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Comparison of Some Notions of Orthogonality in Banach Spaces

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Abstract

Orthogonality concept is a powerful mathematical tool that makes possible the formulation of some mathematical conjectures to prove some important theorems in Mathematics for the solution of real and imaginary problems. There are different notions of orthogonality in normed linear spaces. The concept of orthogonality has certain properties- symmetry, homogeneity, additivity and so on. In this work, the properties of orthogonality as applied to normed linear spaces are investigated and the comparisons between the different notions of orthogonality are shown.

1. Introduction

The notion of orthogonality is an important mathematical concept and the existence of an inner product provided a means of introducing the notion of orthogonality in normed linear spaces. There are several notions of orthogonality in Banach spaces. In this paper, we shall look at these notions, their properties and comparisons. The space E stands for a Banach space or normed linear space and $K \in \mathbb{R}$ or $\mathbb{C}$, the real or complex field. These definitions are all equivalent in real Hilbert spaces. The concept of orthogonality has certain desired properties,

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the most important being that every two dimensional subspace contains non-zero orthogonal elements.

Preliminaries

There are several notions of orthogonality in Banach spaces. In this section, we shall look at some of these notions and their properties.

Definition 1.1. (Pythagorean Orthogonality): An element x is orthogonal to y in E ($x\perp_P y$) if and only if $\|x\|^2 + \|y\|^2 = \|x - y\|^2$.

Definition 1.2. (Isosceles Orthogonality): An element x is orthogonal to y in E ($x\perp_I y$) if and only if

$$\|x + y\| = \|x - y\|.$$

The following are properties of orthogonality as applied to normed linear spaces:
(i) symmetry: If $x\perp y$, then $y\perp x$;
(ii) homogeneity: If $x\perp y$, then $ax\perp by \;\; \forall \;\; a, b \in \mathbb{K}$;
(iii) additivity: If $x\perp y$ and $x\perp z$, then $x\perp y + z$; and
(iv) if x and y are any two elements, then there exists a number a such that $x\perp ax + y$.

Property (iv) is satisfied by isosceles and Pythagorean orthogonality, which are clearly symmetric. However, these types of orthogonality are not homogeneous or additive in a general normed linear space, and the assumption of these properties implies that the space is a real Hilbert space. It follows that symmetry, homogeneity and additivity of these orthogonality in real Hilbert spaces are equivalent to symmetry, homogeneity and additivity of the inner product. Also, $\langle x, ax + y\rangle = a\|x\|^2 + \langle x, y\rangle$. Hence $x\perp ax + y$ if and only if $a\|x\|^2 + \langle x, y\rangle = 0$.

Theorem 1.3. *The orthogonality in real Hilbert space E is symmetric, homogeneous and additive, and for two elements ($x \neq 0$) and y in E there exists a unique number a such that $x\perp ax + y$.*

Isosceles orthogonality is symmetric and for any element $x(\neq 0)$ and y in E, $x\perp_I ax + y$ for some a

Lemma 1.4. [7]. *If x and y are elements in E, then*

$$\lim_{n\to\infty} [\|\|(n + a)x + y\| - \|nx + y\|\|] = a\|x\|.$$

Theorem 1.5. . *If x and y are elements in E, then there exists a number a such that*

$$\|x + (ax + y)\| = \|x - (ax + y)\| \quad i.e. \quad x\perp_I ax + y.$$

By Theorem 1.3, orthogonality in real Hilbert spaces is symmetric, homogeneous and additive, and for any elements $x(\neq 0)$ and y in a real Hilbert space, there is a unique number a such that $x\perp ax + y$. For normed linear spaces, isosceles orthogonality is clearly symmetric and the existence of such a number a is given by Theorem 1.5. Isosceles orthogonality is homogeneous or additive only for real Hilbert spaces.

Theorem 1.6. *(Homogeneity of Isosceles Orthogonality). Let E be a normed linear space. If isosceles orthogonality is homogeneous in E, then E is a real Hilbert space.*

Proof: A normed linear space E is a real inner product space if and only if $\|ax + y\| = \|x + ay\|$ for all numbers a and elements x and y in E for which $\|x\| = \|y\|$. If for elements x and y in E, $\|x\| = \|y\|$, then

$$\|(x + y) + (x - y)\| = \|(x + y) - (x - y)\|$$

and

$$(x + y)\perp_I(x - y).$$

If isosceles orthogonality is homogeneous in E, then

$$\begin{aligned} ||(a+1)(x+y) + (a-1)(x-y)|| &= ||(a+1)(x+y) - (a-1)(x-y)|| \\ \Rightarrow \quad 2||ax+y|| &= 2||x+ay|| \\ \Rightarrow \quad ||ax+y|| &= ||x+ay|| \end{aligned}$$

This implies that E is a real Hilbert space $\square$.

Theorem 1.7. *(Additivity of Isosceles Orthogonality). Additivity of isosceles orthogonality in a normed linear space E implies that E is a real Hilbert space.*

Proof: If $x\perp_I y$,then $x\perp_I -y$ and $y\perp_I x \quad \forall \quad x, y \in E$ because of the nature of the condition for isosceles orthogonality. Hence if isosceles orthogonality is additive, then $nx\perp_I my$ for all integers m and n. Thus,

$$\|nx + my\| = \|nx - my\|$$

and

$$\|x + my/n\| = \|x - my/n\|.$$

Since the norm is continuous, it follows that $\|x + ky\| = \|x - ky\|$ for all numbers k. That is $x\perp_I ky$ for all k. Thus, isosceles orthogonality is homogeneous if it is additive. It follows from Lemma 1.4 that E is a real Hilbert space.

As for isosceles orthogonality, the assumption of homogeneity or additivity of the orthogonality implies that an inner product can be defined so that the space is a real Hilbert space. Thus, homogeneity of the orthogonality is equivalent to its additivity.

Also, Pythagorean orthogonality is obviously symmetric and we shall proceed to show that for any elements x and y in $E, x\perp_p ax + y$ for some number a. That is,

$$\|x\|^2 + \|ax + y\|^2 = \|x - (ax + y)\|^2.$$

Theorem 1.8. *(Symmetry of Pythagorean Orthogonality). Let x and y be elements in a normed linear space E. Then there exists a number a such that*

$$\|x\|^2 + \|ax + y\|^2 = \|x - (ax + y)\|^2, \quad i.e. \quad x \perp_p ax + y.$$

Proof: Define the real-valued function of a real variable, f, as

$$f(n) = \|x\|^2 + \|nx + y\|^2 - \|x - (nx + y)\|^2,$$

or

$$f(n) = \|x\|^2 + \|nx + y\|^2 - \|(n-1)x + y\|^2.$$

Using the identity $\frac{(n-1)^2}{n^2} + \frac{2n-1}{n^2} = 1$, we get:

$$f(n) = \|x\|^2 + \left[\frac{(n-1)^2}{n^2} + \frac{2n-1}{n^2}\right] \|nx + y\|^2 - \|(n-1)x + y\|^2$$

$$= \|x\|^2 + \frac{(n-1)^2}{n^2}\|nx + y\|^2 + \frac{2n-1}{n^2}\|nx + y\|^2 - \|(n-1)x + y\|^2$$

$$= \|x\|^2 + \frac{2n-1}{n^2}\|nx + y\|^2 + \left[\left\|(n-1)x + \left(\frac{n-1}{n}\right) y\right\|^2 - \|(n-1)x + y\|^2\right]$$

$$= \|x\|^2 + (2n-1)\left\|x + \frac{y}{n}\right\|^2 + \left[\|(n-1)x + \left(\frac{n-1}{n}\right) y\| - \|(n-1)x + y\|\right]$$

$$\times \left[\left\|(n-1)x + \left(\frac{n-1}{n}\right) y\right\| + \|(n-1)x + y\|\right]$$

$$\geq \|x\|^2 + (2n-1)\left\|x + \frac{y}{n}\right\|^2 - \left|\frac{1}{n}\right| \|y\| \left[\|(n-1)x + \left(\frac{n-1}{n}\right) y\| + \|(n-1)x + y\|\right]$$

If $n \geq 1$, it follows by using the triangular inequality that

$$\|x + \frac{y}{n}\|^2 \geq [\|x\| - \|\frac{y}{n}\|]^2$$

and

$$\|(n-1)x + \left(\frac{n-1}{n}\right) y\| + \|(n-1)x + y\| \leq 2(n-1)\|x\| + \frac{2n-1}{n}\|y\|.$$

Hence,

$$f(n) \geq 2n\|x\|^2 + \frac{2n-1}{n^2}\|y\|^2 - 2\left(\frac{2n-1}{n}\right)\|x\|\|y\|$$

$$-\|y\| \left[2\left(\frac{n-1}{n}\right)\|x\| + \frac{2n-1}{n^2}\|y\|\right]$$

$$= 2n\|x\|^2 - 2\left(\frac{2n-1}{n}\right)\|x\|\|y\| - 2\left(\frac{n-1}{n}\right)\|x\|\|y\|$$

$$= 2n\|x\|^2 - \left(\frac{6n-4}{n}\right)\|x\|\|y\|$$

$$= 2n\|x\|^2 - 2\left(\frac{3n-2}{n}\right)\|x\|\|y\|$$

$$= 2\|x\|\left[n\|x\| - \left(\frac{3n-2}{n}\right)\|y\|\right],$$

which can be made greater than zero by taking n sufficiently large. Now consider,

$$f(-n) = \|x\|^2 + \|nx-y\|^2 - \|(n+1)x-y\|^2.$$

Using the identity $\frac{(n+1)^2}{n^2} - \frac{(2n+1)}{n^2} = 1$, we get:

$$f(-n) = \|x\|^2 + \left[\frac{(n+1)^2}{n^2} - \frac{2n+1}{n^2}\right]\|nx-y\|^2 - \|(n+1)x-y\|^2$$

$$= \|x\|^2 + \frac{(n+1)^2}{n^2}\|nx-y\|^2 - \frac{2n+1}{n^2}\|nx-y\|^2 - \|(n+1)x-y\|^2$$

$$= \|x\|^2 - \frac{2n+1}{n^2}\|nx-y\|^2 + \left[\left\|(n+1)x - \left(\frac{n+1}{n}\right)y\right\|^2 - \|(n+1)x-y\|^2\right]$$

$$= \|x\|^2 - \frac{2n+1}{n^2}\|nx-y\|^2 + \left[\left\|(n+1)x - \left(\frac{n+1}{n}\right)y\right\| - \|(n+1)x-y\|\right]$$

$$\times \left[\left\|(n+1)x - \left(\frac{n+1}{n}\right)y\right\| + \|(n+1)x-y\|\right]$$

$$\leq \|x\|^2 - (2n+1)\left\|x - \frac{y}{n}\right\|^2 + \|y\|$$

$$\left[\left\|\left(\frac{n+1}{n}\right)x - \left(\frac{n+1}{n^2}\right)y\right\| + \left\|\left(\frac{n+1}{n}\right)x - \frac{y}{n}\right\|\right].$$

If $n > 0$, this can be written:

$$f(-n) \leq -2n\|x\|^2 + 2\left(\frac{2n+1}{n}\right)\|x\|\|y\| - \frac{2n+1}{n^2}\|y\|^2$$

$$+\|y\|\left[\frac{2(n+1)}{n}\|x\| + \frac{2n+1}{n^2}\|y\|\right] = -2\|x\|\left[n\|x\| - \frac{3n+2}{n}\|y\|\right].$$

This $f(-n)$ is negative if n is sufficiently large. Since f is continuous, it follows that $f(n) = 0$ for some value a of n. That is,
$\|x\|^2 + \|ax+y\|^2 = \|x-(ax+y)\|^2$ for some number a
The condition for Pythagorean orthogonality is not equivalent to $\|x\|^2 + \|y\|^2 = \|x+y\|^2$. That is, $x\perp_p y$ does not imply $x\perp_p -y$, as it did with isosceles orthogonality. However, Theorem 1.8 is valid if this change in sign is made.

Corollary 1.9. *(Existence of a number a for Pythagorean Orthogonality). Let x and y be elements in normed linear space E. Then there exists a number a such that*

$$\|x\|^2 + \|ax+y\|^2 = \|x+(ax+y)\|^2.$$

Proof: By Theorem **??** 3.2.8, there exists a number b for the elements x and $-y$ in E such that

$$\|x\|^2 + \|bx-y\|^2 = \|x-(bx-y)\|^2.$$

Let $a = -b$. Then

$$\|x\|^2 + \|ax+y\|^2 = \|x+(ax+y)\|^2. \qquad \diamondsuit$$

Theorem 1.10. *(Homogeneity of Pythagorean Orthogonality). Let E be a normed linear space. Homogeneity of Pythagorean orthogonality in E implies that E is a real Hilbert space.*

Proof: It is sufficient to show that the condition for the existence of an inner product is satisfied i.e. $\|x+y\|^2 + \|x-y\|^2 = 2[\|x\|^2 + \|y\|^2]$ for all x and y in E. By Theorem 1.8, it has been shown that for any elements x and y in E a number a exists such that $x\perp_p ax+y$. Assuming homogeneity of the orthogonality, it follows that
$k^2\|x\|^2 + \|ax+y\|^2 = \|kx-(ax+y)\|^2, \forall k \in K$.
If $k = (a \pm 1)$, this gives

$$\|x \mp y\|^2 = (a\pm 1)^2\|x\|^2 + \|ax+y\|^2.$$

Hence,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|ax+y\|^2 + 2(a^2+1)\|x\|^2.$$

By using homogeneity again, it follows that $ax\perp_p ax+y$, and hence

$$\|ax+y\|^2 + a^2\|x\|^2 = \|y\|^2.$$

Therefore,

$$\|x+y\|^2 + \|x-y\|^2 = 2[\|x\|^2 + \|y\|^2],$$

and we have shown that the condition for the existence of an inner product is satisfied. Thus, E is a real Hilbert space. $\diamondsuit$.
One of the most obvious results of the assumption of homogeneity of Pythagorean orthogonality is that

$$\|x+y\| = \|x-y\| \quad \text{if} \quad x\perp_p y,$$

since in that case $x\perp_p y$ implies $x\perp_p -y$. Thus if Pythagorean orthogonality is homogeneous, and x and y are orthogonal $(x\perp_p y)$, then they are also orthogonal in the isosceles sense $(x\perp_I y)$. The converse is true using Theorems 2.10.
The Pythagorean orthogonality is homogeneous and additive in real Hilbert space.

Definition 1.11. (Birkhoff's Orthogonality). An element x is said to be orthogonal to y in a normed linear space E $(x\perp_B y)$ if and only if $\|x+ky\| \geq \|x\|$ for all $k \in K$.

Definition 1.12. (Semi Inner Product). Let E be a complex (real) vector space. We shall say that a complex (real) semi-inner product is defined on E, if to any $x, y \in E$ there corresponds a complex (real) number $[x, y]$ and the following properties hold:
(i) $[x+y, z] = [x, z] + [y, z]$
$[\lambda x, y] = \lambda[x, y]$, for $x, y, z \in E, \lambda \in K$.
(ii) $[x, x] > 0$, for $x \neq 0$.
(iii) $|[x, y]|^2 \leq [x, x][y, y]$.
Then, the pair $(E, [.,.])$ is called a semi-inner-product space (in short s.i.p.s) and it is a normed linear space with the norm $\|x\| = [x, x]^{1/2}$. [[5] & [12]].

Theorem 1.13. *([12]). A semi-inner-product space E is a normed linear space with the norm $[x, x]^{1/2}$. Every normed linear space can be made into a semi-inner-product space.*

Definition 1.14. (Lumer-Giles Orthogonality). Let $[.,.]$ be a semi-inner-product which generates the norm of E, and let $x, y \in E$. The vector x is orthogonal to y (in the sense of Lumer-Giles relative to a semi-inner-product $[.,.]$) if and only if $[y, x] = 0$. We denote this by $\perp_{LG} y$.

Theorem 1.15. [8]. *For any elements x and y in a normed linear space E, there exists a number a such that $ax+y\perp_B x$. This number a is a value of k for which $\|kx+y\|$ takes on its absolute minimum. If $Ax+y\perp_B x$ and $Bx+y\perp_B x$, then $ax+y\perp_B x$ if a is a number between A and B. For real Hilbert spaces,*

$$\lim_{n\to\infty}\{\|nx+y\|-\|nx\|\}, \quad or \quad N_+(x;y) = \lim_{h\to 0_+}\left[\frac{\|x+hy\|-\|x\|}{h}\right]$$

is zero if and only if $x\perp_B y$. This is not true in general normed linear spaces and $N_+(x;y)$ is closely related to orthogonality and its being zero implies $x\perp_B y$..

Theorem 1.16. [8]. *Let $x(\neq 0)$ and y be any elements in a normed linear space E. Then $\lim_{n\to\infty}\{\|nx+y\|-\|nx\|\} = -A\|x\|$ and $\lim_{n\to\infty}\{\|nx\|-\|nx-y\|\} = -B\|x\|$, where A and B are the algebraically smallest and largest of the numbers a such that $x\perp_B ax+y$.*

Lemma 1.17. [9]. *If x and y are elements of a normed linear space E and $x\perp_B Ax+y$ and $x\perp_B Bx+y$, then $x\perp_B x+y$ if a is a number between A and B.*

Lumer-Giles orthogonality relation is not symmetric since the $s.i.p$ is not commutative, that is, if x is orthogonal to y then y is not necessarily orthogonal to x. However, it follows from propert I of Definition 2.12 that the relation is additive, that is, if x is orthogonal to y and to z then x is orthogonal to $\lambda y + \mu z$ for all complex λ, μ.
Lumer-Giles and Birkhoff's orthogonality are equivalent in continuous semi-inner product spaces.

Definition 1.18. (Homogeneity of an s.i.p Space). An $s.i.p$ space E has the homogeneity property when the $s.i.p$ satisfies:
(iv). $[x, \lambda y] = \bar{\lambda}[x, y]$ for all $x, y \in E$ and all complex λ.

Theorem 1.19. *(A normed Linear Space as an s.i.p Space). If E is a normed linear space, then E can be represented as an s.i.p space with the homogeneity property.*
Proof: *Let E be a normed linear space. By the Hahn-Banach theorem, for each $x \in S$ there exists at least one continuous linear functional, and we choose exactly one, $f_x \in S^*$ such that $f_x(x) = 1$, where*

$$S := \{x \in E : \|x\| = 1\} \quad and \quad S^* := \{f_x \in E^* : \|f_x\| = 1, x \in E\}.$$

For $\lambda x \in E$ where $x \in S$ and any complex λ we choose $f_{\lambda x} \in E^*$ such that $f_{\lambda x} = \bar{\lambda} f_x$.
Given one such mapping from E into E^*, it is readily verified that the function

$$[x, y] = f_y(x)$$

satisfies the properties (i) - (iv) for a s.i.p.

Definition 1.20. (Continuous Semi-inner Product Space).

2.20 Definition (Continuous Semi-inner Product Space). A continuous semi-inner product space is an $s.i.p$ space C where the $s.i.p$ has the additional property:
(v) for every $x, y \in S$ and $\lambda \in \mathbb{R}$,

$$\lim_{\lambda \to 0} Re\{[y, x + \lambda y]\} \longrightarrow Re\{[y, x]\}.$$

The space is a uniformly continuous $s.i.p$ space when it possesses the property:
(vi). The limit (v) is approached uniformly for all $(x, y) \in S \times S$.

[4] gave a characterization of Birkhoff's orthogonality:
Let $\mathcal{J}$ be the normalized duality mapping on a real normed linear space E, that is, let

$$\mathcal{J}(x) = \left\{f \in E^* | f(x) = \|x\|^2 \ \text{ and } \ \|f\| = \|x\|\right\},$$

were E^* is the dual of E.
The norm of E can always be represented through a semi-inner product in the form [12]

$$[x, y] = \tilde{\mathcal{J}}(y), x \quad \text{for all} \quad x, y \in E,$$

where $\tilde{\mathcal{J}}$ is a section of the normalized duality mapping $\mathcal{J}$.

We recall that if $x \perp_B y$, then from the inequality

$$\|x\| \leq \|x + ky\| \leq \|x\| + |k|\|y\|$$

valid for all $k \in \mathbb{R}$, we deduce that $\|x + ky\| \geq \|x\|$ for all $k \in \mathbb{R}$ is equivalent to

$$\inf_{k \in \mathbb{R}} \|x + ky\| = \|x\|;$$

if $\langle y \rangle$ is the subspace of E generated by y, then $\inf_{k \in \mathbb{R}} \|x + ky\| = \|x\|$ means that $d(x, \langle y \rangle) = \|x\|$.

Definition 1.21. $\mathbb{K}$ is the field of real or complex numbers, E is a Banach space over $\mathbb{K}$ with unit ball denoted by B and norm denoted by $\| \cdot \|$, and $(x_n) := (x_n)_{n=1}^N := (x_n)_{n \in L}$ is a finite or infinite sequence in E, where either N is a positive integer and $L := \{1, 2, \ldots, N\}$, or $N = \infty$ and $L := \{1, 2, \ldots\}$. For $J (\neq \phi) \subset L$, the closure of the span of the set $\{x_n : n \in J\}$ is denoted by $[x_n : n \in J]$. The unit ball in $[x_n : n \in J]$ is denoted by B_J.

Definition 1.22. (Semi-orthonormality). A finite or infinite sequence $(x_n)_{n \in L}$ in E is said to be semi-orthonormal if and only if $\|x_n\| = 1$ for all $n \in L$ and

$$\left\| \sum_{n \in L} a_n x_n \right\| \geq \sup_{n \in L} |a_n|, \quad \text{for each} \quad \sum_{n \in L} a_n x_n \in E. \tag{1}$$

If follows that if $(x_n)_{n \in L}$ is semi-orthonormal then $\{x_n : n \in L\}$ is linearly independent and for each $i \in L$, if we set

$$\left(x_i^*, \sum_{n \in L} a_n x_n \right) := a_i. \tag{2}$$

then x_i^* is the unique element in $[x_n : n \in L]^*$ that satisfies, for all $j \in L$,

$$(x_i^*, x_j) = \delta_{ij} := \begin{cases} 1 & \text{if} \quad i = j \\ \\ 0 & \text{if} \quad i \neq j, \end{cases}$$

where $(x_i^*, x_j) := x_i^*(x_j)$. By the Hahn-Banach theorem, each x_n^* can be extended to an element of E^*, denoted also by x_n^*, without changing its norm.
The sequence $(x_n^*)_{n \in L}$ in E^* is called a sequence of associated (or corresponding) coefficient functional for the sequence $(x_n)_{n \in L}$ [18].

Definition 1.23. (Khalil's Orthogonality) . Let $(x_n)_{n\in L}$ be a semi-orthonormal sequence in E and let (x_n^*) be a sequence of corresponding coefficient functionals. The sequence $(x_n)_{n\in L}$ is said to be orthonormal if and only if, for any $(\lambda_n)_{n\in L} \in \ell^\infty$ and any $x \in E, \sum_{n\in L} \lambda_n(x_n^*, x)x_n$ converges and

$$\left\|\sum_{n\in L} \lambda_n(x_n^*, x)x_n\right\| \leq \|x\| \sup_{n\in L} \|\lambda_n\|. \tag{3}$$

The sequence $(x_n)_{n\in L}$ is said to be orthogonal if the sequence $(x_n/\|x_n\|)$ obtained from the sequence $(x_n)_{n\in L}$ after the trivial terms are deleted is orthonormal.

Definition 1.24. (Saidi's Orthogonality). A finite or infinite sequence $(x_n)_{n\in L}$ in E is said to be orthogonal if

$$\left\|\sum_{n\in L} a_n x_n\right\| = \left\|\sum_{n\in L} |a_n| x_n\right\|, \quad \text{for each} \quad \sum_{n\in L} a_n x_n \in E. \tag{4}$$

If in addition, $\|x_n\| = 1$ for all $n \in L$, then $(x_n)_{n\in L}$ is said to be orthonormal. It follows that $(x_n)_{n\in L}$ is orthogonal in E if and only if $(x_n)_{n\in L}$ is orthogonal in $[x_n : n \in L]$. Also, $\left(\sum_{n\in I} a_n x_n\right) \perp \left(\sum_{n\in J} a_n x_n\right)$ whenever I and J are disjoint subsets of L and $(x_n)_{n\in L}$ is orthogonal in E.

Definition 1.25. (S- and K-orthonomality). A finite or infinite sequence $(x_n)_{n\in L}$ in E is said to be S-orthogonal (S-orthonormal) if it is orthogonal (orthonormal) in the sense of Saidi and K-orthogonal (K-orthonormal) if it is orthogonal (orthonormal) in the sense of Khalil.

Theorem 1.26. *Given a sequence $(x_n)_{n\in L}$ in E, the following are equivalent:*

(i) The sequence $(x_n)_{n\in L}$ is S-orthogonal in E;

(ii) For each pair of nonempty and disjoint sets $I, J \in L$,

$$[x_n : n \in I] \perp [x_n : n \in J];$$

and

(iii)] For each $i \in L$,

$$x_i \perp [x_n : n \in L, n \neq i] \tag{5}$$

Proof: (i)$\Rightarrow$ (ii): Suppose $(x_n)_{n\in L}$ is S-orthogonal in E. Let $I, J \subset L$ such that $I \cap J = \phi$ and $i \in I$ and $j \in J$. Since $(x_n)_{n\in L}$ is orthogonal, then $x_i \perp x_j$.

$$\text{Therefore, } [x_n : n \in I] \perp [x_n : n \in J].$$

(ii)$\Rightarrow$ (iii): Suppose $[x_n : n \in I] \perp [x_n : n \in J], I \cap J = \phi$.
Let $i \in L$, and $n \in L$ such that $n \neq i$.
Then,

$$x_i \perp x_n \quad \forall \quad n \in L \quad \text{and} \quad n \neq i$$

$$\Rightarrow x_i \perp [x_n : n \in L, n \neq i].$$

(iii)$\Rightarrow$ (i): Let ℓ and m be any two positive integers satisfying $1 \leq \ell \leq m \leq N$. Using (4) successively for $i = \ell, \ell+1, \ldots, m$, we get

(6)
$$\left\|\sum_{n\in\ell}^{m} a_n x_n\right\| = \left\||a_\ell|x_\ell + \sum_{n\in\ell+1}^{m} a_n x_n\right\| = \cdots = \left\|\sum_{n\in\ell}^{m-1} |a_n|x_n + a_m x_m\right\| = \left\|\sum_{n=\ell}^{m} |a_n|x_n\right\|.$$

This implies, since $\sum_{n\in L} a_n x_n$ converges, that the sequence $(\sum_{n=1}^{m} |a_n|x_n)_{m\in L}$ is Cauchy. Hence, $\sum_{n\in L} |a_n|x_n$ converges and, again by (6),

$$\left\|\sum_{n=1}^{N} a_n x_n\right\| = \left\|\sum_{n=1}^{N} |a_n|x_n\right\|,$$

which implies that the sequence $(x_n)_{n\in L}$ is orthogonal. $\Diamond$

Definition 1.27. . A function g defined on the field $\mathbb{K}$ is said to be radial if $g(z) = g(|z|)$, for all $z \in \mathbb{K}$.

Lemma 1.28. *If g is a convex real-valued function defined on the field $\mathbb{K}$, then g is radial if and only if $g(z)$ is nondecreasing as $|Z|$ increases in $[0, \infty)$.*

Using (1.28), the following theorem shows the necessary and sufficient conditions for $(x_n)_{n\in L}$ to be S-orthogonal [19].

Theorem 1.29. *Given a sequence $(x_n)_{n\in L}$ in E, the following are equivalent:*

(i) The sequence $(x_n)_{n\in L}$ is S-orthogonal in E;

(ii) For each pair of sequences $(b_n)_{n\in L}$ and $(c_n)_{n\in L}$ in $\mathbb{K}$ satisfying $|b_n| \leq |c_n|$ for all $n \in L$, if $\sum_{n\in L} c_n x_n$ converges then $\sum_{n\in L} b_n x_n$ converges, and

$$\left\|\sum_{n=L} b_n x_n\right\| \leq \left\|\sum_{n\in L} c_n x_n\right\| \tag{7}$$

(iii) For each pair of sequences $(b_n)_{n\in L}$ and $(c_n)_{n\in L}$ in $\mathbb{K}$ satisfying $|b_n| = |c_n|$ for all $n \in L, \sum_{n\in L} c_n x_n$. Converges, if and only if $\sum_{n\in L} b_n x_n$ converges, and, if both converge,

$$\left\|\sum_{n=L} b_n x_n\right\| = \left\|\sum_{n\in L} c_n x_n\right\|.$$

Proof: (i)$\Rightarrow$ (ii): Suppose that $(x_n)_{n\in L}$ is S-orthogonal and let $(b_n)_{n\in L}$ and $(c_n)_{n\in L}$ be two sequences in $\mathbb{K}$ satisfying $|b_n| \leq |c_n|$ for all $n \in L$ and such that $\sum_{n\in L} c_n x_n$ converges. For each $i \in L$ and each $v_i \in [x_n : n \in L, n \neq i]$, the function $g(\lambda) := \|\lambda x_i + v_i\|$ is convex and radial, since by Theorem 1.29 (iii), $x_i \perp v_i$. Hence, by (7) and since $|b_i| \leq |c_i|$, we obtained that $g(b_i) \leq g(c_i)$.

In other words, for each $i \in L$ and each $v_i \in [x_n : n \in L, n \neq i]$, we have,

$$\|b_i x_i + v_i\| \leq \|c_i x_i + v_i\| \tag{8}$$

Applying equation (8) successively, we obtain, for any finite set $\{\ell, \ell+1, \ldots, m\} \subset L$, that

$$\left\|\sum_{n\in\ell}^{m} b_n x_n\right\| \leq \left\|c_\ell x_\ell + \sum_{n\in\ell+1}^{m} b_n x_n\right\| \leq \cdots \leq \left\|\sum_{n\in\ell}^{m-1} c_n x_n + b_m x_m\right\| \leq \left\|\sum_{n=\ell}^{m} c_n x_n\right\| \tag{9}$$

From equation (9),

$$\left\|\sum_{n\in\ell} b_n x_n\right\| \leq \left\|\sum_{n\in\ell} c_n x_n\right\|.$$

(ii)⇒ (iii): Suppose (ii) holds and $|b_n| = |c_n|$, then

$$\left\|\sum_{n\in L} b_n x_n\right\| = \left\|\sum_{n\in L} c_n x_n\right\|.$$

(iii)⇒ (i): Suppose (iii) holds, then

$$\left\|\sum_{n\in L} b_n x_n\right\| = \left\|\sum_{n\in L} c_n x_n\right\| = \left\|\sum_{n\in L} |c_n| x_n\right\| = \left\|\sum_{n\in L} |b_n| x_n\right\| \quad \text{forall} \quad \sum_{n\in L} b_n x_n \in E. \tag{10}$$

Therefore, $(x_n)_{n\in L}$ is S-orthogonal in E, hence (i) holds. ◊

2. Main Results

2.1. **Equivalence of Different Notions of Orthogonality.** The different notions of orthogonality are all equivalent in real Hilbert space.

Theorem 2.1. [10] *Let $(E, \langle ., . \rangle)$ be a real Hilbert space and $x, y \in E$. $\langle x, y\rangle = 0$ if and only if:*

(i) $\|x - y\| = \|x + y\|$;
(ii) $\|x - y\|^2 = \|x\|^2 + \|y\|^2$;
(iii) $\|x + \lambda y\| \geq \|x\|$, *for all* $\lambda \in \mathbb{R}$;
(iv) $\|x + \lambda y\| = \|x - \lambda y\|$, *for all* $\lambda \in \mathbb{R}$.

Birkhoff's and Lumer-Giles Orthogonality. Lumer-Giles orthogonality is equivalent to Birkhoff's orthogonality in a continuous *s.i.p* space [5].

Theorem 2.2. *(Equivalence of Lumer-Giles and Birkhoff's Orthogonality). Let E be a continuous semi-inner product space and $x, y \in E$. Then, $x\perp_{LG} y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all complex λ.*

Proof: If $x\perp_{LG} y$, then,

(11)
$\|x+\lambda y\|\|x\| \geq |[x+\lambda y, x]| = |\|x\|^2+\lambda[y, x]| = \|x\|^2$,

therefore, $\|x + \lambda y\| \geq \|x\|$ for all complex λ.
Conversely, if $\|x + \lambda y\| - \|x\| \geq 0$ for all complex λ, then

$$\|x + \lambda y\|^2 - \|x\|\|x + \lambda y\| \geq 0.$$

Therefore $Re\,\{[x, x + \lambda y]\} + Re\{\lambda[y, x + \lambda y]\} - |[x, x + \lambda y]| \geq 0$,
which implies that $Re\{\lambda[y, x + \lambda y]\} \geq 0$ for complex λ. Therefore for real λ,

(12)
$Re\{[y, x+\lambda y]\} \geq 0 \text{ for } \lambda \geq 0 \leq 0 \text{ for } \lambda \leq 0$

By the continuity condition (v), we have for real λ,
$Re\{[y, x + \lambda y]\} \longrightarrow Re\{[y, x]\}$ through positive values for $\lambda \longrightarrow 0_+$ and through negative values for $\lambda \longrightarrow 0_-$.
Therefore, $Re\{[y, x]\} = 0$.
For imaginary λ, say $\lambda = i\lambda_1$, where λ_1 is real,

$$Re\{\lambda[y, x + \lambda y]\} = \lambda, Re\{[iy, x + \lambda, iy]\} \geq 0$$

and again by the continuity condition (v)

$$Re\{[iy, x]\} = 0, \quad \text{i.e.,} \quad Im\{[y, x]\} = 0$$

Therefore, $\quad [y, x] = 0.$ ◊

The following connection between Birkhoff's and Lumer-Giles' orthogonality holds [4].

Theorem 2.3. *(Relationship between Lumer-Giles and Birkhoff's Orthogonality). Let E be a real normed linear space and $x, y \in E$.Then, $x\perp_B y$ if and only if $x\perp_{LG} y$ relative to some semi-inner product $[.,.]$ which generates the norm $\|\cdot\|$.*

Proof: Assume that $[.,.]$ is a semi-inner product which generates the norm of E and that $x\perp_{LG} y$, that is, $[y, x] = 0$. Then,

$$\|x\|^2 = [x, x] = [x + \lambda y, x] \leq \|x\|\|x + \lambda y\|$$

for all $\lambda \in \mathbb{R}$, that is, $\|x\| \leq \|x + \lambda y\|$ for $\lambda \in \mathbb{R}$, which is equivalent to $x\perp_B y$.

Conversely, suppose that $x\perp_B y$. Then

$$\inf_{\lambda\in\mathbb{R}} \|x+\lambda y\| = \|x\|$$

and by the Hahn-Banach theorem there is $f_x \in E^*$ such that

$$f_x(y) = 0,\ f_x(x) = \|x\|^2,\ \|f_x\| = \|x\|.$$

Hence, we can choose a section $\tilde{\mathcal{J}}$ of the normalized duality mapping $\mathcal{J}$ so that $\tilde{\mathcal{J}}(x) = f_x$. The semi-inner product

$$[u,v] = (\tilde{\mathcal{J}}(v), u), \quad u, v \in E,$$

generates the norm of E, and

$$[y,x] = (\tilde{\mathcal{J}}(x), y) = f_x(y) = 0.$$

Consequently, $x\perp_{LG} y$ relative to $[.,.]$. ◊

The following proposition also holds:

Proposition 2.4. *Let E be a real normed linear space and $x, y \in E$. The following statements are equivalent:*

(i) $x\perp_B y$;

(ii) For every semi-inner product $[.,.]$ which generates the norm of E we have the inequalities $[y, x+uy] \leq 0 \leq [y, x+ty]$ for all $u < 0 < t$.

[4].

Isosceles and Birkhoff's Orthogonality. Birkhoff's and isosceles orthogonality can be compared using the following Theorem.

Theorem 2.5. *(Isosceles and Birkhoff's Orthogonality). If $x\perp_I y$ in a normed linear space E, then*

$$\|x+ky\| \geq \|x\| \ \textit{ for } \ |k| \geq 1.$$

Proof: From the identity $2x = (x+y) + (x-y)$, it follows that

$$\|2x\| = \|(x+y)+(x-y)\| \leq \|x+y\| + \|x-y\|. \tag{13}$$

Since,

$$\|x+y\| = \|x-y\|,$$

this gives,

$$\|x\| \leq \|x \pm y\|.$$

But, from ([7], Lemma 41),

$$\|x\pm y\| \leq \|x+ky\| \ \text{ if } \ |k| \geq 1$$

Hence,

$$\|x\| \leq \|x+ky\| \ \text{ if } \ |k| \geq 1. \Diamond$$

Khalil's and Saidi's Orthogonality. There are some comparison between S-orthonormality and semi-orthonormality and between S-orthonormality and K-orthonormality. We now discuss these using the following results [20].

Lemma 2.6. *If $(x_n)_{n\in L}$ is an S-orthonormal sequence in E, then $(x_n)_{n\in L}$ is semi-orthonormal.*

Proof: Let $\sum_{n\in L} a_n x_n \in E$. Then for each $i \in L$ we have, by Theorem 1.26,

$$|a_i| = \|a_i x_i\| \leq \left\| \sum_{n\in L} a_n x_n \right\|.$$

Therefore $\sup_{n\in L} |a_n| \leq \left\|\sum_{n\in L} a_n x_n\right\|$, which ends the proof. ◊

Lemma 2.7. *If the sequence $(x_n)_{n\in L}$ is K-orthonormal with respect to some sequence of corresponding coefficient functional $(x_n^*)_{n\in L}$, then $(x_n)_{n\in L}$ is S-orthonormal.*

Proof: Let $(x_n)_{n\in L}$ be K-orthonormal with respect to a sequence $(x_n^*)_{n\in L}$ of corresponding coefficient functional. Also, let $\sum_{n\in L} a_n x_n, \sum_{n\in L} b_n x_n \in E$ satisfy $|a_n| \leq |b_n|$ for all $n \in L$. For each n there exists $\lambda_n \in \mathbb{K}$ such that

$$a_n = \lambda_n b_n \quad \text{and} \quad |\lambda_n| \leq 1.$$

If $x := \sum_{n\in L} b_n x_n$ then, for all $n \in L$, we have $(x_n^*, x) = b_n$. Therefore, by assumption and since $|\lambda_n| \leq 1$ for all $n \in L$, we get

(14)
$$\left\| \sum_{n\in L} a_n x_n \right\| = \left\| \sum_{n\in L} \lambda_n b_n x_n \right\| = \left\| \sum_{n\in L} \lambda_n (x_n^*, x) x_n \right\| \leq \|x\| \sup_{n\in L} |\lambda_n| \leq \|x\| = \left\| \sum_{n\in L} b_n x_n \right\|.$$

Therefore, by Theorem 1.26, $(x_n)_{n\in L}$ is S-orthonormal. ◊

Theorem 2.8. *Let $(x_n)_{n\in L}$ be a finite or infinite sequence in E. The following are equivalent:*

(i) $(x_n)_{n\in L}$ is S-orthonormal;

(ii) $(x_n)_{n\in L}$ is semi-orthonormal and $(x_n^*)_{n\in L}$ is S-orthonormal in $[x_n : n \in L]^*$.

Proof: (i)⇒(ii): First, by Lemma 2.6, $(x_n)_{n\in L}$ is semi-orthonormal. Also, by the definition of $(x_n^*)_{n\in L}, \|x_n^*\| = 1$ for all $n \in L$.
Now, let

$$x^* := \sum_{n\in L} a_n x_n^* \quad \text{and} \quad y^* := \sum_{n\in L} b_n x_n^*$$

be two elements in $[x_n^* : n \in L]$ satisfying $|a_n| \leq |b_n|$ for $n \in L$. For each n there exists $\lambda_n \in \mathbb{K}$ such that

$$a_n = \lambda_n b_n \;\; \text{and} \;\; |\lambda_n| \leq 1.$$

We need to show that $\|x^*\| \leq \|y^*\|$. Let $x \in [x_n : n \in L]$, $x := \sum_{n\in L} \mu_n x_n$. Then, since $\lambda_n \leq 1$, $\sum_{k\in L} \lambda_k u_k x_k$ converges and

$$\left\| \sum_{k\in L} \lambda_k \mu_k x_k \right\| \leq \left\| \sum_{k\in L} \mu_k x_k \right\|.$$

Therefore, since $\sum_{n\in L} \lambda_n b_n x_n^*$ is a continuous functional and since $(x_i^*, x_j) = \delta_{i,j}$, we get

(15)

$$|(x^*, x)| = \left| \left(\sum_{n\in L} \lambda_n b_n x_n^*, \sum_{k\in L} \mu_k x_k \right) \right| = \left| \sum_{n\in L} \lambda_n b_n \mu_n (x_n^*, x_n) \right| = \left| \left(\sum_{n\in L} b_n x_n^*, \sum_{k\in L} \lambda_k \mu_k x_k \right) \right|$$

$$= \left| \left(y^*, \sum_{k\in L} \lambda_k \mu_k x_k \right) \right| \leq \|y^*\| \left\| \sum_{k\in L} \lambda_k \mu_k x_k \right\| \leq \|y^*\| \left\| \sum_{k\in L} \mu_k x_k \right\| = \|y^*\| \|x\|.$$

Therefore, $\|x^*\| \leq \|y^*\|$ and consequently, $(x_n^*)_{n\in L}$ is S-orthonormal in $[x_n : n \in L]^*$.
(ii) $\Rightarrow$ (i): Let $(x_n)_{n\in L}$ be semi-orthonormal and suppose that $(x_n^*)_{n\in L}$ is S-orthonormal in $[x_n : n \in L]^*$. Now, let $x := \sum_{n\in L} a_n x_n$ and $y := \sum_{n\in L} b_n x_n$ be two elements in E satisfying $|a_n| \leq |b_n|$ for all $n \in L$ and let λ_n be as above. We need to show that $\|x\| \leq \|y\|$. This follows immediately from the proof of $\|x^*\| \leq \|y^*\|$ in (i) $\Rightarrow$ (ii) by interchanging y^* with y, x^* with x, and x_n^* with x_n. $\Diamond$

Theorem 2.9. *Let $(x_n)_{n\in L}$ be an S-orthonormal sequence in E satisfying $[x_n : n \in L] \subset E$.*
Then, we have,

(i) For each sequence of corresponding coefficient functional $(x_n^*)_{n\in L}$ in $E^*, (x_n)_{n\in L}$ is K-orthonormal with respect to $(x_n^*)_{n\in L}$ if and only if the projection

$$P : E \longrightarrow [x_n : n \in L],$$

defined by

$$P(x) := \sum_{n\in L} (x_n^*, x) x_n,$$

is well defined and has norm 1.

(ii) There exists a sequence of corresponding coefficient functional $(x_n^*)_{n\in L}$ in E^* such that $(x_n)_{n\in L}$ is K-orthonormal with respect to $(x_n^*)_{n\in L}$ if and only if there exist a projection $P : E \longrightarrow [x_n : n \in L]$ of norm 1.

Proof: It follows by Lemma 2.6 that the sequence $(x_n)_{n\in L}$ is semi-orthonormal, since it is S-orthonormal.

(i) Let $P(x) := \sum_{n\in L}(x_n^*, x)x_n$. If $(x_n)_{n\in L}$ is K-orthonormal with respect to $(x_n^*)_{n\in L}$, then it follows directly from Definition 1.23, by taking $\lambda_n := 1$ for all $n \in L$, that $P(x)$ is well defined for each $x \in E$, that is, $\sum_{n\in L}(x_n^*, x)x_n$ converges for each $x \in E$, and $\|P\| = 1$.
Conversely, suppose that P is well defined and that $\|p\| = 1$. Then, $\sum_{n\in L}(x_n^*, x)x_n$ is convergent for each $x \in E$.
Therefore, it follows, by Theorem 1.26 and since $(x_n)_{n\in L}$ is S-orthonormal, that, for any $(\lambda_n)_{n\in L} \in \ell^\infty$ and any $x \in E$, $\sum_{n\in L}\lambda_n(x_n^*, x)x_n$ converges, since $|\lambda_n(x_n^*, x)| \leq |(x_n^*, x)|$, and that

(16)
$$\left\|\sum_{n\in L}\lambda_n(x_n^*, x)x_n\right\| \leq \left(\sup_{n\in L}|\lambda_n|\right)\left\|\sum_{n\in L}(x_n^*, x)x_n\right\| = \left(\sup_{n\in L}|\lambda_n|\right)\|P(x)\| \leq \|x\|\left(\sup_{n\in L}|\lambda_n|\right).$$

Therefore, $(x_n)_{n\in L}$ is K-orthonormal with respect to $(x_n^*)_{n\in L}$.

(ii) If there exists a sequence of corresponding coefficient functional $(x_n^*)_{n\in L}$ in E^* such that $(x_n)_{n\in L}$ is K-orthonormal with respect to $(x_n^*)_{n\in L}$, then by part (i),

$$P(x) := \sum_{n\in L}(x_n^*, x)x_n$$

is a well defined projection of norm 1 from E onto $[x_n : n \in L]$.
Conversely, suppose that there exists a projection

$$P : E \longrightarrow [x_n : n \in L]$$

satisfying $\|p\| = 1$. Then, every $x \in E$ can be written uniquely in the form:

$$x = u_x + v_x,$$

where, $u_x \in [x_n : n \in L]$ and $v_x \in$ Ker P. Since $(x_n)_{n\in L}$ is semi-orthonormal, there exists a unique sequence $(x_n^*)_{n\in L}$ in $[x_n : n \in L]^*$ satisfying $(x_i^*, x_j) = \delta_{ij}$. Extend each $x_n^* \in [x_n; n \in L]^*$ to an element of E^* by

$$(x_n^*, x) := (x_n^*, u_x), \tag{17}$$

for every $x \in E$. Then, $(x_n)_{n\in L}$ is S-orthonormal with respect to the sequence of coefficient functional defined by equation 17. Indeed, let $(\lambda_n)_{n\in L} \in \ell^\infty$ and let $x \in E$. Then,

$$P(x) = u_x := \sum_{k\in L} a_k x_k,$$

and for all $n \in L$, we have,

$$(x_n^*, x) := (x_n^*, u_x) = a_n.$$

It follows that, for all $n \in L$,

$$|\lambda_n(x_n^*, x)| \leq \left(\sup_{k\in L} |\lambda_k|\right) |a_n|.$$

Therefore, since $(x_n)_{n\in L}$ is S-orthogonal, we obtian, by Theorem 1.26 and since $\sum_{n\in L} a_n x_n$ converges, that $\sum_{n\in L} \lambda_n(x_n^*, x)x_n$ converges and that

(18)
$$\left\|\sum_{n\in L} \lambda_n(x_n^*, x)x_n\right\| \leq \left\|\sum_{n\in L} \left(\sup_{k\in L} |\lambda_k|\right) a_n x_n\right\| = \left\|\sum_{n\in L} a_n x_n\right\| \sup_{k\in L} |\lambda_k| = \|P(x)\| \sup_{k\in L} |\lambda_k| \leq \|x\| \sup_{k\in L} |\lambda_k|.$$

This completes the proof. ◊

It is possible for $(x_n)_{n\in L}$ to be S-orthonormal, while, for any choice of corresponding coefficient functional, $(x_n)_{n\in L}$ is not K-orthonormal [20].

Example 1. Let $\{x_1, x_2, x_3\}$ be the standard basis in $\mathbb{R}^3$ as a real vector space with norm

$$\|a_1x_1 + a_2x_2 + a_3x_3\| := \max\{|a_1| + |a_2|, |a_3|, \frac{1}{3}|4a_2 - a_3|\}. \tag{19}$$

Setting $a_3 = 0$ in equation 17, we obtain that

$$\|a_1x_1 + a_2x_2\| = \max\{|a_1| + |a_2|, \frac{4}{3}|a_2|\},$$

which clearly implies that the set $\left\{x_1, \frac{3}{4}x_2\right\}$ is orthonormal, since

$$\|x_1\| = \left\|\frac{3}{4}x_2\right\| = \|x_3\| = 1.$$

We claim that for any choice of coefficient functional $\{x_1^*, x_2^*\}$ associated with $\left\{x_1, \frac{3}{4}x_2\right\}$, the set $\left\{x_1, \frac{3}{4}x_2\right\}$ is not K-orthogonal. By Theorem 2.9, it is enough to show that there are no projections of norm 1 on span $\{x_1, x_2\}$. Indeed, let $P : \mathbb{R}^3 \longrightarrow$ span $\{x_1, x_2\}$ be any projection onto span $\{x_1, x_2\}$ and let $u := u_1x_1 + u_2x_2 + u_3x_3$ be a non-zero element of ker P. Then,

$$P(a_1x_1 + a_2x_2 + a_3x_3) = (a_1 - u_1a_3)x_1 + (a_2 - u_2a_3)x_2.$$

Therefore, we have, if $w := a_1x_1 + a_2x_2 + a_3x_3$,

$$\|P(w)\| = \max\{|a_1 - u_1a_3| + |a_2 - u_2a_3|, \frac{4}{3}|a_2 - u_2a_3|\} \tag{20}$$

Case 1: If $u_1 \neq 0$, say $u_1 > 0$ (the case $u_1 < 0$ is similar). Then, setting $a_1 = -1, \quad a_2 = 0$ and $a_3 = 1$ in equations 19 and 20, we have,

$$\| - x_1 + x_3\| = 1$$

and

(21)

$$\|P(-x_1+x_3)\| = \|(-1-u_1)x_1-u_2x_3\| = \max\left\{|-1-u_1|+|-u_2|, \frac{4}{3}|-u_2|\right\} \geq |1+u_1| > 1.$$

Hence, $\|P\| > 1$.

Case 2: If $u_1 = 0$ and $u_2 \neq 0$, say $u_2 > 0$ (the case $u_2 < 0$ is similar). Then, setting $a_1 = 1, a_2 = 0$, and $a_3 = 1$ in equations 17 and 19, we get

$$\|x_1 + x_3\| = 1$$

and

(22)

$$\|P(x_1 + x_3)\| = \|x_1 - u_2x_2\| = \max\left\{|1| + |-u_2|, \frac{4}{3}|-u_2|\right\} \geq |1| + |-u_2| > 1.$$

Hence, $\|P\| > 1$.

Case 3: If $u_1 = u_2 = 0$, then, setting $a_1 = 0$ and $a_2 = a_3 = 1$ in equations 17 and 19, we get,

$$\|x_2 + x_3\| = 1$$

and

$$\|P(x_2 + x_3)\| = \|x_2\| = \frac{4}{3} > 1.$$

Hence, $\|P\| > 1$.

Therefore, in all cases, we obtain that $\|P\| > 1$. This implies that there are no projections of norm 1 onto span $\{x_1, x_2\}$.

Also, we could have $x_i \perp x_j$ for all $i \neq j$, while $(x_n)_{n\in L}$ is not orthogonal.

Example 2. Again let $\{x_1, x_2, x_3\}$ be the standard basis in $\mathbb{R}^3$ as a real vector space with norm

$$\|a_1x_1 + a_2x_2 + a_3x_3\| := \max\left\{|a_1|, |a_2|, |a_3|, \frac{1}{2}|a_1 + a_2 - a_3|\right\}.$$

Then, we have, for all $1 \leq i \neq j \leq 3$,

$$\|a_ix_i + a_jx_j\| = \max\{|a_i|, |a_j|\},$$

and, consequently,

$$x_i \perp x_j \quad \text{for all} \quad 1 \leq i \neq \leq 3.$$

On the other hand, we have

$$\|x_1 + x_2 + x_3\| = 1 \quad \text{and} \quad \|x_1 + x_2 - x_3\| = \frac{3}{2}.$$

Therefore, $\{x_1, x_2, x_3\}$ is not orthogonal.

In general,

$$(x \perp v_1 \ \text{ and } \ x \perp v_2) \not\Rightarrow (x \perp \text{ span } \{v_1, v_2\}),$$

since, in the previous example, we have $x_3 \perp x_1$ and $x_3 \perp x_2$, but x_3 is not orthogonal to $(x_1 + x_2)$, since

$$\|(x_1 + x_2) + x_3\| = 1 \ \text{ and } \ \|(x_1 + x_2) - x_3\| = 3/2$$

$\square$.

Conclusions: In real Hilbert space, the different notions of orthogonality are equivalent. Lumer-Giles orthogonality is equivalent to Birkhoff's orthogonality in a continuous semi-inner product space. Isosceles and Birkhoff's orthogonality can be compared if $|k| \geq 1$, with respect to some sequence of corresponding coefficient functionals, K-orthonormality is equivalent to S-orthonormality. S-orthogonality is the most natural extension of ortogonality to Banach spoace.

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