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Two Modified Hybrid Inertial Algorithms for Bregman Weak Relatively Nonexpansive Mapping in Banach Spaces

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Abstract

This paper formulates two modified hybrid algorithms with inertial parameter corresponding to Bregman distance function for approximating common fixed point of a Bregman weak relatively nonexpansive mappings in a Banach space. The inertial parameter of these algorithms involve a computation of the gradient of the given function at each iterates. We prove a strong convergence theorems for these algorithms under appropriate control conditions. We perform numerical computations for particular examples as an applicable illustration for the theoretical analysis. Our results improves previously cited results of some authors in the literature.

1. Introduction

In this paper, all spaces are taking to be real $\mathbb{R}$, with $\mathbb{R}$ as the set of all real numbers and $\mathbb{N}$ the set of natural numbers. The extended real numbers is denoted by $\overline{\mathbb{R}}$. Let X represents reflexive real Banach space with its dual spaces denoted as

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X^*. Let $||.|| : X \to \mathbb{R}$ represents the norm function. Let $d_h : domh \times int(domh) \to \mathbb{R}^+$ represent a function induced by a convex function $h : X \to (-\infty, +\infty]$.

Definition 1.1. [16, 15] Let $h : X \to \mathbb{R} \cup \{+\infty\}$ be a function.

(a) The domain of h denoted by $dom\, h$ is the set $dom\, h = \{u \in X : h(u) < +\infty\}$
(b) h is called proper if $dom\, h$ is nonempty
(c) h is called convex if

$$h(\lambda u + (1-\lambda)z) \leq \lambda h(u) + (1-\lambda)h(z), \forall u, z \in dom\, h, \lambda \in (0,1). \tag{1}$$

(d) h is called strictly convex if the inequality in (1) is strict.
(e) The domain of h is said to be a convex set if the line segment $\lambda u + (1-\lambda)z \subset dom\, h$ for all $u, z \in dom\, h$ and $\lambda \in (0,1)$.
(f) h is called lower semi-continuous at $u_0 \in domh$ if $h(u_0) \leq \liminf_{u \to u_0} h(u)$
(g) h is called upper semi-continuous at $u_0 \in domh$ if $h(u_0) \geq \limsup_{u \to u_0} h(x)$
(h) The interior domain of h is denoted by $int(dom\, h)$
(i) h is coercive when $\lim_{\|u\| \to \infty} h(u) = +\infty$. It is strongly coercive when $\lim_{\|u\| \to \infty} \frac{h(u)}{\|u\|} = +\infty$.

If for any $u \in int(domh)$ and $z \in X$, we have the right-hand directional derivative given by

$$h^o(u,z) := \lim_{s \to 0^+} \frac{h(u+sz) - h(u)}{s}.$$

The convex function $h : X \to (-\infty, +\infty]$ is said to be Gâteaux differentiable at a point u if

$$\lim_{s \to 0^+} \frac{h(u+sz) - h(u)}{s}$$

exists for any z element of X. By this definition, $h^o(u,z) := \nabla h(u)$ which is the gradient of a convex function h. The function h is said to be Frechet differentiable at x if this limit is attained uniformly in $\|y\| = 1$. The convex function h is said to be uniformly Frechet differentiable on a subset K of X if the limit is attained uniformly for $x \in K$ and $\|y\| = 1$ [16]. Let $h : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semi-continuous (lsc) and convex functions. The function $h^* : X^* \to \mathbb{R}$ defined by

$$h^*(u^*) = \sup_{u \in X} \{\langle u, u^* \rangle - h(u)\} \tag{2}$$

is called conjugate function of h. With this definition, we have that h^* is also proper, lsc and convex (see [15]).

Let h be a Gâteaux differentiable function at u, then the bi-function $d_h : dom\, h \times int(dom\, h) \to \mathbb{R}^+$ defined by

$$d_h(z,u) := h(z) - h(u) - \langle \nabla h(u), z - u \rangle \tag{3}$$

is the Bregman distance function induced by the convex function h [3], where $\langle ., . \rangle$ is the duality pairing. It is easy to see that (3) is not symmetric and does not satisfy the well-known triangle inequality associated with classical distance functions, but has the following nice properties (see [3, 7]):

P1. $d_h(., u)$ is convex
P2. $d_h(z, u)$ is positive if and only if $z \neq u$
P3. $d_h(z, u) = d_h(z, v) + d_h(v, u) + \langle \nabla h(v), z - v\rangle - \langle \nabla h(u), z - v\rangle$
P4. $d_h(u, v) + d_h(v, u) := \langle \nabla h(u), u - v\rangle - \langle \nabla h(v), u - v\rangle$
P5. $d_h(u, v) \leq ||u||.||\nabla h(u) - \nabla h(v)|| + ||v||.||\nabla h(u) - \nabla h(v)||$
P6. $d_h(u, u) = 0$.

Let K represent a non-void, closed and convex subset of $int(dom\, h)$. Let $G : K \to K$ represent a self-map. The self-map G on K is said to be nonexpansive if $||Gu - Gz|| \leq ||u - z||$, $\forall u \in K$, $z \in K$. Similarly, the self-map G on K is said to be quasi-nonexpansive if $||Gu - z^o|| \leq ||u - z^o||$, $\forall u \in K$, $z^o \in Fix(G)$, where $Fix(G) := \{z^o \in K : Gu = u\}$ is the fixed point set of the self-map G on K. A point u^* is called asymptotic fixed point of a self-map G on K if there exist $u_n \subset K$ which converges weakly to u^* ($u_n \rightharpoonup u^*$) so that $||u_n - Gu_n|| \to 0$ as $n \to \infty$. The asymptotic fixed point (see [20] and the references therein) is represented by $\hat{Fix}(G)$. A point u^* is called strong asymptotic fixed point of a self-map G on K if there exist $u_n \subset K$ which converges strongly to u^* ($u_n \to u^*$) so that $||u_n - Gu_n|| \to 0$ as $n \to \infty$. The strong asymptotic fixed point [14] is represented by $\widehat{Fix}(G)$.

A map $G : K \to int(dom h)$ with respect to a convex function $h : X \to (-\infty, +\infty]$ is called

(1) Bregman relatively nonexpansive (shortly,(BRNE)) [16, 17] if the following conditions holds

$$d_h(z^o, Gu) \leq d_h(z^o, u), \forall u \in K, \forall z^o \in Fix(G) = \hat{Fix}(G)$$

(2) Bregman weak relatively nonexpansive (shortly,(BWRNE)) [8] if the following conditions holds

$$d_h(z^o, Gu) \leq d_h(z^o, u), \forall u \in K, \forall z^o \in Fix(G) = \widehat{Fix}(G)$$

Example 1.2. Let X represent a sequence space ℓ^2, $h(x) := \frac{1}{p}||x||^p$, where $p = 2, \forall x \in X$.

Let $X = \ell^2 = \{x = \{x_n\}_{n=1}^{\infty} \in R : \|x\|^2 \leq \infty\}, \|x\|^2 = \sum_{n=1}^{\infty} |x_n|^2, \forall x \in \ell^2, \langle x, y\rangle = \sum_{n=1}^{\infty} x_n y_n$. Let $\{x_n\} \subset X$ be defined by $x_n := (x_1, x_2, ...) = (1, \frac{1}{2}, ...)$. Next we define a mapping $T : X \to X$ by

$$G(x) = \begin{cases} \frac{n}{n+1}x \text{ if } x = x_n \\ -x \text{ if } x \neq x_n, \end{cases} \qquad \text{for all } n \geq 1. \tag{4}$$

Then it is clear that $x_n \in \ell^2$, x_n converges weakly to 0 and G is BRNE mapping.

Proof. Indeed for any $\gamma = (\gamma_1, \gamma_2, ...) \in \ell^2$, we have $\gamma(x_n - x_0) = \langle x_n - x_0, \gamma\rangle = \sum_{n=1}^{\infty} x_n\gamma_n = y_{n+1}$. But from definition $\|x\|^2 = \sum_{n=1}^{\infty} |x_n|^2 < \infty$, and consequently, $\lim_{n\to\infty} y_{n+1} = 0$. Moreover, $\lim_{n\to\infty}(\gamma(x_n - x_0)) = \lim_{n\to\infty}(\langle x_n - x_0, \gamma\rangle) = \lim_{n\to\infty}(y_{n+1}) = 0$. Hence $x_n \rightharpoonup x_0 = 0$.
The map G is from X into X and, $G(p) = p = 0$, hence $Fix(G) = \{0\}$, $n \in N$. Now for $x = x_n$, we have from Bregman distance (see (3)) that

$$\begin{aligned} d_h(0, Gx) &= h(0) - h(\frac{n}{n+1}x) - \left\langle \nabla h(\frac{n}{n+1}x), 0 - (\frac{n}{n+1}x) \right\rangle \\ &= 0 - \frac{1}{2}(\frac{n}{n+1}x)^2 + \left\langle \frac{n}{n+1}x, \frac{n}{n+1}x \right\rangle \\ &= \frac{1}{2}(\frac{n}{n+1}x)^2, \end{aligned}$$

and

$$\begin{aligned} d_h(0, x) &= h(0) - h(x) - \langle \nabla h(x), 0 - (x)\rangle \\ &= 0 - \frac{1}{2}(x)^2 + \langle x, x\rangle \\ &= \frac{1}{2}(x)^2. \end{aligned}$$

Thus

$$d_h(0, Gx) < d_h(0, x), \tag{5}$$

for all $x = x_n$, and for each $n \geq 1$.
For $x \neq x_n$, we compute

$$\begin{aligned} d_h(0, Gx) &= h(0) - h(-x) - \langle \nabla h(-x), 0 - (-x)\rangle \\ &= 0 - \frac{1}{2}(-x)^2 + \langle -x, x\rangle \\ &= \frac{1}{2}(x)^2 = d_h(0, x). \end{aligned}$$

Thus

$$d_h(0, Gx) = d_h(0, x), \tag{6}$$

for all $x \neq x_n$, and for each $n \geq 1$. Combining 5 and 6 we conclude that

$$d_h(0, Gx) \leq d_h(0, x), \tag{7}$$

for all x, and for each $n \geq 1$.
Therefore, G is a BRNE mapping and $Fix(G) = \overline{Fix}(G) = \hat{Fix}(G)$. Thus it implies BWRNE mapping. □

Example 1.3. cf.([8, 14]) Let X be a sequence space ℓ^2, $h(u) := \dfrac{1}{p}||u||^p$, where $p = 2, \forall u \in X$.
Let $X = \ell^2 = \{u = \{u_i\}_{i=1}^{\infty} \in R : \sum_{i=1}^{\infty} |u_i|^2 \leq \infty\}$. Let $\{u_n\} \subset X$ be defined by $u_n := (0, 0, ..., 1, 0, 0, 0, ...)$ where 1 is the n^{th} position, and $||u_n|| = 1$, $\forall n \in N$. It is easy to see $\forall n, m(n \neq m)$ that $||u_n - u_m|| = \sqrt{2}$ as $n \to \infty$. $\{u_n\}_{n=1}^{\infty}$ is not Cauchy, and hence not strongly convergent but weakly. Let $G : X \to X$ be defined by

$$G(u) := \begin{cases} \frac{n}{n+1}u & \text{if } u = u_n \\ -u & \text{if } u \neq u_n, \end{cases} \tag{8}$$

$\forall n \geq 1$.
Then G is BWNE mapping but not BRNE mapping.

Proof. Following the lines of proofs in ([14]), we assume given the sequence $\{w_n\}$ in X, which converges strongly to w_0 $(w_n \to w_0)$ so that $||w_n - Gw_n|| \to 0$ as $n \to \infty$. From the construction of the example, $\{u_n\}$ is a not a Cauchy sequence, and we can find a number $M \in N$ large enough so that $w_n \neq u_m, n, m > M$. Supposing we have $m \leq M$ such that $w_n = u_m$, then there exists a subsequence w_{n_j} satisfying $w_{n_j} = u_m$ so that $w_0 = \lim_{j\to\infty} w_{n_j} = u_m$ and $w_0 = \lim_{j\to\infty} Gw_{n_j} = Gu_m = \frac{m}{m+1}u_m$. This is not possible. It means that $Gw_n = -w_n$, hence $||w_n - Gw_n|| \to 0$ as $n \to \infty$. Consequently $2w_n \to 0$ and thus $w_n \to w_0 = 0$. Now, since $w_0 = 0 \in Fix(G)$, we then conclude that G is Bregman weak relatively nonexpansive mapping which is not Bregman relatively nonexpansive mapping. □

This completes the example.

Let $P_K^h : int(dom\ ) \to K$ be a mapping such that $P_K^h(u) \in K$ satisfying

$$d_h(P_K^h(u), u) := inf\,\{d_h(z, u) : z \in K\}, \tag{9}$$

is the Bregman Projection (see[16]) of $u \in int(dom\ h)$ onto a nonempty closed and convex set $K \subset dom\ h$.

Remark. We remark here that, if X is a smooth and strictly convex Banach spaces and $h(u) =: \|u\|^2$, $\forall u \in X$, then we have that $\nabla h(u) = 2Ju$, $\forall u \in X$, where J is the normalized duality mapping. Clearly, we obtain that

$$\begin{aligned} d_h(z, u) &:= h(z) - h(u) - \langle \nabla h(u), z - u \rangle \\ &= \|z\|^2 - \|u\|^2 - 2\langle z, Ju \rangle + 2\|u\|^2 \\ &= \|u\|^2 - 2\langle z, Ju \rangle + \|z\|^2 \\ &= \phi(z, u), \forall u, z \in X, \end{aligned}$$

which is the Lyapunov function introduced by [16] and has extensively been used by various authors (see for example [9, 13] and the references they contain). We clearly see that $P_K^h(u)$ reduces to the generalized projection [1] given as

$$\Pi_K(u) := argmin_{z\in K}\phi(z,u).$$

In addition, if X coincides with H, in Hilbert space then $J = I$ which is the identity map and

$$\begin{aligned} d_h(z,u) &:= h(z) - h(u) - \langle \nabla h(u), z-u\rangle \\ &= \|u\|^2 - \|z\|^2 - 2\langle u,z\rangle + 2\|z\|^2 \\ &= \|u\|^2 + \|z\|^2 - 2\langle u,z\rangle \\ &= \|u-z\|^2, \forall u,z \in X. \end{aligned}$$

Hence the Bregman Projection $P_K^h(x)$ reduces to metric projection of H onto K, $P_K(u)$.

A map $G : X \to X$ with respect to $h(u) = ||u||^2$ is called relatively nonexpansive [9] if the following conditions holds:

$$\phi(z^o, Gu) \le \phi(z^o, u), \forall u \in X, \forall z^o \in Fix(G) = \hat{Fix}(G).$$

Remark.

(1) We note that weak convergence of sequence need not imply strong convergence of the sequence to a given point,

(2) If a sequence $\{u_n\}$ in a nonempty, closed and convex subset of X converges strongly to a point $u \in K$, then $\{u_n\}$ also converges weakly to u,

(3) Every relatively nonexpansive mapping is Bregman relatively nonexpansive mapping with respect to $h(u) := \|u\|^2$, $\forall\ u \in X$

(4) Bregman weak relatively nonexpansive mapping includes Bregman relatively nonexpansive and relatively nonexpansive mappings and $\widehat{Fix}(G) \subset \hat{Fix}(G)$. Thus this class of Bregman mapping is a generalization of other mappings previously introduced and studied.

A function $h : X \to (-\infty, +\infty]$ is said to be a Legendre function [6], if it satisfies the following two conditions:

(1) $int(dom\ h) \neq \emptyset$, h is Gâteaux differentiable on $int(dom\ h)$ and $dom\ f = int(dom\ h)$,

(2) $int(dom h^*) \neq \emptyset$, h^* is Gâteaux differentiable on $int(dom h^*)$ and $dom h^* = int(dom\ h^*)$.

Remark. (See for e.g [4, 2, 15, 5] Since X is reflexive, then we have that $(\partial h^{-1}) := \partial h^*$ and since h is Legendre, then ∂h is a bijection which satisfies $\nabla h = (\nabla h^*)^{-1}$, $ran\nabla h = dom\nabla h^* = int(dom\ h^*)$ and $ran\nabla h^* = dom\nabla h =$

$int(dom\, h)$. h and h^* are strictly convex on their $int(dom\, h)$. If the subdifferential of h is single valued, it coincides with the gradient of h, that is $\partial h := \nabla h$. Example of a Legendre function is $h(u) := \frac{1}{p}||u||^p$, $(1 < p < \infty)$. If X is smooth and strictly convex Banach spaces, then in this case the gradient ∇h coincides with the generalised duality mapping of X, that is $\nabla h = J_p$. If the space is a Hilbert space, H then $\nabla h = I$, where I is the identity mapping in H. Throughout this paper, we assumed that h is Legendre.

Let $h : X \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The modulus of total convexity of h at $u \in int(dom\ h)$ is the function $V_h(u, .) : int(dom\ h) \times [0, +\infty) \to [0, +\infty)$ defined by

$$V_h(u,t) := inf\left\{d_h(z,u) : z \in dom\, h, \|z-u\| = t\right\}. \tag{10}$$

The function h is called totally convex (see [5] and the references it contains) at u if $V_h(u, t) > 0$ whenever $t > 0$. The function h is called totally convex if it is totally convex at any point $u \in int(dom\ h)$. The function is said to be totally convex on bounded sets if $V_h(B, t) > 0$ for any non-void bounded subset B of X and $t > 0$, where the modulus of total convexity of the function h on the set B is the function $V_h : int(dom\, h) \times [0, +\infty) \to [0, +\infty)$ defined by

$$V_h(B,t) := inf\left\{V_h(u,t) : u \in B \cap dom\, h\right\}. \tag{11}$$

Let $V_h : X \times X^* \to [0, +\infty)$ associated with h ([14, 13]) be defined by

$$V_h(u,u^*) := h(u) - \langle u, u^*\rangle + h^*(u^*), \forall u \in X, u^* \in X^*. \tag{12}$$

We see that $V_h(,) \geq 0$ and the relation

$$V_h(u,u^*) := d_h(u, \nabla h^*(u^*)), \tag{13}$$

holds.

The distance function d_h introduced by Bregman [3] with respect to convex function have been intensively used and studied by many authors over the past years to develop effective, efficient and implimentatble algorithms instead of norm. It has become an area of research and used among other things for solving numerious problems in common fixed point problems; nonlinear operator theory; convex feasibility problems; optimization problems; Nash equilibrium problems, smooth convex minimization problems, etc (see for example [16, 17, 8, 11, 14, 20] and the references therein).

Remarkably, smooth convex minimization problems involving generalized nonexpansive and Bregman nonexpansive-type operators have attracted the interest of many researchers and authors alike seeking existence of solutions to different common problems. It is a fact that most published research works on these operators has been the iterative approximation of (common) fixed points of the operators. Furthermore, most of the results obtained only focused on the strong or weak convergence of sequences generated by the formulated schemes to the fixed point

sets of the various mappings either in Hilbert spaces or general Banach spaces (see for example [16, 8, 14, 20] and the references therein).
However, very few authors have rapidly paid attention to the speed/rate of convergence of the generated sequence of iterates of some operators to their (common) fixed point sets when found to exists. Based on this, a two-step iterative method with an inertial parameter known to improve the rate of convergence when compared with iterative methods without it was used in the works of [10, 9, 12]. For the purpose of completeness, we have the following reviews for our motivation for the study.
Using iterative method corresponding to the Lyapunov functional of equation, [13] formulated the following algorithm:

$$\begin{cases} u_0 \in K, \\ y_n = J^{-1}(\alpha_n J u_n + (1-\alpha_n) JGu_n), \\ K_n = u \in K : \phi(u, y_n) \leq \phi(u, u_n), \\ Q_n = u \in K : \langle u - u_n, Ju_n - Ju \rangle \geq 0 \\ u_{n+1} = \Pi_{K_n \cap Q_n}(u_0), n \geq 1, \end{cases} \tag{14}$$

where $\alpha_n \in (0,1)$, $G : K \to K$ is a relatively nonexpansive map. The authors showed their sequence converge strongly to a fixed point set of their mapping which satisfies some convergence conditions.
Similarly, [9] formulated and studied the following algorithm generated by the sequence $\{u_n\}$ as follows: $u_0, u_1 \in X$ and

$$\begin{cases} K_0 = X, \\ z_n = u_n + \alpha_n(u_n - u_{n-1}), \\ y_n = J^{-1}((1-\beta) J z_n + \beta JGz_n), \\ K_{n+1} = u \in K_n : \phi(u, y_n) \leq \phi(u, z_n), \\ u_{n+1} = \Pi_{K_{n+1}}(u_0), n \geq 1, \end{cases} \tag{15}$$

where $\alpha_n \in (0,1)$, $\beta \in (0,1)$,$G : X \to X$ is a relatively nonexpansive map. The authors showed that their sequence converge strongly to a fixed point set of their mapping.
Also, by using the Lyapunov functional of equation, Wei et al. [18] presented some new modified inertial-type multi-choice CQ-algorithms for approximating common fixed point of countable weakly relatively non-expansive mappings in a real uniformly convex and uniformly smooth Banach space. They authors deployed a new proof techniques to prove for strong convergence theorems of their problems with numerical illustration to support the theoretical analysis.
Using iterative algorithm with computational errors corresponding to Bregman distance function, [8] was the first to introduce class of Bregman weak relatively

nonexpansive mapping in a real reflexive Banach space. Below is their theorem for this class of mapping:

Theorem 1.4. *([8] Theorem 3.1) Let K be a non-empty closed and convex subset of a real reflexive Banach space X and let $h : X \to R$ be a Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of X. Let $G : K \to K$ be a weak Bregman relatively nonexpansive mapping such that $Fix(G) \neq \emptyset$. Define a sequence $\{u_n\} \in K$ by the following algoithm:*

$$(16)\qquad \begin{cases} u_0 \in K, Q_0 = K, \\ z_n = \nabla h^*(b_n \nabla h(G(u_n + e_n)) + (1 - b_n)\nabla h(u_n + e_n)), \\ y_n = \nabla h^*(a_n \nabla h(u_n + e_n) + (1 - a_n)\nabla h(z_n)), \\ K_n = \{u^* \in K_{n-1} \cap Q_{n-1} : d_h(u^*, y_n) \leq d_h(u^*, u_n + e_n)\}, \\ K_0 = \{u^* \in K : d_h(u^*, y_0) \leq d_h(u^*, u_0)\}, \\ Q_n = \{u^* \in K : \langle \nabla h(u_0) - \nabla h(u_n), u_0 - u_n\rangle \leq 0\}, \\ u_{n+1} = P^h_{K_n \cap Q_n}(u_0), n \geq 0. \end{cases}$$

Then the sequence $\{u_n\}$ generated by their algorithm (16) converge strongly to a point set of their mapping.

Following from the above work of [8], [14] introduce a hybrid method for finding a common fixed point of finite family of Bregman weak relatively nonexpansive self-mappings on X. The authors proved a strong convergence of their sequences to a common point set of their mappings. Below is their algorithm:

$$(17)\qquad \begin{cases} x_0 = x \in X, \\ K_0 = X, \\ z_n = \nabla h^*[b_{n,0}\nabla h(x_n) + \sum_{j=1}^N b_{n,j}\nabla h(G_j(x_n + e^j_n))], \\ K_{n+1} = \{u^* \in K_n : d_h(u^*, z_n) \leq d_h(u^*, x_n) \\ + \sum_{j=1}^N b_{n,j} d_h(x_n, x_n + e^i_n) \\ + \sum_{j=1}^N b_{n,j}\langle u^* - x_n, \nabla h(x_n) - \nabla h(x_n + e^j_n\rangle\}, \\ x_{n+1} = P^h_{C_{n+1}}(x), n \geq 0. \end{cases}$$

where for each $j = \{0, 1, 2, ..N\}$, $\{e^j_n\}$ is the error sequence and $\{b_{n,j} : n \geq 0, j = \{0, 1, 2, ..N\}\}$, is a scalar sequence in $(0, 1)$ satisfying the following conditions:

(a) $\sum_{j=0}^N b_{n,j} = 1$
(b) There exists $i \in \{1, 2, ..., N\}$ such that $\liminf_{n\to\infty} b_{n,i}b_{n,j} > 0, \forall j \in \{1, 2, ..., N\}$.

Using a modified Mann iteration corresponding to Bregman distance function, [20] formulated the following algorithms:

Theorem 1.5. *([20] Theorem 13) Let $G_i : K \to K$ $\forall i = 1, 2, ..., N$ be a finite family of Bregman weak relatively nonexpansive mappings. Assume that the interior of $\bigcap_{i=1}^{N} Fix(G_i) \neq \emptyset$. For $u_o \in K$, $\{u_n\}$ be a sequence generated by*

$$u_{n+1} = P_K^h \nabla h^*(b_0 \nabla h(u_n) + \sum_{i=1}^{N} b_i \nabla h(T_i u_n), n \geq 0. \tag{18}$$

Then the sequence $\{u_n\}$ generated by their algorithm (18) converge strongly to a point set of their mapping provided that the interior of $\bigcap_{i=1}^{N} Fix(G_i) \neq \emptyset$.

Remark.

(1) Algorithm (14) and (16) are implementable. However, it contains two half-spaces K_n and Q_n which must be computed at each stage of the iteration and the next iterate taken as the generalised or Bregman projection of initial guess onto the intersection of two half space which is taken to be the feasible solution. The computations of these algorithms may require more computer time in application.

(2) The strong convergence of sequences generated by algorithm (18) is guaranteed by imposing the condition; "interior of the mappings is not empty-set", that is, $\bigcap_{i=1}^{N} Fix(G_i) \neq \emptyset, N \in \mathbb{N}$. This condition is noted to be highly restrictive in analysis.

(3) Algorithm (15) combines the Mann iteration with the inertial parameter. Furthermore, the algoorithm is formulated for a countable family of relatively generalised nonexpansive mappings in a uniformly smooth an uniformly convex Banach spaces.

Motivated by the results of [8, 14, 10, 9, 12, 13, 20, 18], our aim is to study two modified hybrid algorithms with inertial parameter for the problem of approximating the common fixed point of a Bregman weak relatively nonexpansive mappings in real reflexive Banach spaces. To achieve this aim, we prove two strong convergence theorems for such problem, which finds a common fixed point for a finite family of this class of generalized mappings under appropriate and very less restrictive control conditions. In addition, we perform numerical computations with particular examples as an applicable illustration for the theoretical analysis. The intended results will improve previously cited results for this class of generalized Bregman mapping in the literature (see [8, 14, 20]).

2. Preliminaries

Lemma 2.1. [5] *The function $h : X \to (-\infty, +\infty]$ is totally convex on bounded sets if and only if given two sequences $\{u_n\}$ and $\{z_n\}$ in X such that either $\{u_n\}$ or $\{z_n\}$ is bounded, then*

$$\lim_{n\to\infty} d_h(z_n, u_n) = 0 \Rightarrow ||z_n - u_n|| = 0$$

Lemma 2.2. [14] *Let $h : X \to (-\infty, +\infty]$ be a Legendre function. Let K be a non-empty, closed and convex subset of $int(dom\ h)$. Let $T : K \to K$ be a Bregman weak relatively nonexpansive map with respect to $h : X \to (-\infty, +\infty]$. Then $Fix(T)$ is closed and convex.*

Lemma 2.3. [19] *Let X be a reflexive real Banach space. Let $h : X \to (-\infty, +\infty]$ be a continuous convex function which is super coercive. Then the following assertions are equivalent:*

(1) $h : X \to (-\infty, +\infty]$ is bounded on bounded subsets and uniformly smooth on bounded subsets of X
(2) $h : X \to (-\infty, +\infty]$ is Frechet differentiable and ∇h^* is uniformly norm-to-norm continuous on bounded subsets of X^*
(3) dom $h^* = X^*$, h^* is super coercive and uniformly convex on bounded subsets of X^*

Lemma 2.4. [16] *Let $h : X \to (-\infty, +\infty]$ be a differentiable function on $int(domh)$ such that ∇h^* is bounded on bounded subsets of dom h^*. Let $u_0 \in X$ and $\{u_n\}$ is a sequence in X. If $\{d_h(u_0, u_n)\}$ is bounded, then $\{u_n\}$ is bounded.*

Lemma 2.5. [14, 20] *Let X be a Banach space. Let $r > 0$ be a constant and let $h : X \to (-\infty, +\infty]$ be a continuous and convex function which is uniformly convex on bounded subsets of X. Then*

$$h(\sum_{k=0}^{\infty} \alpha_k u_k) \leq \sum_{k=0}^{\infty} \alpha_k u_k h(u_k) - \alpha_i \alpha_j \rho_r(||u_i - u_j||), \forall i, j \in N \cup 0$$

, $u_k \in B_r$, $\alpha_k \in (0, 1)$ with $\sum_{k=0}^{\infty} \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of h, B_r is an open ball with radius r.

Lemma 2.6. [15]. *Let $h : X \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function, then $h^* : X^* \to (-\infty, +\infty]$ is a proper, weak* lower semi-continuous and convex function. Thus, for all $z \in X$, we have*

$$d_h(z, \nabla h^*(\sum_{i=1}^{N} t_i \nabla h(x_i))) \leq \sum_{i=1}^{n} t_i d_h(z, x_i).$$

Lemma 2.7. [19] *Let K be anon-empty, closed and convex subsets of X. Let $h : X \to (-\infty, +\infty]$ be a differentiable and totally convex function and let $u \in X$. Then we have the following equivalent conditions:*

(1) $P_K^h(u) = u_0$ *if and only if* $\langle \nabla h(u) - \nabla h(u_0), z - u_0 \rangle \leq 0, \forall z \in K$
(2) $d_h(z, P_K^h(u)) + d_h(P_K^h(u), u) \leq d_h(z, u), \forall z \in K$.

3. Main Results

Theorem 3.1. *Let X represent real reflexive Banach space with its dual space X^*. Let $h : X \to (-\infty, +\infty]$ represent a strongly coercive Legendre function which is bounded, uniform Frechet differentiable and totally convex on bounded subsets of X. Let $G_i : X \to X, \forall i = 1, 2, \ldots, N$ represent finite family of Bregman weak relatively nonexpansive mappings such that $\Omega = \bigcap_{i=1}^N Fix(G_i) \neq \emptyset$. Initialize $x_0, x_1 \in X$ arbitrarily. Define the sequence $\{x_n\}$ by the following manner below:*

$$(19) \qquad \begin{cases} x_0 \in K_1 = X \\ z_n = \nabla h^*(\nabla h(x_n) + \alpha_n(\nabla h(x_n) - \nabla h(x_{n-1}))), \\ y_n = \nabla h^*(b_0 \nabla h(x_n) + \sum_{i=1}^N b_i \nabla h(G_i z_n)), \\ K_{n+1} = \{u \in K_n : d_h(u, y_n) \leq b_0 d_h(u, x_n) + \sum_{i=1}^N b_i d_h(u, z_n)\}, \\ x_{n+1} = P_{K_{n+1}}^h(x_0), n \geq 1, \end{cases}$$

where $\{\alpha_n\} \in (0,1)$, $\{b_i\} \in (0,1)$ such that $\sum_{i=0}^N b_i = 1$. Then the sequences $\{x_n\}, \{z_n\}, \{y_n\}$ in (19) converges strongly to $\overline{x} = P_\Omega^h(x_0)$ nearest to x_0.

Proof. We divide the proof into various steps.

Step 1. The scheme (19) is well-defined.
We observe from Lemma 2.2 that $Fix(G)$ is closed and convex. Next we demonstrate that the half-set K_n is closed and convex for each positive integer. To realize this, we observe from our setting in (19) that K_n is closed. Similarly, since $d_h(u, y_n) \leq b_0 d_h(u, x_n) + \sum_{i=1}^N b_i d_h(u, z_n)$, we let $u_1, u_2 \in K_{n+1}$ be given. Consider $u = \lambda u_1 + (1-\lambda) u_2, \forall \lambda \in [0,1]$. We show that $\lambda u_1 + (1-\lambda) u_2 \in K_{n+1} \subset K_n$. Since $u_1, u_2 \in K_n$, we observe that

$$(20) \qquad d_h(u_1, y_n) \leq b_0 d_h(u_1, x_n) + \sum_{i=1}^N b_i d_h(u_1, z_n)$$

and

$$(21) \qquad d_h(u_2, y_n) \leq b_0 d_h(u_2, x_n) + \sum_{i=1}^N b_i d_h(u_2, z_n).$$

Applying the definition of Bregman distance function as defined by (3), we have that (20) and (21) are respectively equivalent to

(22)
$$b_0\langle\nabla h(x_n), u_1-x_n\rangle+\sum_{i=1}^{N} b_i\langle\nabla h(z_n), u_1-z_n\rangle-\langle\nabla h(y_n), u_1-y_n\rangle \leq h(y_n)-h(x_n)-\sum_{i=1}^{N} b_i h(z_n),$$

and

$$\begin{aligned} b_0\langle\nabla h(x_n), u_2-x_n\rangle+ \\ \sum_{i=1}^{N} b_i\langle\nabla h(z_n), u_2-z_n\rangle- \\ \langle\nabla h(y_n), u_2-y_n\rangle \leq \\ h(y_n)-h(x_n)-\sum_{i=1}^{N} b_i h(z_n). \end{aligned} \tag{23}$$

We now multiply both sides of (22) and (23) with λ and $(1-\lambda)$ respectively to get:

$$\begin{aligned} d_h(\lambda u_1+(1-\lambda)u_2, y_n) \leq \\ b_0 d_h(\lambda u_1+(1-\lambda)u_2, x_n)+ \\ \sum_{i=1}^{N} b_i d_h(\lambda u_1+(1-\lambda)u_2, z_n). \end{aligned}$$

This shows that $\lambda u_1+(1-\lambda)u_2 \in K_{n+1} \subset K_n$. Thus, K_{n+1} is closed and convex for all positive integers.

In addition, we demonstrate that $\Omega \subset K_{n+1}$, for all $n \in N$. Let $q \in \Omega$ and since G_i is a finite family of Bregman weak relatively nonexpansive mappings, we obtain using (12) and (13) that:

$$\begin{aligned}
d_h(q, y_n) &= d_h(q, \nabla h^*(b_0 \nabla h(x_n) + \sum_{i=1}^{N} \nabla h(G_i z_n))) \\
&= V_h(q, b_0 \nabla h(x_n) + \sum_{i=1}^{N} \nabla h(G_i z_n)) \\
&= h(q) - \langle q, b_0 \nabla h(x_n) + \sum_{i=1}^{N} \nabla h(G_i z_n) \rangle + h^*(b_0 \nabla h(x_n) \\
&+ \sum_{i=1}^{N} \nabla h(G_i z_n)) \\
&= h(q) - b_0 \langle q, \nabla h(x_n) \rangle + \sum_{i=1}^{N} \langle q, \nabla h(G_i z_n) \rangle + h^*(b_0 \nabla h(x_n) \\
&+ \sum_{i=1}^{N} \nabla h(G_i z_n)) \\
&\leq b_0 h(q) + \sum_{i=1}^{N} h(q) - b_0 \langle q, \nabla h(x_n) \rangle + \sum_{i=1}^{N} \langle q, \nabla h(G_i z_n) \rangle \\
&+ b_0 h^*(\nabla h(x_n)) + \sum_{i=1}^{N} h^*(\nabla h(G_i z_n)) \\
&= b_0 V_h(q, \nabla h(x_n)) + \sum_{i=1}^{N} V_h(q, \nabla h(G_i z_n)) \\
&= b_0 d_h(q, x_n) + \sum_{i=1}^{N} d_h(q, G_i z_n) \\
&\leq b_0 d_h(q, x_n) + \sum_{i=1}^{N} d_h(q, z_n). \quad (24)
\end{aligned}$$

Thus,

$$d_h(q, y_n) \leq b_0 d_h(q, x_n) + \sum_{i=1}^{N} d_h(q, z_n). \quad (25)$$

Here, $q \in K_{n+1}$ and $K_{n+1} \subset K_n$. This implies by set induction that $\Omega \in K_n$. Thus, the scheme defined by (19) is well-defined for all natural numbers. This completes the proof of **Step 1**.

Step 2. Let $\{x_n\}$ be the sequence generated by (19), then the following assertions hold

(1) $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$,
(2) $\lim_{n\to\infty} ||x_n - z_n|| = 0$,
(3) $\lim_{n\to\infty} ||x_{n+1} - y_n|| = 0$,
(4) $\lim_{n\to\infty} ||z_n - y_n|| = 0$,
(5) $\lim_{n\to\infty} ||z_n - G_i z_n|| = 0$.

To justify the above assertions, notice from our setting in (19), that $x_n = P^h_{K_n}(x_0)$ and $x_{n+1} = P^h_{K_{n+1}}(x_0) \in K_{n+1} \subset K_n$. Thus, we obtain using Lemma 2.7 that

$$\begin{aligned} d_h(x_n, x_0) &\le d_h(x_{n+1}, x_0) - d_h(x_{n+1}, x_n) \\ d_h(x_{n+1}, x_0) &\ge d_h(x_n, x_0). \end{aligned} \tag{26}$$

This shows that $\{d_h(x_n, x_0)\}$ is monotone non-decreasing sequence. Again using Lemma 2.7, we obtain $\forall n \in N, q \in Fix(G)$ that

$$\begin{aligned} d_h(x_n, x_0) &= d_h(P^h_{K_n}(x_0), x_0) \\ &\le d_h(q, x_0) - d_h(q, P^h_{K_n}(x_0)) \\ &\le d_h(q, x_0). \end{aligned} \tag{27}$$

This shows that $\{d_h(x_n, x_0)\}$ is bounded. From Lemma 2.4, $\{x_n\}$ is bounded. Consequently, $\{z_n\}$ bounded. Following from the fact that $d_h(q, G_i z_n) \le d_h(q, z_n), \forall q \in \Omega, \forall i = 1, 2, \ldots, N$, we get that $\{G_i z_n\}$ is bounded $\forall i = 1, 2, \ldots, N$. Since $d_h(q, x_n)$ is bounded, we set $M = \sup_n\{d_h(q, x_n)\}$ for some n. Furthermore, we assume that $d_h(q, z_n) \le M$. It then follows from our setting that

$$\begin{aligned} d_h(q, y_n) &\le b_0 M + \sum_{i=1}^{N} b_i M \\ &= (1 - \sum_{i=1}^{N} b_i)M + \sum_{i=1}^{N} b_i M \\ &= M. \end{aligned} \tag{28}$$

Hence $\{d_h(q, y_n)\}$ is bounded and thus $\{y_n\}$ is bounded. Combining (26) and (27) shows that $\lim_{n\to\infty} d_h(x_n, x_0)$ exist. Now, without loss of generality, let

$$\lim_{n\to\infty} d_h(x_n, x_0) = l. \tag{29}$$

In addition to (29) and Lemma 2.7, we get for any positive integer t and as $n \to \infty$, that

$$\begin{aligned} d_h(x_{n+t}, x_n) &= d_h(x_{n+t}, P^h_{K_n}(x_0)) \\ &\le d_h(x_{n+t}, x_0) - d_h(x_n, x_0) \to 0. \end{aligned} \tag{30}$$

So that $\lim_{n\to\infty} d_h(x_{n+t}, x_n) = 0$. In particular,

$$\lim_{n\to\infty} d_h(x_{n+1}, x_n) = 0. \tag{31}$$

Therefore, we obtain using Lemma 2.1 that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{32}$$

From (32), assertion (1) is justified. This also implies that $\{x_n\}$ is a Cauchy sequence. Hence $x_n \to \overline{x} \in X$. Now since ∇h is uniformly continuous, we obtain that

$$\lim_{n\to\infty} ||\nabla h(x_{n+1}) - \nabla h(x_n)|| = 0. \tag{33}$$

Furthermore, we obtain from the definition of z_n and together with (33) as $n \to \infty$ that

$$\begin{aligned} ||\nabla h(x_n) - \nabla h(z_n)|| &= ||\nabla h(x_n) - \nabla h(x_n) - \alpha_n \nabla h(x_n - x_{n-1})|| \\ &= ||\alpha_n \nabla h(x_{n-1}) - \nabla h(x_n)|| \\ &\leq ||\nabla h(x_{n-1}) - \nabla h(x_n)|| \to 0. \end{aligned}$$

This implies that

$$\lim_{n\to\infty} ||\nabla h(x_n) - \nabla h(z_n)|| = 0. \tag{34}$$

Using Lemma 2.3, we obtain that

$$\lim_{n\to\infty} ||x_n - z_n|| = 0. \tag{35}$$

From (35), assertion (2) is justified. In view of $x_n \to \overline{x} \in K$, we obtain that $z_n \to \overline{x} \in K$. More so, since $\{z_n\}$ is bounded, we obtain from (34), (35) and $(P5)$ as $n \to \infty$ that

$$\begin{aligned} d_h(x_n, z_n) + d_h(z_n, x_n) &= \langle \nabla h(x_n) - \nabla h(z_n), x_n - z_n \rangle \\ d_h(x_n, z_n) + d_h(z_n, x_n) &\leq ||x_n - z_n||.||\nabla h(x_n) - \nabla h(z_n)|| \end{aligned}$$

$$d_h(x_n, z_n) \leq ||x_n - z_n||.||\nabla h(x_n) - \nabla h(z_n)|| \to 0. \tag{36}$$

In addition to the above and since $x_{n+1} \in K_{n+1} \subset K_n$, we obtain from the definition of half-set in (19) that

$$d_h(x_{n+1}, y_n) \leq b_0 d_h(x_{n+1}, x_n) + \sum_{i=1}^{N} b_i d_h(x_{n+1}, z_n). \tag{37}$$

Moreover by combining (31), (34) and $(P5)$ we obtain as $n \to \infty$ that

$$\begin{aligned} 0 \leq d_h(x_{n+1}, z_n) &= d_h(x_{n+1}, x_n) + d_h(x_n, z_n) + \langle \nabla h(x_n) - \nabla h(z_n), x_{n+1} - x_n \rangle \\ &\leq d_h(x_{n+1}, x_n) + d_h(x_n, z_n) \\ &+ ||\nabla h(x_n) - \nabla h(z_n)||.||x_{n+1} - x_n|| \to 0. \end{aligned}$$

Hence

$$\lim_{n\to\infty} d_h(x_{n+1}, z_n) = 0. \tag{38}$$

Consequently, we obtain that

$$\lim_{n\to\infty} d_h(x_{n+1}, y_n) = 0. \tag{39}$$

Thus, by Lemma 2.1 we obtain that

$$\lim_{n\to\infty} ||x_{n+1} - z_n|| = 0, \tag{40}$$

and

$$\lim_{n\to\infty} ||x_{n+1} - y_n|| = 0. \tag{41}$$

From (41), assertion (3) is justified. Combining (32) and (41) we also obtain as $n \to \infty$ that

$$||x_n - y_n|| \leq ||x_n - x_{n+1}|| + ||x_{n+1} - y_n|| \to 0.$$

This implies that

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \tag{42}$$

Since $x_n \to \overline{x} \in X$, we obtain that $y_n \to \overline{x} \in X$. Thus from (40) and (41) we obtain as $n \to \infty$ that

$$||z_n - y_n|| \leq ||z_n - x_{n+1}|| + ||x_{n+1} - y_n|| \to 0.$$

This implies

$$\lim_{n\to\infty} ||z_n - y_n|| = 0. \tag{43}$$

From (42), assertion (4) is justified. Since ∇h is uniformly continuous, we then get using (42) that

$$\lim_{n\to\infty} ||\nabla h(x_n) - \nabla h(y_n)|| = 0. \tag{44}$$

Furthermore, using the definition of y_n in (19) we obtain that

$$\begin{aligned} ||\nabla h(z_n) - \nabla h(y_n)|| &= ||\nabla h(z_n) - \nabla h(\nabla h^*(b_0 \nabla h(x_n) + \sum_{i=1}^{N} b_i \nabla h(G_i z_n)))|| \\ &\geq \sum_{i=1}^{N} b_i ||\nabla h(z_n) - \nabla h(G_i z_n))|| - b_0 ||\nabla h(z_n) - \nabla h(x_n)||, \end{aligned}$$

this becomes

$$\sum_{i=1}^{N} b_i ||\nabla h(z_n) - \nabla h(G_i z_n)|| \leq ||\nabla h(z_n) - \nabla h(w_y)|| + b_0 ||\nabla h(z_n) - \nabla h(x_n)||. \tag{45}$$

Thus, by (44) and (35), as $n \to \infty$, we get from (45) that

$$\sum_{i=1}^{N} b_i||\nabla h(z_n) - \nabla h(G_i z_n)|| \to 0.$$

Since $\sum_{i=1}^{N} b_i > 0$, we then have that

$$\lim_{n\to\infty} ||\nabla h(z_n) - \nabla h(G_i z_n)|| = 0. \tag{46}$$

Using by Lemma 2.3, we get that

$$\lim_{n\to\infty} ||z_n - G_i z_n|| = 0. \tag{47}$$

From (47), assertion (5) is justified. This completes the proof of **Step 2**.

Step 3. The sequence $\{x_n\}$ generated by algorithm (19) converges strongly to the element of Ω nearest to x_0. We show that $x_n \to \overline{x} = P^h_\Omega(x_0)$ as $n \to \infty$. In **Step 1**, we demonstrated that $\Omega \subset K_{n+1}$. Since $P^h_\Omega(x_0) \in \Omega$, we get that $P^h_\Omega(x_0) \subset K_{n+1}$. It then follows from our setting with $x_n = P^h_{K_{n+1}}(x_0)$ that $d_h(x_n, x_0) \leq d_h(\overline{x}, x_0)$. Since $\{x_n\}$ is a convergent sequence, we obtain from Lemma 2.7 that $x_n \to \overline{x}$ as $n \to \infty$. Consequently,$\{z_n\}, \{y_n\}$ converges strongly to $\overline{x} = P^h_\Omega(x_0)$. This completes the proof of **Step 3** and consequently the proof our Theorem 3.1. □

Theorem 3.2. *Let X represent real reflexive Banach space with its dual space X^*.. Let $h : X \to (-\infty, +\infty]$ represent a Legendre function which is bounded, uniform Frechet differentiable and totally convex on bounded subsets of X. Let $G_i : X \to X, \forall i = 1, 2, \ldots, N$ represent finite family of Bregman relatively nonexpansive mappings such that $\Omega = \bigcap_{i=1}^{N} Fix(G_i) \neq \emptyset$. Then the sequences $\{x_n\}, \{z_n\}, \{y_n\}$ in (19) converges strongly to $\overline{x} = P^h_\Omega(x_0)$ nearest to x_0*

Proof. For any subsequence z_{n_j} of z_n which converges weakly to $\overline{x}$, we get

$$\lim_{n\to\infty} ||z_{n_j} - G_i z_{n_j}|| = 0. \tag{48}$$

Since $G_i, \forall i = 1, 2, ..., N$ is a finite family of Bregman relatively nonexpansive mappings, we get that $\bigcap_i^N Fix(G_i) = \bigcap_i^N \hat{Fix}(G_i)$. Using the same method of proof as above, we conclude that sequences $\{x_n\}, \{z_n\}, \{y_n\}$ converge strongly to $\overline{x} = P^h_\Omega(x_0)$ nearest to x_0. □

If in Theorem 3.1, we take $N = 1$, then we have

Corollary 3.3. *Let X represent real reflexive Banach space with its dual space X^*.. Let $h : X \to (-\infty, +\infty]$ represent a Legendre function which is bounded, uniform Frechet differentiable and totally convex on bounded subsets of X. Let*

$G : X \to X$, represent a Bregman relatively nonexpansive self-mapping such that $\Omega = Fix(G) \neq \emptyset$. Define the sequence $\{x_n\}$ by the following manner below:

$$(49)\qquad \begin{cases} x_0 \in K_1 = X \\ z_n = \nabla h^*(\nabla h(x_n) + \alpha_n(\nabla h(x_n) - \nabla h(x_{n-1}))), \\ y_n = \nabla h^*(b\nabla h(x_n) + (1-b)\nabla h G z_n), \\ K_{n+1} = \{u \in K_n : d_h(u, y_n) \leq b d_h(u, x_n) + (1-b) d_h(u, z_n)\}, \\ x_{n+1} = P^h_{K_{n+1}}(x_0), n \geq 1, \end{cases}$$

where $\{\alpha_n\} \in (0,1)$, $b \in (0,1)$. Then the sequences $\{x_n\}, \{z_n\}, \{y_n\}$ in (49) converges strongly to $\overline{x} = P^h_{\Omega}(x_0)$ nearest to x_0.

If $h(u) = ||u||^2$, $\forall u \in X$, then from Remarks 1 and 2, and Theorem 3.1, we have the following strong convergence result in smooth and strictly convex Banach space.

Corollary 3.4. *Let X represent real reflexive, smooth, and strictly convex Banach space with its dual space X^*. Let $G : X \to X$, represent a relatively nonexpansive self-mapping such that $\Omega = Fix(G) \neq \emptyset$. Define the sequence $\{x_n\}$ by the following manner below:*

$$(50)\qquad \begin{cases} x_0 \in K_1 = X \\ z_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = J^{-1}(bJx_n + (1-b)J(Gz_n)), \\ K_{n+1} = \{u \in K_n : \phi(u, y_n) \leq b\phi(u, x_n) + (1-b)\phi(u, z_n)\}, \\ x_{n+1} = \Pi_{K_{n+1}}(x_0), n \geq 1, \end{cases}$$

where $\{\alpha_n\} \in (0,1)$, $b \in (0,1)$. Then the sequences $\{x_n\}, \{z_n\}, \{y_n\}$ in (50) converge strongly to $\overline{x} = \Pi_{\Omega}(x_0)$ nearest to x_0.

If $h(u) = ||u||^2$, $\forall u \in X$, and $N = 1$ then from Remarks 1, and Theorem 3.1, we have the following strong convergence result in Hilbert space.

Corollary 3.5. *Let K represent a non-empty, closed and convex subset of a real Hilbert space X with its dual space X. Let $G : X \to X$, represent a nonexpansive self-mapping such that $\Omega = Fix(G) \neq \emptyset$. Define the sequence $\{x_n\}$ by the following manner below:*

$$(51)\qquad \begin{cases} x_0 \in K_1 = X \\ z_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = bx_n + (1-b)Gz_n, \\ K_{n+1} = \{u \in K_n : ||y_n - u|| \leq b||x_n - u|| + (1-b)||z_n - u||\}, \\ x_{n+1} = P_{K_{n+1}}(x_0), n \geq 1, \end{cases}$$

where $\{\alpha_n\} \in (0,1)$, $b \in (0,1)$. Then the sequences $\{x_n\}, \{z_n\}, \{y_n\}$ in (51) converge strongly to $\overline{x} = P_\Omega(x_0)$ nearest to x_0.

Following the approach used in [14], we have the following result

Theorem 3.6. *Let X represent real reflexive Banach space with its dual space X^*. Let $h : X \to (-\infty, +\infty]$ represent a Legendre function which is bounded, uniform Frechet differentiable and totally convex on bounded subsets of X. Let $G_i : X \to X, \forall i = 1, 2, \ldots, N$ represent finite family of Bregman weak relatively nonexpansive mappings such that $\Omega = \bigcap_{i=1}^N Fix(G_i) \neq \emptyset$. Initialize $x_0, x_1 \in X$ arbitrarily. Define the sequence $\{x_n\}$ by the following manner below:*

$$\begin{cases} x_0 \in K_1 = X \\ z_n = \nabla h^*(\nabla h(x_n) + \alpha_n(\nabla h(x_n) - \nabla h(x_{n-1}))), \\ y_n = \nabla h^*(b_0 \nabla h(x_n) + \sum_{i=1}^N b_i \nabla h(G_i z_n)), \\ K_{n+1} = \{u \in K_n : d_h(u, y_n) \leq d_h(u, x_n) + \sum_{i=1}^N b_i d_h(x_n, z_n) \\ + \sum_{i=1}^N b_i \langle \nabla h(z_n - \nabla h(x_n), x_n - u\rangle\}, \\ x_{n+1} = P^h_{K_{n+1}}(x_0), n \geq 1, \end{cases} \tag{52}$$

where $\{\alpha_n\} \in (0,1)$, $\{b_i\} \in (0,1)$ such that $\sum_{i=0}^N b_i = 1$. Then the sequences $\{x_n\}, \{z_n\}, \{y_n\}$ in (52) converges strongly to $\overline{x} = P^h_\Omega(x_0)$ nearest to x_0.

Proof. The scheme (52) is well-defined. We demonstrate that the half-set K_n is closed and convex for each positive integer. To realize this, we observe from our setting in (52) that K_n is closed. Similarly, since $d_h(u, y_n) \leq b_0 d_h(u, x_n) + \sum_{i=1}^N b_i d_h(x_n, z_n) + \sum_{i=1}^N b_i \langle \nabla h(z_n - \nabla h(x_n), x_n - u\rangle\}$, we let $u_i, u_2 \in K_{n+1}$ be given. Consider $u = \lambda u_1 + (1-\lambda)u_2, \forall \lambda \in [0,1]$. We show that $\lambda u_1 + (1-\lambda)u_2 \in K_{n+1} \subset K_n$. Since $u_1, u_2 \in K_n$ and following similar approach from the proof of Theorem 3.1, we get that $d_h(u, y_n) \leq b_0 d_h(u, x_n) + \sum_{i=1}^N b_i d_h(x_n, z_n) + \sum_{i=1}^N b_i \langle \nabla h(z_n - \nabla h(x_n), x_n - u\rangle\}$ is equivalent to

$$\begin{aligned} h(x_n) - h(y_n) + \langle \nabla h(x_n), \lambda u_1 + (1-\lambda)u_2 - x_n\rangle \\ -\langle \nabla h(y_n), \lambda u_1 + (1-\lambda)u_2 - y_n\rangle \\ \leq \sum_{i=1}^N b_i d_h(x_n, z_n) \\ + \sum_{i=1}^N b_i \langle \nabla h(z_n) - \nabla h(x_n), x_n - \lambda u_1 + (1-\lambda)u_2\rangle \end{aligned}$$

This shows that $\lambda u_1 + (1-\lambda)u_2 \in K_{n+1} \subset K_n$. Thus, by an easy argument, K_{n+1} is closed and convex for all positive integers.

In addition, we demonstrate that $\Omega \subset K_{n+1}$, for all $n \in N$. Let $q \in \Omega$ and since G_i is a finite family of Bregman weak relatively nonexpansive mappings, we obtain using (12) and (13) that:

$$\begin{aligned}
d_h(q, y_n) &= d_h(q, \nabla h^*(b_0 \nabla h(x_n) + \sum_{i=1}^{N} \nabla h(G_i z_n)) \\
&= V_h(q, b_0 \nabla h(x_n) + \sum_{i=1}^{N} \nabla h(G_i z_n)) \\
&= h(q) - \langle q, b_0 \nabla h(x_n) + \sum_{i=1}^{N} \nabla h(G_i z_n) \rangle + h^*(b_0 \nabla h(x_n) \\
&+ \sum_{i=1}^{N} \nabla h(G_i z_n)) \\
&= h(q) - b_0 \langle q, \nabla h(x_n) \rangle + \sum_{i=1}^{N} \langle q, \nabla h(G_i z_n) \rangle + h^*(b_0 \nabla h(x_n) \\
&+ \sum_{i=1}^{N} \nabla h(G_i z_n)) \\
&\leq b_0 h(q) + \sum_{i=1}^{N} h(q) - b_0 \langle q, \nabla h(x_n) \rangle + \sum_{i=1}^{N} \langle q, \nabla h(G_i z_n) \rangle \\
&+ b_0 h^*(\nabla h(x_n)) + \sum_{i=1}^{N} h^*(\nabla h(G_i z_n)) \\
&= b_0 V_h(q, \nabla h(x_n)) + \sum_{i=1}^{N} V_h(q, \nabla h(G_i z_n)) \\
&= b_0 d_h(q, x_n) + \sum_{i=1}^{N} d_h(q, G_i z_n) \\
&\leq b_0 d_h(q, x_n) + \sum_{i=1}^{N} d_h(q, z_n) \\
&= b_0 d_h(q, x_n) + \sum_{i=1}^{N} d_h(q, x_n) + \sum_{i=1}^{N} b_i d_h(x_n, z_n) \qquad (53)
\end{aligned}$$

$$+\sum_{i=1}^{N} b_i\langle\nabla h(z_n-\nabla h(x_n),x_n-q\rangle\}$$
$$=d_h(q,x_n)+\sum_{i=1}^{N} b_i d_h(x_n,z_n)+\sum_{i=1}^{N} b_i\langle\nabla h(z_n-\nabla h(x_n),x_n-q\rangle\}.$$

Thus,

(54)
$$d_h(q,y_n)\le d_h(q,x_n)+\sum_{i=1}^{N} b_i d_h(x_n,z_n)+\sum_{i=1}^{N} b_i\langle\nabla h(z_n-\nabla h(x_n),x_n-q\rangle\}.$$

So $q\in K_{n+1}$ and $K_{n+1}\subset K_n$. This implies by set induction that $\Omega\in K_n$. Thus, the scheme defined by (52) is well-defined for all natural numbers. Following similar approach from the proof of Theorem 3.1, then the conclusion hold true and this completes the proof of Theorem 3.6. □

4. Numerical examples and Implementation of (19) and (52)

In this section, we would demonstrate the effectiveness, implementation and convergence of our schemes (19) and (52). We do this by considering an example below. We also use Python software and run on HP PC with Intel(R)Core(TM)i7-3520M CPU @ 2.90GHz to perform a side by side simulations of our algorithms (19) and (52) with that of (17) in ([14]).

Example 4.1. Let $X=(-\infty,+\infty)$, $h(x):=x^2$. Clearly, $\nabla h(x):=x$. Further, using the definition on conjugate function, we arrive that $h^*(x^*):=\dfrac{1}{2}x^{*2}$ and $\nabla h^*(x^*):=x^*$. By this, the assumptions on h in our theorem are clearly satisfied with $\nabla h^*(\nabla h(x)):=x$.Next, Let $G_i:X\to X$ be defined by $G_i(x)=\frac{5}{3^{i+1}}x$ for $i\in N$. Then it is easy to see that $G_i:i=1,2,...,N$ is BWRNE mappings with $\cap_{i=1}^{N}Fix(G_i)=\{0\}$. Let $b_i=\frac{1}{2^{i+1}}$ for $i\in N\cup\{0\}$, $a_n=0.4$, then our assumptions in Theorem 3.1 are satisfied for this special case example. In addition, by using Maple 18 computational software, our algorithms (19) and (52) are simplified respectively thus:

(55)
$$\begin{cases} x_0,x_1\in X,\\ z_n:=x_n+a_n(x_n-x_{n-1}),\\ y_n:=b_0x_n+\sum_{i=1}^{N}\frac{5}{3^{i+1}}z_n,\\ C_{n+1}:=\{u\in C_n:u\le\frac{8}{7}x_n+\frac{16}{81}z_n-\frac{1}{2268}\times\sqrt{-2099952x_n^2+6450192x_nz_n-3493945z_n^2}\}\\ x_{n+1}:=P^h_{C_{n+1}}(x_0),n\ge 1.\end{cases}$$

and

$$(56)\qquad \begin{cases} x_0, x_1 \in X, \\ z_n := x_n + a_n(x_n - x_{n-1}), \\ y_n := b_0 x_n + \sum_{i=1}^{N} \frac{5}{3^{i+1}} z_n, \\ C_{n+1} := \{u \in C_n : u \leq \frac{1}{2592} \times \frac{524880x_n^2 - 589680x_n z_n + 527807z_n^2}{81x_n + 111z_n}\} \\ x_{n+1} := P^h_{C_{n+1}}(x_0), n \geq 1. \end{cases}$$

Next, with codes written in Python software, we get the table and the corresponding figure below.

Table 1. Selected values of the sequences $\{x_n\}$ and $\{z_n\}$

in the experiment of our schemes (55) and (56) and that of (17).

n	0	5	10	50	70	100	150	200
$x_n(EO)$	2.000000	0.716039	0.450542	0.011067	0.001735	0.000108	0.000000	0.000000
$z_n(EO)$	0.000000	0.688270	0.433045	0.010638	0.001667	0.000103	0.000000	0.000000
$x2_n(NY)$	2.000000	1.624844	1.162052	0.0059675	0.013042	0.001317	0.000026	0.000001
$x3_n(EO)$	2.000000	0.744536	0.496767	0.0019510	0.003866	0.000341	0.000001	0.000000
$x4_n(NY)$	2.000000	1.358333	0.922534	0.041763	0.008886	0.000872	0.000018	0.000001

Key: $x_n(EO)$ and $z_n(EO)$ stands for the sequences generated by our Algorithm (55), $x3_n(EO)$ stands for the sequences generated by our Algorithm (56) and while $x2_n(NY)$, $x4_n(NY)$ stands for the sequences generated by the Algorithm (17) studied in ([14]).

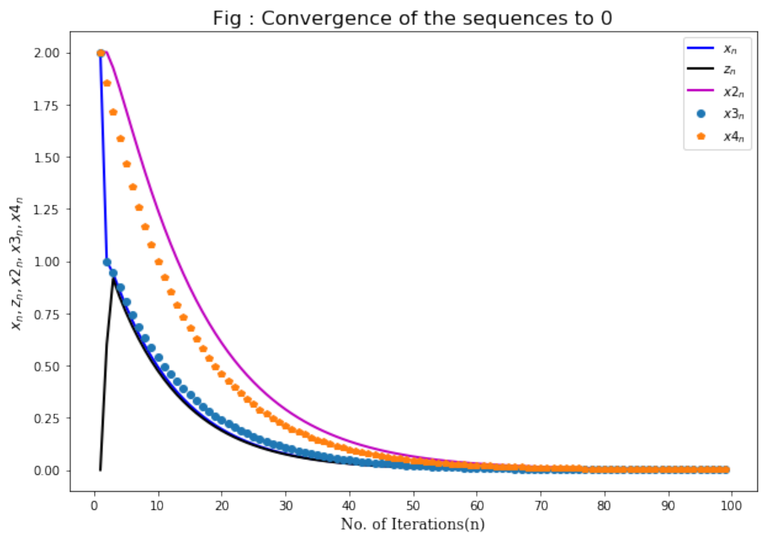

Figure 1: Sequence values versus number of iterations showing faster convergence of our Algorithms (55) and (56) indicated in black colour line for z_n, blue colour line for x_n and dotted blue colour line for $x3_n$ over that of Algorithm (17) indicated in pink and dotted orange colour lines

Discussion:

(a). Example (1.3) shows the generality of Bregman weak relatively nonexpansive mapping over Bregman relatively nonexpansive mapping given by Example (1.2) defined in a sequence space (also known to be a Banach space).

(b). Our main results in Theorems (3.1) and (3.6) extends the recent results of Chidume et al. [9] a from the class of relatively nonexpasive mappings to the much general class of Bregman weak relatively nonexpansive mappings (see Remark (1)). Similarly, our results compliments that of Wei et al. [18] when the Legendre function $h(.) := \|.\|^2$.

(c). Algorithm (17), though without inertial parameter was used in the literature for comparison because it is the only algorithm closely related to our considered Algorithms (19) and (52) with inertial parameters for Bregman weak relatively nonexpansive mappings. Numerical illustration was displayed to show strong convergence of the algorithms. In the table and figure shown above, it was observed that Algorithms (19) and (52) converges faster than Algorithm (17). Indeed, Algorithms (19) and (52) reaches the approximate common solution set faster by number of iterations, than Algorithm (17), hence Algorithms (19) and (52) improves Algorithm (17).

(d). We considered this problem in the framework of reflexive Banach space, for which other ideas and facts used in this paper constitute integral parts to, mainly in application. This ideas are necessary in approximating the common solution of the considered problem.

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