifying (9) and (10), we obtained

(10)
$$y_{n+1}(t) = y_n(t) - \frac{1}{\Gamma(4)\Gamma(\alpha-3)} \int_0^t (t-x)^{\alpha-4}\{(t-x)^{4-1} \times \left(y_n^{\alpha}(x) - p(x)\bar{y}_n(x) - \int_0^x k(s,t)\left(y_n(s) - F(\bar{y}_n(s)ds \right)\right) dx$$

such that

(11)
$$y_{n+1}(t) = y_n(t) - \frac{1}{\Gamma(4)\Gamma(\alpha-3)} \int_0^t (t-x)^{\alpha-1} \left(y_n^{\alpha}(x) - p(x)\bar{y}_n(x) - \int_0^x k(s,t)\left(y_n(s) - F(\bar{y}_n(s)ds \right)\right) dx$$

simplification of (11) and (12) yields

(12)
$$y_{n+1}(t) = y_n(t) - \frac{(\alpha-3)(\alpha-2)(\alpha-1)}{6\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} \left(y_n^{\alpha}(x) - p(x)\bar{y}_n(x) - \int_0^x k(s,t)\left(y_n(s) - F(\bar{y}_n(s)ds \right)\right) dx$$

letting operator

$$I^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1}$$

.

(13)
$$y_{n+1}(t) = y_n(t) - \frac{(\alpha-3)(\alpha-2)(\alpha-1)}{6} I^{\alpha} \left(y_n^{\alpha}(x) - p(x)\bar{y}_n(x) - \int_0^x k(s,t)\left(y_n(s) - F(\bar{y}_n(s)ds \right)\right) dx$$

An approximate series solution of the problem can be obtained by constructing a power series using the given initial or boundary conditions.

so that

(14)
$$y_0(x) = \beta_0 + x\beta_1 + \frac{x^2}{2}\beta_2 + \frac{x^3}{3!}\beta_3.$$

To evaluate an unknown initial condition β_n, the boundary conditions is imposed on the obtained solution.

3.2 Modified Homotopy Perturbation Method. The method of modified homotopy perturbation can be extended to obtain the approximate solution of (1). According to 'He', a homotopy can be constructed for (1) such that:

$$(1-p)y^{\alpha}(t)+p\left(y^{\alpha}(t)-p(t)y(t)-\int_0^t[k(x)y(x)-f(y(x))dx\right)=0 \tag{15}$$

if $p=0$,

$$y^{\alpha}(t)=0 \tag{16}$$

if p=1,

$$y^{\alpha}(t)-p(t)y(t)+\int_0^t[k(x)y(x)-f(y(x)]\,dx=0 \tag{17}$$

the solution of the problem can be expressed in series form as

$$y(t)=p^0y_0(t)+p^1y_1(t)+p^2y_2(t)+p^3y_3(t)+\cdots \tag{18}$$

substituting (18) into (17) and equating coefficients of equal order of P, the following system of fractional order integro-differential equation is obtained:

$$\begin{array}{lcll}
p^0 & : & y_0^{\alpha}(t)=0; & \\
p^1 & : & y_1^{\alpha}(t)= & p(t)y_0(t)+\int_0^t k(x)y_0(x)-F(y_0(x))dx; \\
p^2 & : & y_2^{\alpha}(t)= & p(t)y_1(t)+\int_0^t k(x)y_1(x)-F(y_1(x))dx; \\
p^3 & : & y_3^{\alpha}(t)= & p(t)y_2(t)+\int_0^t k(x)y_2(x)-F(y_2(x))dx; \\
\vdots & & & \\
p^n & : & y_n^{\alpha}(t)= & p(t)y_n(t)+\int_0^t k(x)y_n(x)-F(y_n(x))dx;
\end{array} \tag{19}$$

According to Liao homotopy analysis, as long as a given solution series converges, it must be one of the solutions. This deduction is also satisfied by non-linear fractional order integro-differential equations. Thus, the non linear term is decomposed to its series form to make it fractionally integrable and operator I^{α} is applied on both sides of (19) to obtain $y_0(t), y_1(t), y_2(t), \cdots$ so that the approximate series solution of the problem is

$$\phi_N(x)=y_0(t)+y_1(t)+y_2(t)+y_3(t)+\cdots \tag{20}$$

4. Application

In this section, we present some examples to establish that the modified variational iteration method and homotopy perturbation methods are effective in obtaining the solution of both linear and non-linear fractional order integro-differential equations as in (1). The computation process was carried out using Maple 18.

4.1. **Modified variational iteration method.**

Example 4.1. Consider the following linear fractional order integro-differential equation.

$$D^{\alpha}y(x) = 2 - x - e^{x}(5 + x) - y(x) + \int_0^x y(t)dt \tag{21}$$

subject to the following initial conditions:

$$y(0) = 1 = \beta_0, \quad y''(0) = -2 = \beta_2 \tag{22}$$

and boundary condition

$$y(1) = 1 - e, \quad y''(1) = -3e, \tag{23}$$

The exact value at $\alpha = 4$ is

$$y(x) = 1 - xe^{x}$$

According to the MVIM, the first approximation at $n = 0$ is

$$y_1(x) = y_0(x) - \mu I^{\alpha}\left[D^{\alpha}y(x) - 2 + x + e^{x}(5 + x) + y_0(x) - \int_0^x y_0(t)dt\right] \tag{24}$$

To make e^x fractionally integrable, it is converted to it's series form

$$e^{x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} \tag{25}$$

applying (14), the approximate power series solution to the problem is

$$y_0(x) = 1 + x\beta_1 - x^2 + \beta_3\frac{x^3}{6} \tag{26}$$

where $\beta_1 = y'(0)$ and $\beta_3 = y'''(0)$ are unknowns to be determined. Substituting (25) and (26) into (24),

$$\begin{aligned} &y_1(x) = 1 + x\beta_1 - x^2 + \beta_3\tfrac{x^3}{6} - \mu I^{\alpha} \\ &\left[D^{\alpha}y(x) + 4 + x(6 + \beta_1) + (5 - \beta_1)\tfrac{x^2}{2} + (10 + \beta_3)\tfrac{x^3}{6} + (4 - \beta_3)\tfrac{x^4}{24}\ \right] \end{aligned} \tag{27}$$

expanding the bracket by operator $I^{(\alpha)}$ the resulting equation is obtained

$$y_1(x) = 1 + \beta_1(x) - x^2 + \beta_3\tfrac{x^3}{3} - \tag{28}$$

$$\mu\left[4\frac{x^{\alpha}}{\Gamma(\alpha+1)} + (6 + \beta_1)\frac{x^{\alpha+1}}{\Gamma(\alpha+2)} + (5 - \beta_1)\frac{x^{\alpha+2}}{\Gamma(\alpha+3)} + (10 + \beta_3)\frac{x^{\alpha+3}}{\Gamma(\alpha+4)} + (4 - \beta_3)\frac{x^{\alpha+4}}{\Gamma\alpha+5}\right]$$

The boundary conditions are imposed on (28) to obtain the values of β_1 and β_3 and the obtained values are presented in Table 1. These values of β_1 and β_3 were in turn substituted into (28) to obtain the approximate solution of the problem and numerical Table 2 shows that the solution is in good agreement with the exact solution.

Example 4.2. Consider the following equation:

$$D^{\alpha}y(x) = -\frac{1}{4} - \frac{1}{2}x + \frac{5}{4}e^{2x} - \int e^{2(x-t)}y(t)dt \tag{29}$$

subject to the following initial conditions

$$y(0) = 1 = \beta_0, \quad y'(0) = 2 = \beta_1, \tag{30}$$

and boundary conditions

$$y(1) = 1 + e, \quad y'(1) = 1 + e \tag{31}$$

The exact solution is $y(x) = x + e^x$ when $\alpha = 4$. Applying (14) to construct the initial approximation,

$$y_0(x) = 1 + 2x + \beta_2\frac{x^2}{2} + \beta_3\frac{x^3}{6} \tag{32}$$

equation (32) is simplified by substituting the series of e^{2x}, e^{-2t} and $y_0(x)$. Applying the MHPM,

$$\begin{aligned} &y_1(x) = 1 + 2x + \tfrac{1}{2}\beta_2x^2 + \tfrac{1}{6}\beta_3x^3 + \mu[-\tfrac{x^{\alpha}}{\Gamma(\alpha+1)} - \tfrac{x^{\alpha+1}}{\Gamma(\alpha+2)} - \tfrac{x^{\alpha+2}}{\Gamma(\alpha+3)} \\ &+\tfrac{(\beta_2-2)x^{\alpha+3}}{\Gamma(\alpha+4)} + \tfrac{(2\,\beta_2+\beta_3+16)x^{\alpha+4}}{\Gamma(\alpha+5)} + 2\,\tfrac{(2\,\beta_2+\beta_3-32)x^{\alpha+5}}{\Gamma(\alpha+6)} + 4\,\tfrac{(2\,\beta_2+\beta_3-112)x^{\alpha+6}}{\Gamma(\alpha+7)} \\ &-8\,\tfrac{(98\,\beta_2-\beta_3+112)x^{\alpha+7}}{\Gamma(\alpha+8)} + 128\,\tfrac{(14\,\beta_2-13\,\beta_3-224)x^{\alpha+8}}{\Gamma(\alpha+9)} \\ &-3840\,\tfrac{(14\,\beta_2-\beta_3)x^{\alpha+9}}{\Gamma(\alpha+10)} - 153600\,\tfrac{x^{\alpha+10}}{\Gamma(\alpha+11)}] \end{aligned} \tag{33}$$

4.2. **Modified homotopy perturbation method.**

Example 4.3. Consider the following linear fractional integro-differential equation

$$D^{\alpha}y(x) = 2 - x - e^x\,(5 + x) - y(x) + \int_0^x y(t)dt \tag{34}$$

subject to the following set of initial and boundary conditions:

$$\begin{aligned} y(0) &= 1 = \beta_0, & y''(0) &= -2 = \beta_2, \\ y(1) &= 1 - e, & y''(1) &= -3e \end{aligned} \tag{35}$$

The exact solution at $\alpha = 4$ *is* $y(x) = 1 - xe^x$

The series of e^x is substituted into the problem to make it fractionally integrable. By simplifying the resulting equation, (36) is obtained;

$$D^{\alpha}y(x) = -3 - 7x - \frac{7}{2}x^2 - \frac{4}{3}x^3 - \frac{1}{6}x^4 - y(x) + \int_0^x y(t)dt. \tag{36}$$

Constructing a homotopy for (36) yields (37):

$$D^{\alpha}y(x) = p\left(-3 - 7x - \frac{7}{2}x^2 - \frac{4}{3}x^3 - \frac{1}{6}x^4 - y(x) + \int_0^x y(t)dt\right) \tag{37}$$

substituting (18) into (37) and equating terms with equal powers of p:

$$(38)\qquad \begin{array}{l} p^0 : [D^\alpha y_0] = 0; \\ p^1 : [D^\alpha y_1] = -3 - 7x - \frac{7}{2}x^2 - \frac{4}{3}x^3 - \frac{1}{6}x^4 - y_0(x) + \int_0^x y_0(t)dt; \\ p^2 : [D^\alpha y_2] = -y_1(x) + \int_0^x y_1(t)dt; \\ p^3 : [D^\alpha y_3] = -y_2(x) + \int_0^x y_2(t)dt; \\ \vdots \\ p^n : [D^\alpha y_n] = -y_n(x) + \int_0^x y_n(t)dt; \end{array}$$

applying the operator I^α to (38),

$$(39)\qquad y_0(x) = 1$$

(40)

$$y_1(x) = \beta_1(x) - x^2 + \beta_3\frac{x^3}{6} - 4\frac{x^\alpha}{\Gamma(\alpha+1)} - 6\frac{x^{\alpha+1}}{\Gamma(2+\alpha)} - 7\frac{x^{\alpha+2}}{\Gamma(3+\alpha)} - 8\frac{x^{\alpha+3}}{\Gamma(4+\alpha)} - 4\frac{x^{\alpha+4}}{\Gamma(5+\alpha)}$$

$$(41)\qquad y_2(x) = -\beta_1\frac{x^{\alpha+1}}{\Gamma(\alpha+2)} + (2+\beta_1)\frac{x^{(\alpha+2)}}{\Gamma(\alpha+3)} - (2+\beta_3)\frac{x^{(\alpha+3)}}{\Gamma(\alpha+4)} + \beta_3\frac{x^{(\alpha+4)}}{\Gamma(\alpha+5)}$$

$$+4\frac{x^{(2\alpha)}}{\Gamma(2\alpha+1)} + 2\frac{x^{(2\alpha+1)}}{\Gamma(2\alpha+2)} + \frac{x^{(2\alpha+2)}}{\Gamma(2\alpha+3)} + \frac{x^{(2\alpha+3)}}{\Gamma(2\alpha+4)}$$

The approximate solution of the problem is obtained by adding the three iterations obtained in (39) to (41).

$$(42)\qquad \begin{array}{l} \phi_N(x) = 1 + \beta_1(x) - x^2 + \beta_3\frac{x^3}{6} - 4\frac{x^\alpha}{\Gamma(\alpha+1)} - 6\frac{x^{\alpha+1}}{\Gamma(2+\alpha)} - 7\frac{x^{\alpha+2}}{\Gamma(3+\alpha)} \\ -8\frac{x^{\alpha+3}}{\Gamma(4+\alpha)} - 4\frac{x^{\alpha+4}}{\Gamma(5+\alpha)} - \beta_1\frac{x^{\alpha+1}}{\Gamma(\alpha+2)} + (2+\beta_1)\frac{x^{(\alpha+2)}}{\Gamma(\alpha+3)} - (2+\beta_3)\frac{x^{(\alpha+3)}}{\Gamma(\alpha+4)} \\ +\beta_3\frac{x^{(\alpha+4)}}{\Gamma(\alpha+5)} + 4\frac{x^{(2\alpha)}}{\Gamma(2\alpha+1)} + 2\frac{x^{(2\alpha+1)}}{\Gamma(2\alpha+2)} + \frac{x^{(2\alpha+2)}}{\Gamma(2\alpha+3)} + \frac{x^{(2\alpha+3)}}{\Gamma(2\alpha+4)} \end{array}$$

Example 4.4. Consider the nonlinear fractional integro-differential equation

$$(43)\qquad D^\alpha y(x) = -\frac{1}{4} - \frac{1}{2}x + \frac{5}{4}e^{2x} - \int e^{2(x-t)}y(t)dt$$

subject to the following initial conditions

$$(44)\qquad y(0) = 1 = \beta_0, \quad y'(0) = 2 = \beta_1$$

and boundary conditions

$$(45)\qquad y(1) = 1 + e, \quad y'(1) = 1 + e$$

The exact solution is

$$y(x) = x + e^x \quad at \ \ \alpha = 4.$$

Substituting the series of e^{2x} into (43) gives

$$(46)\qquad D^\alpha y(x) = 1 + 2x + \frac{5}{2}x^2 + \frac{5}{3}x^3 - \int e^{2(x-t)}y(t)dt.$$

Constructing homotopy for (46) gives

$$p\left[D^{\alpha}y(x)\right]=p\left[1+2x+\frac{5}{2}x^{2}+\frac{5}{3}x^{3}-\int e^{2(x-t)}y(t)dt\right] \tag{47}$$

substituting (18) into (47) and comparing coefficients gives the following system of equations in (48)

$$\begin{aligned} &p^{0}: D^{\alpha}y_{0}(x)=0\\ &p^{1}: D^{\alpha}y_{1}(x)=1+2x+\tfrac{5}{2}x^{2}+\tfrac{5}{3}x^{3}-\textstyle\int e^{2(x-t)}y_{0}(t)dt\\ &p^{2}: D^{\alpha}y_{2}(x)=\textstyle\int e^{2(x-t)}y_{1}(t)dt \end{aligned} \tag{48}$$

applying the operator I^{α} on (48),

$$y_{0}(x)=1 \tag{49}$$

also,

$$\begin{aligned} y_{1}(x)&=2\,x+\tfrac{1}{2}x^{2}\beta_{2}+\tfrac{1}{6}x^{3}\beta_{3}+\tfrac{x^{\alpha}}{\Gamma(\alpha+1)}+\tfrac{x^{\alpha+1}}{\Gamma(\alpha+2)}+3\,\tfrac{x^{\alpha+2}}{\Gamma(\alpha+3)}+6\,\tfrac{x^{\alpha+3}}{\Gamma(\alpha+4)}\\ &8\,\tfrac{x^{\alpha+4}}{\Gamma(\alpha+5)}+80\,\tfrac{x^{\alpha+5}}{\Gamma(\alpha+6)}-160\,\tfrac{x^{\alpha+6}}{\Gamma(\alpha+7)}+2240\,\tfrac{x^{\alpha+7}}{\Gamma(\alpha+8)} \end{aligned} \tag{50}$$

Solving for y_2 in the same vein,

(51)

$$\begin{aligned} y_{2}(x)=&-2\,\tfrac{x^{\alpha+1}}{\Gamma(\alpha+2)}-\tfrac{\beta_{2}x^{\alpha+2}}{\Gamma(\alpha+3)}-\tfrac{\beta_{3}x^{\alpha+3}}{\Gamma(\alpha+4)}+320\,\tfrac{x^{\alpha+5}}{\Gamma(\alpha+6)}+480\,\tfrac{\beta_{2}x^{\alpha+6}}{\Gamma(\alpha+7)}+1120\,\tfrac{x^{\alpha+7}(16+\beta_{3})}{\Gamma(\alpha+8)}\\ &+35840\,\tfrac{\beta_{2}x^{\alpha+8}}{\Gamma(\alpha+9)}+107520\,\tfrac{\beta_{3}x^{\alpha+9}}{\Gamma(\alpha+10)}-\tfrac{x^{2\,\alpha}}{\Gamma(2\,\alpha+1)}-\tfrac{x^{2\,\alpha+1}}{\Gamma(2\,\alpha+2)}-3\,\tfrac{x^{2\,\alpha+2}}{\Gamma(2\,\alpha+3)}-6\,\tfrac{x^{2\,\alpha+3}}{\Gamma(2\,\alpha+4)}\\ &+\tfrac{4}{3}\,\tfrac{\left(\alpha^{4}+10\,\alpha^{3}+35\,\alpha^{2}+50\,\alpha+30\right)x^{2\,\alpha+4}}{\Gamma(2\,\alpha+5)}+\tfrac{4}{3}\,\tfrac{\left(\alpha^{4}+14\,\alpha^{3}+71\,\alpha^{2}+154\,\alpha+60\right)x^{2\,\alpha+5}}{\Gamma(2\,\alpha+6)}+160\,\tfrac{x^{2\,\alpha+6}}{\Gamma(2\,\alpha+7)}\\ &+\tfrac{8}{9}\,\tfrac{\left(2\,\alpha^{6}+54\,\alpha^{5}+599\,\alpha^{4}+3528\,\alpha^{3}+11819\,\alpha^{2}+21798\,\alpha+15120\right)x^{2\,\alpha+7}}{\Gamma(2\,\alpha+8)}+\tfrac{16}{3}\,\tfrac{x^{2\,\alpha+8}(\alpha+8)(\alpha+7)(\alpha+6)(\alpha+2)(\alpha+5)^{2}}{\Gamma(2\,\alpha+9)}\\ &+\tfrac{32}{3}\,\tfrac{\left(\alpha^{2}+9\,\alpha+30\right)(\alpha+6)(\alpha+7)(\alpha+8)(\alpha+9)x^{2\,\alpha+9}}{\Gamma(2\,\alpha+10)}-\tfrac{128}{9}\,\tfrac{\left(\alpha^{2}+11\,\alpha+45\right)(\alpha+7)(\alpha+8)(\alpha+9)(\alpha+10)x^{2\,\alpha+10}}{\Gamma(2\,\alpha+11)}\\ &+\tfrac{1280}{9}\,\tfrac{\left(\alpha^{2}+13\,\alpha+63\right)(\alpha+8)(\alpha+)(\alpha+10)(\alpha+11)x^{2\,\alpha+11}}{\Gamma(2\,\alpha+12)}-\tfrac{2560}{9}\,\tfrac{x^{2\,\alpha+12}(\alpha+12)(\alpha+11)(\alpha+7)(\alpha+8)(\alpha+9)(\alpha+10)}{\Gamma(2\,\alpha+13)}\\ &+\tfrac{35840}{9}\,\tfrac{x^{2\,\alpha+13}(\alpha+13)(\alpha+12)(\alpha+8)(\alpha+9)(\alpha+10)(\alpha+11)}{\Gamma(2\,\alpha+14)} \end{aligned}$$

5. Numerical tables

5.1. **Numerical results For Example 4.1 and Example 4.2:** From Example 4.1 and Example 4.2, the given values of $\beta_0 = 1$ and $\beta_2 = -2$. Imposing the boundary conditions on the results obtained using modified variational iteration method and modified homotopy perturbation method respectively, the values of β_1 and β_3 as obtained are shown on Tables 1 and 2.

Table 1. Numerical values of β_1 and β_3 $3 < \alpha \leq 4$ MVIM

constant	$\alpha = 3.25$	$\alpha = 3.5$	$\alpha = 3.75$	$\alpha = 4.0$
β_1	-0.7404837902	-0.8176244610	-0.9094372723	-0.999480418
β_3	-5.396873656	-4.529003651	-3.693700603	-3.003415366

Table 2. Numerical values of β_1 and β_3 for $3 < \alpha \leq 4$ MHPM

constant	$\alpha = 3.25$	$\alpha = 3.5$	$\alpha = 3.75$	$\alpha = 4.0$
β_1	-1.099904662	-1.089826751	-1.050863268	-0.9985952594
β_3	0.2616916705	-0.9916977748	-2.085217410	-3.0093257030

Tables 3 and 4 show the numerical results of Example 4.1 and Example 4.2 at different levels of α using the modified variational iteration and modified homotopy perturbation method respectively. Table 5 shows the comparison of error between the two methods applied.

Table 3. Numerical results of MVIM for $3 < \alpha \leq 4$

x	$\alpha = 3.25$	$\alpha = 3.5$	$\alpha = 3.75$	$\alpha = 4.0$
0	1.000000000	1.000000000	1.000000000	1.000000000
0.5	-1.718281829	-1.718281829	-1.718281829	-1.718281828
1	-1.718281829	-1.718281828	-1.718281829	-1.718281828

Table 4. Numerical results of MHPM for $3 < \alpha \leq 4$

x	$\alpha = 3.25$	$\alpha = 3.5$	$\alpha = 3.75$	$\alpha = 4.0$
0	1.000000000	1.000000000	1.000000000	1.000000000
0.5	-1.718281830	-1.718281830	-1.718281830	-1.718281829
1	-1.718281828	-1.718281829	-1.718281827	-1.718281828

TABLE 5. Comparison of errors of MVIM and MHPM at $\alpha = 4$.

x	VIM	HPM
0	0.000000000	0.000000000
0.5	0.000000001	0.000000000
1	0.000000001	0.000000000

5.2. **Numerical results for Example 4.3 and Example 4.4:** From Example 4.3 and Example 4.4, the given values of $\beta_0 = 1$ *and* $\beta_1 = 2$. The values of β_2 and β_3 after imposing the boundary conditions on the results obtained using modified variational iteration method and modified homotopy perturbation method respectively are shown on Tables 6 and 7.

TABLE 6. Numerical values of β_2 and β_3 for $3 < \alpha \leq 4$ MVIM

constant	$\alpha = 3.25$	$\alpha = 3.5$	$\alpha = 3.75$	$\alpha = 4.0$
β_2	0.853127320	0.820348279	0.781971369	0.745636143
β_3	1.859552690	2.053046871	2.236082300	2.381949209

TABLE 7. Numerical values of β_2 and β_3 for $3 < \alpha \leq 4$ MHPM

constant	$\alpha = 3.25$	$\alpha = 3.5$	$\alpha = 3.75$	$\alpha = 4.0$
β_2	1.053499078	1.022210547	0.9946782832	0.970270368
β_3	0.451721289	0.753958813	0.9866622002	1.166634504

Tables 8 and 9 show the numerical results of Example 4.3 and Example 4.4 at different levels of α using the modified variational iteration and modified homotopy perturbation method respectively. Table 10 shows the comparison of error between the two methods applied.

TABLE 8. Numerical results of MVIM for $3 < \alpha \leq 4$

x	$\alpha = 3.25$	$\alpha = 3.5$	$\alpha = 3.75$	$\alpha = 4.0$
0	1.000000000	1.000000000	1.000000000	1.000000000
0.5	2.143703059	2.142652469	2.141326871	2.139941702
1	3.718281828	3.718281827	3.718281828	3.718281829

TABLE 9. Numerical results of MHPM for $3 < \alpha \leq 4$

x	$\alpha = 3.25$	$\alpha = 3.5$	$\alpha = 3.75$	$\alpha = 4.0$
0	1.000000000	1.000000000	1.000000000	1.000000000
0.5	2.152642509	2.150427345	2.149000484	2.147985614
1	3.718281828	3.718281828	3.718281830	3.718281829

TABLE 10. Comparison of errors of MVIM and HPM at $\alpha = 4$.

x	VIM	HPM
0	0.000000000	0.000000000
0.5	0.000000001	0.000000001
1	0.000000001	0.000000001

5.3. **Graphical interpretation.** It was evident on the numerical tables that the two methods were elegant in solving both linear and non linear problems of fractional order integro-differential equations presented. In this section, to ascertain the potency of these methods in solving these class of problems, graphical demonstrations are presented by plotting the approximate and exact solutions against x on the interval $0 \leq x \leq 1$ at α equals 4.

5.3.1. *Graph of Example 4.1 and Example 4.2.* In Figures 1 and 2, the graphical interpretation of the results obtained for problem one using the modified variational iteration method (MVIM) and the modified homotopy perturbation method (MHPM) are displayed. Due to cumbersome number of terms, only one iteration is performed using the MVIM, hence it's curve shows a divergence from the exact solution as the value of x approaches 1.

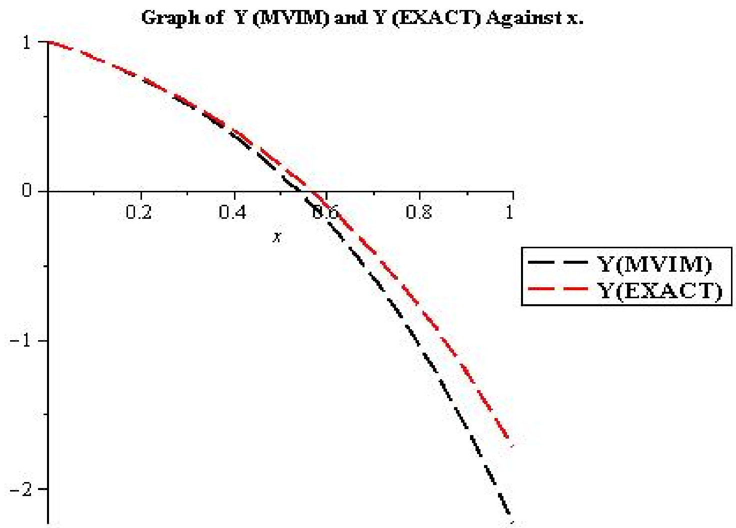

Figure 1: The graph of Y(MVIM) and Y(Exact) in Example 1 against $0 \leq x \leq 1$.

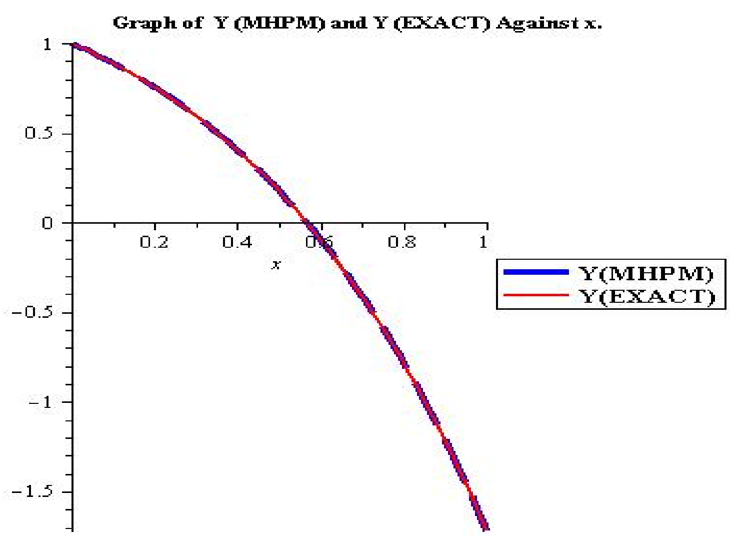

Figure 2: The graph of Y(MHPM) and Y(Exact) in Example 1 against $0 \leq x \leq 1$.

5.3.2. *Graph of Example 4.3 and Example 4.4.* Figures 3 and 4 reveals that the two methods applied are competent in handling the class of problem solved as their results are in good agreement with the exact solution.

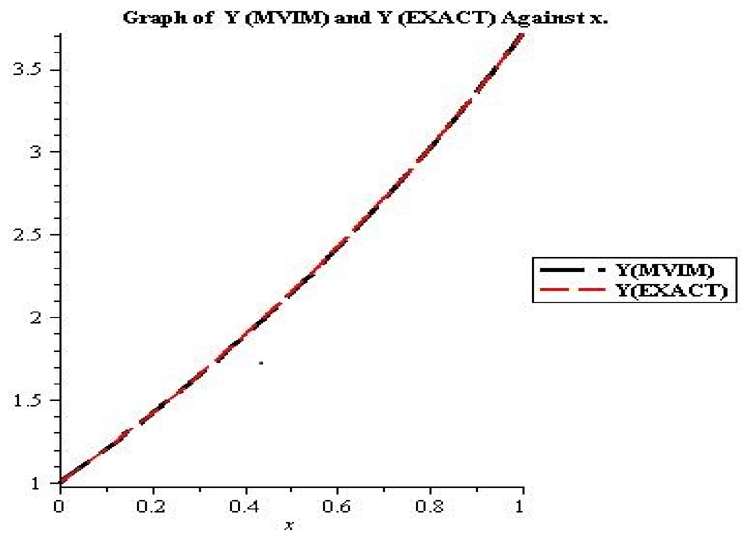

Figure 3: The graph of Y(MVIM) and Y(Exact) in Example 2 against $0 \leq x \leq 1$.

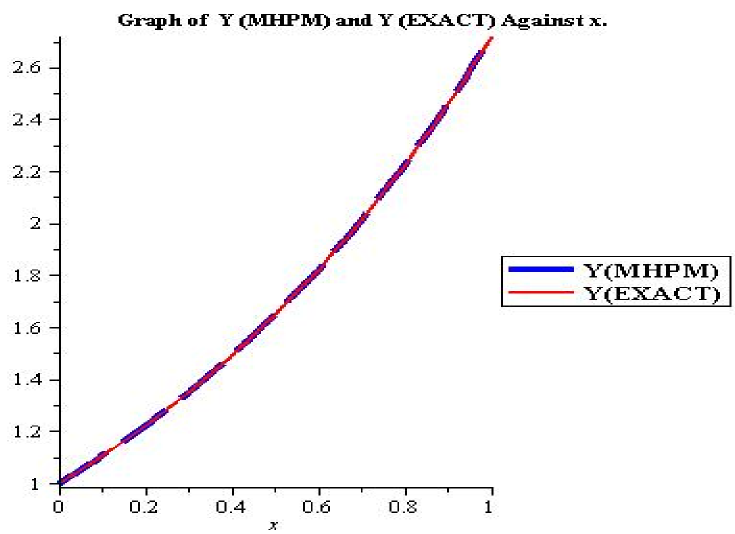

Figure 4: The graph of Y(MHPM) and Y(Exact) in Example 2 against $0 \leq x \leq 1$.

Remark. Figure 1 shows that the curve of Y(MVIM) diverges from the exact solution curve as x approaches 1; Figure 2 reveals that the result of the modified homotopy perturbation method is in good agreement with the exact solution after performing two iterations; Figure 3 indicates that the approximate solution of the modified variational iteration method is in line with the exact solution as there is no deflection on the curves as x approach 1; and Figure 4 displays that the approximate solution of the modified homotopy perturbation method agree with the exact solution as there is no deflection on the curves as x approach 1.

Conclusions: It could be observed that the modified variational iteration method and modified homotopy perturbation method were successfully applied to obtain the numerical solutions of linear and nonlinear fractional order integro-differential equations. These methods give more feasible series solutions that converge very rapidly to the exact solution. The efficiency of the methods in solving this type of problems were demonstrated with the presented examples. Their accuracy were compared with the exact solution on numerical tables and graphs to verify their validity and it is worth mentioning that: in both cases of the problem solved, the variational iteration method converges after the first iteration while the homotopy perturbation method converges after two iterations; the computer implementation of the methods are feasible as the computation process was carried out using MAPLE 18; the methods can easily handle both boundary value and initial value fractional order integro-differential equations.

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