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Lower estimates on the principal eigenvalue of Witten q-Laplacian on Smooth metric measure spaces

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Abstract

Some eigenvalue inequalities in terms of generalized Mckean bound and the weighted Cheeger's constant are proved for Witten q-Laplacian on smooth metric measure spaces with smooth boundary imposing certain restrictions on geometric quantities such as mean curvatute and sectional curvature of the domains. On the other hand, a clamped plate problem involving Witten bi-Laplacian is considered on a weighted manifold in the regime of positive generalized Ricci curvature, while lower bound estimates on its principal frequency are established. Indeed, the principal frequency of the problem is shown to be bounded from below by a double of McKean bound, provided the generalized curvature is nonnegative. As an application clamped plate eigenvalue lower bound is derived on the weighted geodesic ball having nonnegative generalized Ricci curvature.

1. Introduction and Preliminaries

A smooth metric measure space (SMMS) is denoted by $(\mathscr{M}, \langle\cdot,\cdot\rangle, \phi)$, where $\mathscr{M}$ is a complete differentiable manifold of dimension m, $\langle\cdot,\cdot\rangle$ is the inner product induced

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by the Riemannian metric, and a real function $\phi \in C^{\infty}(\mathscr{M})$ is the potential. In other words, a SMMS is a generalization of Riemannian manifold. These spaces and their variants have been applied in various contexts where they play crucial roles, e.g. in probability theory, quantum field theory, diffusion processes, PDEs and geometric analysis, having close links with Markov diffusion operators, generalized curvature and geometry [7, 19, 20, 21, 24].
Associated with $(\mathscr{M}, \langle\cdot,\cdot\rangle, \phi)$ is a self adjoint second order differential operator called Witten (or drifting) Laplacian and defined by

$$\mathscr{L}_{\phi}u := \mathrm{div}_{\phi}(e^{-\phi}\nabla\phi) = \Delta_{\mathscr{M}}u - \langle\nabla\phi, \nabla u\rangle,$$

where $\mathrm{div}_{\phi} := e^{\phi}\mathrm{div}$, and div, ∇ and $\Delta_{\mathscr{M}}$ respectively stand for divergence, gradient and Laplace operators with respect to the Riemannian inner product on $\mathscr{M}$. Setting ϕ to a constant, $\mathscr{L}_{\phi}$ becomes $\Delta_{\mathscr{M}}$. Note also that symmetry and self adjointness properties of $\mathscr{L}_{\phi}$ yield the integration by parts formula

$$\int_{\mathscr{M}} \mathscr{L}_{\phi}uwe^{-\phi}dv = -\int_{\mathscr{M}} \langle\nabla u, \nabla w\rangle e^{-\phi}dv = \int_{\mathscr{M}} u\mathscr{L}_{\phi}we^{-\phi}dv$$

for all $u, w \in C^2(\mathscr{M})$, where $e^{-\phi}dv$ is the weighted measure on $(\mathscr{M}, \langle\cdot,\cdot\rangle, \phi)$, while dv is the correspoding Riemannian measure.
A generalized Ricci curvature is also defined on $(\mathscr{M}, \langle\cdot,\cdot\rangle, \phi)$ by

$$Ric^{\phi} := Ric_{\mathscr{M}} + \mathrm{Hess}\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n}, \qquad m \geq n.$$

Here $Ric_{\mathscr{M}}$ is the Ricci curvature of $\mathscr{M}$ and Hessϕ is the Hessian of function ϕ. The critical case $m = n$ is attainable only for a constant ϕ in which case $Ric^{\phi} \equiv Ric_{\mathscr{M}}$, while the case $m = \infty$ yields $Ric^{\phi} := Ric_{\mathscr{M}} + \mathrm{Hess}\phi$ appearing in quasi-Einstein and gradient Ricci soliton equations [3, 21].
Our attention in this paper is basically focused on the eigenvalue problems on curved domains so as to determine the effect of curvatures (mean curvature, sectional curvature, Ricci curvature) on positive lower bounds obtainable for the principal eigenvalue. By the principal eigenvalue, we mean the first nonzero Dirichlet eigenvalue. Let $\mathscr{M}_{\phi} := (\mathscr{M}, \langle\cdot,\cdot\rangle, \phi)$ be a complete noncompact SMMS and let $\mathbb{X}$ be an open connected subdomain with compact closure in $\mathscr{M}_{\phi}$. The Dirichlet eigenvalue problem on $\mathbb{X}$ consists in finding the real numbers λ (eigenvalues) and nonzero C^2-functions u (eigenfunctions) such that

$$\begin{aligned} \mathscr{L}_{\phi}u &= -\lambda u \quad \text{in} \quad \mathbb{X} \\ u &= 0 \qquad \text{on} \quad \partial\mathbb{X}. \end{aligned} \tag{1}$$

It is a known fact that all eigenvalues can be arranged in an increasing order according to their multiplicities. By the minimax Rayleigh princple, the principal eigenvalue $\lambda_1(\mathbb{X})$ is characterized by

$$\lambda_1(\mathbb{X}) = \inf_{0\neq u}\left\{\frac{\int_{\mathbb{X}}|\nabla u|^2 e^{-\phi}dv}{\int_{\mathbb{X}}|u|^2 e^{-\phi}dv} : u \in W_0^{1,2}(\mathbb{X})\right\}. \tag{2}$$

The domain monotonicity property of $\mathbb{X}$ with respect to $\lambda_1(\mathbb{X})$ implies that $\lambda_1(\mathbb{X}_1) \geq \lambda_1(\mathbb{X}_2)$ if $\mathbb{X}_1 \subset \mathbb{X}_2$. Thus if $\{\mathbb{X}_j\}_{j\geq 1}$ is an expanding sequence of open connected domains exhausting $\mathscr{M}_\phi$, then the sequence $\{\lambda_1(\mathbb{X}_j)\}_{j\geq 1}$ decreases and has a well defined nonnegative limit independent of the choice of the sequence $\{\mathbb{X}_j\}_{j\geq 1}$, meaning that the principal eigenvalue (spectrum bottom) of $-\mathscr{L}_\phi$ on $\mathscr{M}_\phi$ is given by

$$\lambda_1(\mathscr{M}_\phi) := \inf \sigma(\mathscr{M}, \langle\cdot,\cdot\rangle, \phi) = \lim_{j\to\infty} \lambda_1(\mathbb{X}_j). \tag{3}$$

Finding the condition under which $\lambda_1(\mathscr{M}) \geq c > 0$ has been intensively researched and many important special geometric properties have been found to be embedded in $\lambda_1(\mathscr{M}) > 0$ [1, 9, 10, 15, 18, 22, 23] and see [3, 5, 6] for evolving manifolds. It is not an easy task in general to obtain lower bounds on the principal eigenvalue. Two important estimates in this direction, nicknamed Cheeger isoperimetric inequality [10] and McKean lower bound [18] on compact or noncompact Riemannian manifolds are examined below

Theorem 1.1. *(Cheeger isoperimetric inequality [10]) Let M be a compact Riemannian manifold. Then*

$$\lambda_1(\Delta_M) \geq \frac{1}{4}\mathcal{C}(M)^2,$$

where

$$\mathcal{C}(M) := \inf_E \frac{Area(\partial E)}{Vol(E)} : E \subset\subset M \text{ with sufficiently smooth boundary } \partial E$$

is the usual Cheeger's constant.

Interested readers can see [4, 22] and the references therein for applications of Cheeger's constant.

Theorem 1.2. *(Mckean lower bound [18]) Let M be an m-dimensional complete noncompact, simply connected Riemannian manifold whose sectional curvature $\mathscr{K}_M \leq -\theta^2$, $\theta > 0$. Then*

$$\lambda_1(\Delta_M) \geq \frac{1}{4}(m-1)^2\theta^2.$$

McKean lower bound has been extended to the setting of submanifolds of bounded mean curvature by Cheung and Leung [12] as follows.

Theorem 1.3. *(Extended Mckean lower bound [12]) Let M be an m-dimensional complete noncompact, submanifold of hyperbolic space $\mathbb{H}$ with sectional curvatute*

$\mathscr{K}_{\mathbb{H}} = -1$. *If further, the mean curvature* $\mathbf{m}_c$ *of* M *in* $\mathbb{H}$ *satisfies* $\alpha := \sup_M \|\mathbf{m}_c\| < m-1$, *then*

$$\lambda_1(\Delta_M) \geq \frac{1}{4}(m-1-\alpha)^2.$$

In the first part of this paper (Section 2), we study the weighted form of Mckean type lower estimates on the principal eigenvalue of the Witten q-Laplacian $1 < q < \infty$. The Witten q-Laplacian is defined by

$$\mathscr{L}_{\phi,q} u := \mathrm{div}_\phi(\|\nabla u\|^{q-2}\nabla u e^{-\phi}) = \mathrm{div}(\|\nabla u\|^{q-2}\nabla u) - \|\nabla u\|^{q-2}\langle\nabla\phi, \nabla u\rangle,$$

For $q = 2$, $\mathscr{L}_{\phi,q} u = \mathscr{L}_\phi u$ (the Witten Laplacian) while for constant ϕ, $\mathscr{L}_{\phi,q} u = \mathrm{div}(\|\nabla u\|^{q-2}\nabla u)$ (the usual q-Laplacian).
Define $\mathbb{X}$ as before, the Witten q-Laplacian Dirichlet eigenvalue problem consists in solving for eigenvalues μ_q and eigenfunctions w satisfying

$$\begin{aligned} \mathscr{L}_{\phi,q} w &= -\mu_q |w|^{q-2} w \quad \text{in} \quad \mathbb{X} \\ w &= 0 \qquad\qquad\quad\; \text{on} \quad \partial\mathbb{X}. \end{aligned} \tag{4}$$

It is important to note that problem (4) is understood in the distributional sense, meaning that, for $w \in W_0^{1,q}(\mathbb{X})$ and every test function $\psi \in C_0^\infty(\mathbb{X})$ must satisfy

$$\int_{\mathbb{X}} \|\nabla u\|^{q-2}\langle\nabla w, \nabla\psi\rangle e^{-\phi} dv = \mu_q \int_{\mathbb{X}} |w|^{q-2} w\psi e^{-\phi} dv,$$

where by $W_0^{1,q}(\mathbb{X})$ we mean the completion of $C_0^\infty(\mathbb{X})$, the space of smooth compactly supported function on $\mathbb{X}$ with respect to the Sobolev norm

$$\|w\|_{1,q,\mathbb{X}} = \left(\int_{\mathbb{X}} (\|\nabla w\|^q + |w|^q) e^{-\phi} dv\right)^{1/q}.$$

In a similar vein, the principal Dirichlet eigenvalue $\mu_{1,q}(\mathbb{X})$ of $-, \mathscr{L}_{\phi,q}$ on $\mathbb{X}$ is characterized by

$$\mu_{1,q}(\mathbb{X}) = \inf_{0 \neq w} \left\{ \frac{\int_{\mathbb{X}} \|\nabla w\|^q e^{-\phi} dv}{\int_{\mathbb{X}} |w|^q e^{-\phi} dv} : u \in W_0^{1,q}(\mathbb{X}) \right\}. \tag{5}$$

with the constraint $\int_{\mathbb{X}} |w|^{q-2} w e^{-\phi} dv = 0$. Furthermore, the domain monotonicity property also holds and then we have

$$\mu_{1,q}(\mathscr{M}_\phi) = \lim_{j\to\infty} \mu_{1,q}(\mathbb{X}_j). \tag{6}$$

In Section 2, we will generalize Cheeger isoperimetric inequality and Mckean lower bounds (Theorems 1.1, 1.2 and 1.3) to the setting of Witten q-Laplacian on SMMS. Our results (Theorems 2.1 and 2.7) are generalization of some existing results, see for instance [8, 12, 13, 14, 16, 17].
In the second part (Section 3), we establish a similar result in the regime of positive generalized curvature for the principal eigenvalue of a weighted clamped

plate problem (see (26)) involving the Witten bi-Laplacian, $\mathscr{L}_\phi^2 = \mathscr{L}_\phi\mathscr{L}_\phi$, on $(\mathscr{M}, \langle\cdot,\cdot\rangle, \phi)$. Note that this problem (clamped plate problem) describes the characteristic vibration in a manner different from that of a fixed membrane, evident in the way the plate is tighten to the boundary. Here, the principal frequency(eigenvalue) is equivalent to the fundamental tone of the domain. Indeed, we obtain that the principal frequency of the problem is bounded from below by a double of McKean bound, provided the generalized curvature is nonnegative. This result can be compared with [11, 26, 27].

2. Generalized McKean bounds and Cheeger inequality

Let $\kappa : \mathscr{M} \to \widetilde{\mathscr{M}}$ be an isometric immersion, the ϕ-mean (or weighted mean) curvature $\mathbf{m}_\phi$ is defined by

$$\mathbf{m}_\phi = \mathbf{m}_c + \tilde{\nabla}\phi^\perp,$$

where $\mathbf{m}_c$ is the mean curvature of the submanifold $\mathscr{M}$, $\tilde{\nabla}$ is the connection on $\widetilde{\mathscr{M}}$ and $^\perp$ is the orthogonal projection onto the normal bundle $T\mathscr{M}^\perp$. Suppose ν is a unit normal vector field on $\mathscr{M}$ and $\mathbb{A} = \nabla_{(\cdot)}\nu$ is the shape operator on $\mathscr{M}$. Then

$$\mathbf{m}_\phi = \mathbf{m}_c - \langle\tilde{\nabla}\phi, \nu\rangle.$$

For the weighted minimal submanifold, the ϕ-mean curvature $\mathbf{m}_\phi = 0$ everywhere and hence, minimal submanifolds carry many of the properties of the ambient spaces.

2.1. Generalized McKean lower bounds.

Theorem 2.1. *Let $\mathscr{M}_\phi = (\mathscr{M}, \langle\cdot,\cdot\rangle, \phi)$ be a complete noncompact subspace of the weighted hyperbolic manifold $(\mathbb{H}^n, \phi)$ with sectional curvature $\mathscr{K}_\mathscr{M} = -1$. Suppose the ϕ-mean curvature $\mathbf{m}_\phi$ and the potential function ϕ of $\mathscr{M}_\phi$ in $(\mathbb{H}^n, \phi)$ verify $\sup_{\mathscr{M}_\phi} \|\mathbf{m}_\phi - \nabla\phi\| < m-1$. Then*

$$\mu_{1,q}(\mathscr{M}_\phi) \geq \frac{1}{q^q}[(m-1) - \sup_{\mathscr{M}_\phi}\|\mathbf{m}_\phi - \nabla\phi\|]^q. \tag{7}$$

Furthermore, if $\mathscr{M}$ is a ϕ-minimal submanifold of $\mathbb{H}^n$, then

$$\mu_{1,q}(\mathscr{M}_\phi) \geq \frac{1}{q^q}[(m-1) - \sup_{\mathscr{M}_\phi}\|\nabla\phi\|]^q. \tag{8}$$

Remark. (1) If ϕ is a constant then (7) becomes [13, Theorem 1.3]. If in addition $q = 2$ (7) becomes [12, Theorem 2].

(2) If ϕ is a constant and $\mathscr{M}$ is a minimal submanifold of $\mathbb{H}^n$, then $\mu_{1,2} \geq \frac{1}{4}(m-1)^2$. McKean's result [18] for this case asserts that the bound is sharp when $\mathscr{M} = \mathbb{H}^n$ as indeed $\mu_{1,2} = (m-1)^2/4$.

(3) Theorem 2.1 generalizes [16, Theorem 1.1] by Huang and Ma.

Now referring to the definition of $\mu_{1,q}$ (5) we note that estimate (7) is a direct consequence of the following q-Hardy-Poincaré type inequality to be proved in this section.

Proposition 2.2. *Suppose $\mathscr{M}_\phi = (\mathscr{M}, \langle\cdot,\cdot\rangle, \phi)$ is a complete noncompact subspace of the weighted hyperbolic manifold $(\mathbb{H}^n, \phi)$ of sectional curvature $\mathscr{K}_{\mathscr{M}} = -1$. If $\sup_{\mathscr{M}_\phi} \|\mathbf{m}_\phi - \nabla\phi\| < m-1$, then the following Hardy-Poincaré inequality holds*

$$\int_{\mathscr{M}_\phi} \|\nabla w\|^q e^{-\phi} dv \geq q^{-q}[m-1-\sup_{\mathscr{M}_\phi} \|\mathbf{m}_\phi - \nabla\phi\|]^q \int_{\mathscr{M}_\phi} |w|^q e^{-\phi} dv \tag{9}$$

for all $w \in C_0^\infty(\mathscr{M}_\phi)$ and $q > 1$.

In order to prove the last Proposition we will make use of the following Lemmas.

Lemma 2.3. [12, 16] *Suppose that $\mathscr{M}_\phi$ is an m-dimensional submanifold of $(\mathbb{H}^n, \phi)$. Then*

$$\mathscr{L}_\phi(\cosh\rho) = m\cosh\rho + \sinh\rho\langle \mathbf{m}_\phi, \tilde{\nabla}\rho\rangle\Big|_{\mathscr{M}_\phi} - \sinh\rho\langle\tilde{\nabla}\phi, \tilde{\nabla}\rho\rangle\Big|_{\mathscr{M}_\phi}, \tag{10}$$

where ρ is the distance function from a base point in $\mathbb{H}^n \backslash \mathscr{M}_\phi$.

Lemma 2.4. [12, 16] *Suppose that $\mathscr{M}_\phi$ is an m-dimensional submanifold of $(\mathbb{H}^n, \phi)$. Then*

$$\mathscr{L}_\phi\rho = (m - \|\nabla\rho\|^2)\coth\rho + \langle \mathbf{m}_\phi, \tilde{\nabla}\rho\rangle\Big|_{\mathscr{M}_\phi} - \langle\tilde{\nabla}\phi, \tilde{\nabla}\rho\rangle\Big|_{\mathscr{M}_\phi}, \tag{11}$$

where ρ is as defined before.

Proof. By direct computation

$$\Delta(\cosh\rho) = \sinh\rho\Delta\rho + \cosh\rho\|\nabla\rho\|^2.$$

It then follows that

$$\begin{aligned}\mathscr{L}_\phi(\cosh\rho) &= \Delta(\cosh\rho) - \langle\nabla\phi, \nabla\cosh\rho\rangle \\ &= \sinh\rho\Delta\rho + \cosh\rho\|\nabla\rho\|^2 - \sinh\rho\langle\nabla\phi, \nabla\rho\rangle \\ &= \sinh\rho\mathscr{L}_\phi\rho + \cosh\rho\|\nabla\rho\|^2.\end{aligned}$$

Applying Lemma 2.3 we arrive at

$$\sinh\rho\mathscr{L}_\phi\rho = m\cosh\rho + \sinh\rho\langle\mathbf{m}_\phi, \tilde{\nabla}\rho\rangle\Big|_{\mathscr{M}_\phi} - \sinh\rho\langle\tilde{\nabla}\phi, \tilde{\nabla}\rho\rangle\Big|_{\mathscr{M}_\phi} - \cosh\rho\|\nabla\rho\|^2.$$

Divide through the last equation by $\sinh\rho$, we obtain (11).

□

Proof of Proposition 2.2. Let $\rho = \rho(x)$ be the distance function from a base point in $\mathbb{H}^n \backslash \mathscr{M}_\phi$, we are to use the fact tha $\|\tilde{\nabla}\rho\|^2 = 1$ to obtain $\|\nabla\rho\|^2 \leq \|\tilde{\nabla}\rho\|^2 = 1$. By considering Lemma 2.4 we have

$$\begin{aligned} \mathscr{L}_\phi \rho &= (m - \|\nabla\rho\|^2)\coth\rho + \langle \mathbf{m}_\phi, \tilde{\nabla}\rho\rangle|_{\mathscr{M}_\phi} - \langle \tilde{\nabla}\phi, \tilde{\nabla}\rho\rangle|_{\mathscr{M}_\phi} \\ &\geq (m-1)\coth\rho - \|\mathbf{m}_\phi - \tilde{\nabla}\phi\|\|\tilde{\nabla}\rho\||_{\mathscr{M}_\phi} \\ &\geq (m-1)\coth\rho - \sup_{\mathscr{M}_\phi}\|\mathbf{m}_\phi - \tilde{\nabla}\phi\|. \end{aligned} \tag{12}$$

Picking any $w \in C_0^\infty(\mathscr{M})$, a direct computation shows that

$$\mathrm{div}_\phi(|w|^q \nabla\rho e^{-\phi}) = q|w|^{q-2} w\langle \nabla w, \nabla\rho\rangle + |w|^q \mathscr{L}_\phi \rho. \tag{13}$$

Applying the inequality $\|\nabla\rho\|^2 \leq 1$ and estimate (12) into (13) gives

$$\mathrm{div}_\phi(|w|^q \nabla\rho e^{-\phi}) \geq -q|w|^{q-1}\|\nabla w\| + [m - 1 - \sup_{\mathscr{M}_\phi}\|\mathbf{m}_\phi - \tilde{\nabla}\phi\|]|w|^q. \tag{14}$$

There is a need to estimate the first term on RHS of (14) before we proceed. So we recall that Young's inequality for $\Phi > 0$, $\Psi > 0$ and the constant $\alpha > 0$ implies $\Phi\Psi \leq q^{-1}(\Phi/\alpha)^q + s^{-1}(\alpha\Psi)^s$, where q and $s = q/(q-1)$ are conjugate exponents. Choosing $\Phi = q\|\nabla w\|$ and $\Psi = |w|^{q-1}$ yields the estimate

$$-q|w|^{q-1}\|\nabla w\| \geq -\frac{q^{q-1}}{\alpha^q}\|\nabla w\|^q - \frac{q-1}{q}\alpha^{q/(q-1)}|w|^q. \tag{15}$$

Since $w \in C_0^\infty(\mathscr{M}_\phi)$, $\mathrm{supp}(|w|^q\nabla\rho e^{-\phi}) \subset\subset \mathscr{M}_\phi$ and then divergence theorem gives

$$\int_{\mathscr{M}_\phi} \mathrm{div}_\phi(|w|^q \nabla\rho e^{-\phi}) e^{-\phi} dv = 0. \tag{16}$$

Integrating (14) with respect to $e^{-\phi}dv$, combining with Young's inequality (15) and the divergence theorem (16), and then rearranging we arrive at

$$\int_{\mathscr{M}_\phi} \|\nabla w\|^q e^{-\phi} dv \geq \frac{\alpha^q}{q^{q-1}}\Big[m - 1 - \sup_{\mathscr{M}_\phi}\|\mathbf{m}_\phi - \tilde{\nabla}\phi\| - \frac{(q-1)}{q}\alpha^{\frac{q}{q-1}}\Big]\int_{\mathscr{M}_\phi}|w|^q e^{-\phi}dv. \tag{17}$$

Prompted by the above we consider the task of maximizing for $\alpha > 0$ the scalar function $\vartheta(\alpha)$ given by $\alpha \mapsto \vartheta(\alpha) = \alpha^q\left(A - \frac{(q-1)}{q}\alpha^{\frac{q}{q-1}}\right)$, where $A := [m - 1 - \sup_{\mathscr{M}_\phi}\|\mathbf{m}_\phi - \tilde{\nabla}\phi\|] > 0$. To do so, we find the first and second derivatives of $\vartheta(\alpha)$ respectively as $\vartheta'(\alpha) = q\alpha^{q-1}(A - \alpha^{q/(q-1)})$ and $\vartheta''(\alpha) = q\alpha^{q-2}\Big((q-1)A - [\frac{q-1)^2+q^2}{q-1}]\alpha^{\frac{q}{q-1}}\Big)$. Clearly, the critical point of $\vartheta(\alpha), \alpha > 0$ occurs at the point $\alpha_\star = A^{(q-1)/q}$, implying that $\vartheta''(\alpha_\star) = -q^3/(q-1)A(\alpha_\star)^{q-2} < 0$ for $q > 1$.

Consequently, $\alpha_\star$ is the maximum point achieved by $\max \vartheta(\alpha) = \vartheta(\alpha_\star) = 1/qA^q$. Therefore

$$\frac{1}{q^{q-1}} \max_{\alpha>0} \vartheta(\alpha) = \frac{1}{q^q} A^q. \tag{18}$$

Substituting (18) back into (17) gives

$$\int_{\mathscr{M}_\phi} \|\nabla w\|^q e^{-\phi} dv \geq \frac{1}{q^q} A^q \int_{\mathscr{M}_\phi} |w|^q e^{-\phi} dv,$$

where $A := [m - 1 - \sup_{\mathscr{M}_\phi} \|\mathbf{m}_\phi - \tilde{\nabla}\phi\|] > 0$. This completes the proof q-Hardy-Poincaré inequality (9).

□

Proof of Theorem 2.1. Applying the definition of the principal eigenvalue of $-\mathscr{L}_{\phi,q}$ on $\mathscr{M}_\phi$ (see (5) and (6)), the required estimate (7) now follows from the q-Hardy-Poincaré inequality (9) by writing

$$\mu_{1,q}(\mathscr{M}_\phi) = \inf_{0\neq w} \left\{ \frac{\int_{\mathscr{M}} \|\nabla w\|^q e^{-\phi} dv}{\int_{\mathscr{M}} |w|^q e^{-\phi} dv} : u \in W_0^{1,q}(\mathbb{X}) \right\} \geq \frac{1}{q^q} A^q. \tag{19}$$

This immediately implies (7). If furthermore, $\mathscr{M}_\phi$ is a ϕ-minimal submanifold in $\mathbb{H}^n$, then the ϕ-mean curvature vanishes everywhere and (7) results into (8).

□

2.2. Weighted Cheeger's constant and McKean type estimates.

Definition 2.5. (Weighted Cheeger's Constant [2]) Let $\mathscr{M}_\phi$ be a smooth metric measure space. The ϕ-Cheeger's (or weighted Cheeger's) constant $\mathcal{C}_\phi(\mathbb{X})$ of a domain $\mathbb{X} \subset \mathscr{M}_\phi$ is defined by

$$\mathcal{C}_\phi(\mathbb{X}) := \inf_{\mathbb{X}'} \frac{Area_\phi(\partial\mathbb{X}')}{\mathrm{V}ol_\phi(\mathbb{X}')}, \tag{20}$$

where $\mathbb{X}'$ ranges over all sufficiently regular $\mathbb{X}' \subset\subset \mathbb{X}$ with smooth boundary $\partial\mathbb{X}'$. Here $\mathrm{V}ol_\phi$ and $Area_\phi$ stand for the weighted volume and Area with respect to the volume and surface measures $e^{-\phi}dv$ and $e^{-\phi}dA$ on $\mathscr{M}_\phi$, respectively.

Lemma 2.6. *(Weighted Cheeger inequality) For a bounded domain $\mathbb{X}$ having smooth boundary $\partial\mathbb{X} \neq \varnothing$ in $\mathscr{M}_\phi$, the following inequality holds*

$$\mu_{1,q}(\mathbb{X}) \geq \frac{1}{q^q} \mathcal{C}_\phi^q(\mathbb{X}). \tag{21}$$

Proof: Suppose $0 < \xi \in C_0^\infty(\mathbb{X})$. Define $\Xi(s) := \{y \in \mathbb{X} : \xi(y) > s\}$ and $Area_\phi(\Xi(s)) := \{y \in \mathbb{X} : \xi(y) = s\}$. By the co-area formula

(22)
$$\begin{aligned}\int_{\mathbb{X}} \|\nabla \xi\| e^{-\phi} dv &= \int_{-\infty}^{\infty} Area_\phi(\Xi(s)) ds \\ &\geq \inf_{\mathbb{X}} \frac{Area_\phi(\Xi(s))}{\mathrm{Vol}_\phi(\Xi(s))} \int_{-\infty}^{\infty} \mathrm{Vol}_\phi(\Xi(s)) ds = \mathcal{C}_\phi(\mathbb{X}) \int_{\mathbb{X}} \xi(y) e^{-\phi} dv.\end{aligned}$$

Picking any $h \in W^{1,q}(\mathbb{X})$, $q > 1$ and applying Hölder's inequality gives

(23)
$$\int_{\mathbb{X}} \|\nabla h^q\| e^{-\phi} dv = q \int_{\mathbb{X}} |h|^{q-1} \|\nabla h\| e^{-\phi} dv \leq q \Big(\int_{\mathbb{X}} |h|^q e^{-\phi} dv \Big)^{\frac{q-1}{q}} \Big(\int_{\mathbb{X}} \|\nabla h\|^q e^{-\phi} dv \Big)^{\frac{1}{q}}.$$

Combining (22) and (23) by setting $\xi = h^q$ we arrive at

$$\mathcal{C}_\phi(\mathbb{X}) \leq \frac{(\int_{\mathbb{X}} \|\nabla h\|^q e^{-\phi} dv)^{\frac{1}{q}}}{(\int_{\mathbb{X}} |h|^q e^{-\phi} dv)^{\frac{1}{q}}}.$$

Since h was arbitrarily chosen in $W_0^{1,q}(\mathbb{X})$ we obtain the desired inequality by invoking the definition of $\mu_{1,q}(\mathbb{X})$. □

The weighted Cheeger inequality obtained in the last lemma will be used to prove a generalized Cheung-Leung estimate [12].

Theorem 2.7. *Let $\mathscr{M}_\phi$ be an m-dimensional complete noncompact simply connected SMMS with sectional curvature $\mathscr{K}_{\mathscr{M}} \leq -\theta^2$, $\theta > 0$. Then*

(24)
$$\mu_{1,q}(\mathscr{M}_\phi) \geq \frac{1}{q^q} \Big[(m-1)\theta - \sup_{\mathscr{M}_\phi} \|\nabla \phi\| \Big]_+^q,$$

where $[Y]_+ := \max\{Y, 0\}$ for $Y \in \mathbb{R}$.

Proof. Consider a proper subspace $\mathbb{X} \subset \mathscr{M}_\phi$ and the distance function ρ satisfying $\|\nabla \rho\| = 1$ and $\Delta \rho \geq (m-1)\theta$ by the Laplacian comparison theorem [22]. Note that $\rho = \rho(x)$ is differentiable since $\mathscr{M}_\phi$ is simply connected with negative curvature. We deduce that

$$\mathscr{L}_\phi \rho \geq (m-1)\theta - \sup_{\mathscr{M}_\phi} \|\nabla \phi\| > [(m-1)\theta - \sup_{\mathscr{M}_\phi} \|\nabla \phi\|]_+ > 0.$$

We also compute

$$\begin{aligned}\mathrm{Area}_\phi(\partial \mathbb{X}) &= \int_{\partial \mathbb{X}} e^{-\phi} dA \geq \int_{\partial \mathbb{X}} \langle \rho, \nu \rangle e^{-\phi} dA = \int_{\mathbb{X}} \mathscr{L}_\phi \rho e^{-\phi} dv \\ &\geq [(m-1)\theta - \sup_{\mathscr{M}_\phi} \|\nabla \phi\|]_+ \int_{\mathbb{X}} e^{-\phi} dv = [(m-1)\theta - \sup_{\mathscr{M}_\phi} \|\nabla \phi\|]_+ \mathrm{Vol}_\phi(\mathbb{X}).\end{aligned}$$

Taking infimum we thus obtain the following lower bound on ϕ-Cheeger's constant

$$\inf_{\mathbb{X}} \frac{\text{Area}_\phi(\partial\mathbb{X})}{\text{Vol}_\phi(\mathbb{X})} \geq [(m-1)\theta - \sup_{\mathscr{M}_\phi} \|\nabla\phi\|]_+. \tag{25}$$

Applying weighted Cheeger inequality (21) and lower bound (25) we have

$$\mu_{1,q}(\mathbb{X}) \geq \frac{1}{q^q}\mathcal{C}^q_\phi(\mathbb{X}) = \frac{1}{q^q}\left(\inf_{\mathbb{X}} \frac{Area_\phi(\partial\mathbb{X})}{Vol_\phi(\mathbb{X})}\right)^q \geq \frac{1}{q^q}[(m-1)\theta - \sup_{\mathscr{M}_\phi}\|\nabla\phi\|]^q_+.$$

Since the bound on the RHS does not depend on $\mathbb{X}$, the required conclusion (i.e., (24)) follows at once by taking an increasing sequence of domains $\mathbb{X}_j, j \geq 1$ exhausting $\mathscr{M}_\phi$ and then passing to the limit with the fact that (6) holds.

□

3. Eigenvalue problem for the Witten-biLaplacian

A fourth order eigenvalue problem involving the Witten bi-Laplacian is considered on a compact SMMS, $\mathscr{M} = (\mathscr{M}, \langle\cdot,\cdot\rangle, \phi)$, namely, the weighted clamped plate problem

$$\begin{aligned} \mathscr{L}^2_\phi w = \lambda w \quad & \text{in } \mathscr{M}, \\ w = \tfrac{\partial w}{\partial\nu} = 0 \quad & \text{on } \partial\mathscr{M}, \end{aligned} \tag{26}$$

where ν stands for the outer unit normal vector field to the boundary $\partial\mathscr{M}$. The principal frequency of the problem can be variationally characterized as

$$\lambda_1(\mathscr{M}) = \inf_{0\neq w}\left\{\frac{\int_{\mathscr{M}}(\mathscr{L}_\phi w)^2 e^{-\phi}dv}{\int_M |w|^2 e^{-\phi}dv} : w \in W^{2,2}_0(M)\right\}. \tag{27}$$

The main result of this section is stated as follows

Theorem 3.1. *Let $\mathscr{M} = (\mathscr{M}, \langle\cdot,\cdot\rangle, \phi)$ be an m-dimensional compact SMMS with smooth boundary $\partial\mathscr{M}$ and $Ric^\phi(\mathscr{M}) \geq m\kappa$, $\kappa \geq 0$. Let $\lambda_1 = \lambda_1(\mathscr{M})$ be the first eigenvalue of*

$$\begin{aligned} \mathscr{L}^2_\phi w = \lambda w \quad & \textit{in } \mathscr{M}, \\ w = \tfrac{\partial w}{\partial\nu} = 0 \quad & \textit{on } \partial\mathscr{M}, \end{aligned}$$

$$\lambda_1(\mathscr{M}) \geq \left(\frac{1}{4}\mathsf{h}^2 + m\kappa\right)\frac{1}{4}\mathsf{h}^2,$$

where

$$\mathsf{h} = \mathsf{h}(\phi) := \sup_X\left\{\frac{\inf_{\mathscr{M}}[div_\phi(Xe^{-\phi})]}{\|X\|_\infty} : \ X \in \mathcal{X}(\mathscr{M})\right\},$$

$\mathcal{X}(\mathscr{M})$ denotes the set of all nonzero smooth vector fields satisfying $\|X\|_\infty := \sup_{\mathscr{M}}\|X\| < \infty$ and $\inf_{\mathscr{M}}[div_\phi(Xe^{-\phi})] > 0$.

Moreover, if $\mathscr{M}$ is of nonnegative generalized curvature (i.e., $Ric^{\phi} \geq 0$), then

$$\lambda_1(\mathscr{M}) \geq \frac{1}{16}\mathsf{h}^4.$$

Quickly recall the weighted Reilly formula which is a very important tool in proving the above theorem. Define the second fundamental form of $\partial\mathscr{M}$ for any vector fields X, $Y \in \Gamma(T\partial\mathscr{M})$ by

$$\mathbf{II}(X,Y) = g(\nabla_X \nu, Y),$$

while the mean curvature and ϕ-mean curvature are respectively defined by

$$\mathbf{m}_c(x) = \mathrm{tr}\mathbf{II} \quad \text{and} \quad \mathbf{m}_\phi(x) := \mathbf{m}_c(x) - \langle \nabla\phi, \nu(x)\rangle. \tag{28}$$

Now denote by h_ν the normal derivative of h on $\partial\mathscr{M}$ and dA as the weighted $(m-1)$-dimensional Riemannian volume measure on $\partial\mathscr{M}$.

Weighted Reilly formula. Let $\mathscr{M} = (\mathscr{M}, \langle\cdot,\cdot\rangle, \phi)$ be an m-dimensional compact SMMS with smooth boundary. Then

$$\begin{aligned}&\int_{\mathscr{M}} \left((\mathscr{L}_\phi h)^2 - \left[|\mathrm{Hess} h|^2 + Ric_\phi(\nabla h, \nabla h)\right]\right)e^{-\phi}dv \\ &= \int_{\partial\mathscr{M}} \left(\mathbf{m}_\phi h_\nu + \mathscr{L}_{\phi,\partial} h\right) h_\nu dA + \int_{\partial\mathscr{M}} \left(\mathbf{II}(\nabla_\partial h, \nabla_\partial h) - \langle \nabla_\partial h, \nabla_\partial h_\nu\rangle\right) dA,\end{aligned} \tag{29}$$

where $\mathscr{L}_{\phi,\partial}\cdot := \Delta_\partial \cdot - \langle\nabla_\partial\phi, \nabla_\partial\cdot\rangle$ and ∇_∂ are weighted Laplacian and covariant derivative with respect to the induced metric on $\partial\mathscr{M}$. See [25, 26] for the proof.

Proof of Theorem 3.1. Proceeding as in previous section (see proof of Theorem 2.1).

$$\mathrm{div}_\phi(|w|^2 X e^{-\phi}) \geq -2|w|\|\nabla w\|\|X\| + \inf_{\mathscr{M}}[\mathrm{div}_\phi(Xe^{-\phi})]|w|^2. \tag{30}$$

By Young's inequality for $\alpha > 0$

$$-2|w|\|\nabla w\|\|X\| \geq -\alpha^{-2}\|\nabla w\|^2 - \alpha^2|w|^2\|X\|^2. \tag{31}$$

The divergence theorem also implies

$$\int_{\mathscr{M}} \mathrm{div}_\phi(|w|^2 X e^{-\phi})e^{-\phi}dv = \int_{\partial\mathscr{M}} |w|^2\langle X, \nu\rangle dA = 0 \tag{32}$$

since $w = w_\nu = 0$ on $\partial\mathscr{M}$. Integrating (30) with respect to $e^{-\phi}dv$ and then combining with (31) and (32) yields

$$\int_{\mathscr{M}} \|\nabla w\| e^{-\phi}dv \geq \alpha^2\left(\inf_{\mathscr{M}}[\mathrm{div}_\phi(Xe^{-\phi})] - \alpha^2\|X\|_\infty^2\right)\int_{\mathscr{M}} |w|e^{-\phi}dv. \tag{33}$$

By the maximaization procedure as used before

$$\max_{\alpha>0}\left\{\alpha^2\left(\inf_{\mathscr{M}}[\mathrm{div}_\phi(Xe^{-\phi})] - \alpha^2\|X\|_\infty^2\right)\right\} = \frac{\inf_{\mathscr{M}}[\mathrm{div}_\phi(Xe^{-\phi})]^2}{4\|X\|_\infty^2} = \frac{1}{4}\mathsf{h}^2. \tag{34}$$

Using (33) and (34) we have

$$\frac{\int_{\mathscr{M}} \|\nabla w\|^2 e^{-\phi} dv}{\int_{\mathscr{M}} |w|^2 e^{-\phi} dv} \geq \frac{1}{4}\mathsf{h}^2. \tag{35}$$

Now applying the weighted Reilly formula (29) together with the boundary condition $w = w_\nu = 0$ on $\partial\mathscr{M}$, we have

$$\int_{\mathscr{M}} (\mathscr{L}_\phi w)^2 e^{-\phi} dv \geq \int_{\mathscr{M}} \left[|\mathrm{H}ess w|^2 + m\kappa\|\nabla w\|^2\right] e^{-\phi} dv,$$

where we have used the condition $\mathbf{II} \geq 0$ and $\mathbf{m}_\phi \geq 0$. Then

$$\frac{\int_{\mathscr{M}} (\mathscr{L}_\phi w)^2 e^{-\phi} dv}{\int_{\mathscr{M}} \|\nabla w\|^2 e^{-\phi} dv} \geq \frac{\int_{\mathscr{M}} |\mathrm{H}ess w|^2 e^{-\phi} dv}{\int_{\mathscr{M}} \|\nabla w\|^2 e^{-\phi} dv} + m\kappa. \tag{36}$$

Notice that

$$|\mathrm{H}ess w|^2 \geq |\nabla|\nabla w||^2. \tag{37}$$

To show (37), we choose normal coordinate for any $p \in \mathscr{M}$. The following identity holds

$$|\nabla|\nabla w||^2(p) = \sum_{\beta=1}^{m}\left(\frac{(\sum_{\alpha=1}^{m} w_\alpha w_{\alpha\beta})^2}{\sum_{\alpha=1}^{m} w_\alpha^2}\right).$$

By the Cauchy-Schwarz inequality we have

$$\sum_{\beta=1}^{m}\sum_{\alpha=1}^{m}(w_\alpha w_{\alpha\beta})^2 \leq \sum_{\beta=1}^{m}(\sum_{\alpha=1}^{m} w_\alpha^2)(\sum_{\alpha=1}^{m} w_{\alpha\beta}^2) = (\sum_{\alpha=1}^{m} w_\alpha^2)(\sum_{\alpha\beta=1}^{m} w_{\alpha\beta}^2).$$

Combining the last two identities we have

$$|\nabla|\nabla w||^2(p) \leq \sum_{\alpha\beta=1}^{m} w_{\alpha\beta}^2 = |\mathrm{H}ess w|^2,$$

which proves (37) for any $w \in W^{2,2}(\mathscr{M})$.
Similarly to (35), one can show that

$$\frac{\int_{\mathscr{M}} |\nabla|\nabla w||^2 e^{-\phi} dv}{\int_{\mathscr{M}} \|\nabla w\|^2 e^{-\phi} dv} \geq \frac{1}{4}\mathsf{h}^2 \tag{38}$$

by replacing $|w|^2$ with $|\nabla w|^2$ in (30) and repeating the computation process for $\mathrm{div}_\phi(|\nabla w|^2 X e^{-\phi})$, noting that $|\nabla w|^2 X$ has compact support on $\mathscr{M}$ since $w \in C_0^\infty(\mathscr{M})$. One easily sees that divergence theorem holds in the sense that $\int_{\mathscr{M}} \mathrm{div}_\phi(|\nabla w|^2 X e^{-\phi}) e^{-\phi} dv = 0$.

Now combine (36), (37) and (38) and obtain

$$\begin{aligned}\frac{\int_{\mathscr{M}}(\mathscr{L}_\phi w)^2 e^{-\phi}dv}{\int_{\mathscr{M}}\|\nabla w\|^2 e^{-\phi}dv} &\geq \frac{\int_{\mathscr{M}}|\mathrm{H}ess w|^2 e^{-\phi}dv}{\int_{\mathscr{M}}\|\nabla w\|^2 e^{-\phi}dv} + m\kappa \\ &\geq \frac{\int_{\mathscr{M}}|\nabla|\nabla w||^2 e^{-\phi}dv}{\int_{\mathscr{M}}\|\nabla w\|^2 e^{-\phi}dv} + m\kappa \geq \frac{1}{4}\mathsf{h}^2 + m\kappa\end{aligned} \tag{39}$$

Finally, using (27), (35) and (39), we have

$$\begin{aligned}\lambda_1(\mathscr{M}) = \frac{\int_{\mathscr{M}}(\mathscr{L}_\phi w)^2 e^{-\phi}dv}{\int_M |w|^2 e^{-\phi}dv} &= \frac{\int_{\mathscr{M}}(\mathscr{L}_\phi w)^2 e^{-\phi}dv}{\int_{\mathscr{M}}\|\nabla w\|^2 e^{-\phi}dv}\frac{\int_{\mathscr{M}}\|\nabla w\|^2 e^{-\phi}dv}{\int_{\mathscr{M}}|w|^2 e^{-\phi}dv} \\ &\geq \left(\frac{1}{4}\mathsf{h}^2 + m\kappa\right)\frac{1}{4}\mathsf{h}^2.\end{aligned}$$

Moreover, we set $\kappa = 0$ for the case $Ric^\phi \geq 0$ and arrive at $\lambda_1 \geq 1/16\mathsf{h}^4$.

□

We remark that

$$\inf_{\mathscr{M}}[\mathrm{div}_\phi(Xe^{-\phi})] = \inf_{\mathscr{M}}[\mathrm{div}(X) - \langle\nabla\phi, X\rangle] \geq [\inf_{\mathscr{M}}\mathrm{div}(X) - \sup_{\mathscr{M}}\|\nabla\phi\|\|X\|_\infty]_+ > 0$$

Taking $X = \nabla\rho$, where $\rho = \rho(x) = d(x, x_0)$ is the distance from a fixed point $x_0 \in \mathscr{M}$, which can be assumed to be differentiale almost everywhere. Then $\|X\|_\infty = \|\nabla\rho\|_\infty = 1$ and then

$$\inf_{\mathscr{M}}[\mathrm{div}_\phi(Xe^{-\phi})] = \inf_{\mathscr{M}}[\mathrm{div}_\phi(\nabla\rho e^{-\phi})] \geq [\inf_{\mathscr{M}}\Delta\rho - \sup_{\mathscr{M}}\|\nabla\phi\|]_+ > 0.$$

By the Hessian comparison theorem [8, 14] for a function $\eta(\rho) := k\cot(k\rho)$ whenever $\sup_\gamma \mathscr{K}_{\mathscr{M}} = +k^2$ and $\rho < \frac{\pi}{2k}$, where $\gamma : [0, \rho(x_1)] \to \mathscr{M}$ is a minimizing geodesic joining x and $x_0 \in \mathscr{M}$, and $\mathscr{K}_{\mathscr{M}}$ is the sectional curvature of $\mathscr{M}$. We have

$$\text{Hess }\rho(x)(X, X) \geq \eta(\rho(x))\|X\|^2 \quad \text{and} \quad \text{Hess }\rho(x)(\gamma', \gamma') = 0,$$

where X is any vector in $T_x\mathscr{M}$ perpendicular to $\gamma'(\rho(x))$. This leads to the lower bound on Laplacian of the distance function as follows

$$\Delta\rho \geq (m-1)k\cot(k\rho).$$

Then clearly $\inf_{\mathscr{M}}\Delta\rho \geq (m-1)k$. Therefore

$$\begin{aligned}\mathsf{h}(\phi) &= \sup_X \frac{\inf_{\mathscr{M}}[\mathrm{div}_\phi(Xe^{-\phi})]}{\|X\|_\infty} \\ &\geq [\inf_{\mathscr{M}}\Delta\rho - \sup_{\mathscr{M}}\|\nabla\phi\|]_+ \geq [(m-1)k - \sup_{\mathscr{M}}\|\nabla\phi\|]_+ > 0.\end{aligned}$$

Following the above argument we can easily prove the following Corollary

Corollary 3.2. *Let $\mathscr{B}(p,r)$ denotes the geodesic ball centred at $p \in \mathscr{M}$ with radius $r < inj(p)$. If in addition to the hypothesis of Theorem 3.1, the sectional curvature $\mathscr{K}_{\mathscr{M}}$ of $\mathscr{M}$ at $x \in \mathscr{B}(p,r)$ satisfies $\sup_{\mathscr{M}} \mathscr{K}_{\mathscr{M}} = +k^2 \quad r < \frac{\pi}{2k}$. Then*

$$\lambda_1(\mathscr{B}(p,r)) \geq \frac{1}{16}[(m-1)k - \sup_{\mathscr{M}} \|\nabla\phi\|]_+^4$$

whenever $Ric^{\phi}(\mathscr{B}(p,r)) \geq 0$.

Conclusion:

This paper presented generalized weighted Cheeger isoperimetric inequality and Mckean lower bounds for the principal eigenvalue of the Witten q-Laplacian $1 < q < \infty$ on Smooth metric measure spaces based on certain restrictions on curvatures. Furthermore, the principal frequency of the Witten bi-Laplacian of the clamped plate problem on a weighted manifold was proved to be bounded from below by a double of McKean bound within non-negativity of the generalized Ricci curvature.

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