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# Idempotent and Nilpotent Elements in Subsemigroups of Order-Preserving or Order-Reversing Partial Transformation Semigroups and their Combinatorial Properties

I. A. Ogunleke[1*], K. Rauf[2], S. O. Makanjuola[3], O. A. Ojo[4], M. M. Mogbonju[5] and M. A. Asiru[6]

Abstract

The work generated the possible elements for $n = 1, 2, 3, 4$ and 5, by using th GAP software. The number of idempotent and nilpotent elements in subsemigroups of partial transformation semigroups is obtained. The elements of order-decreasing and order-preserving or order-reversing that are contraction transformations are equally studied. Their combinatorial properties are established.

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[1]Department of Mathematics, Alvan Ikoku Federal College of Education, P.M.B. 1033, Owerri, Imo State, Nigeria
[2&3]Department of Mathematics, University of Ilorin, Ilorin, Nigeria
[4]Department of Statistics, Oyo State College of Agriculture and Technology, P.M.B. 10, Igbo-Ora
[5]Department of Mathematics, University of Abuja, P.M.B. 117, FCT, Nigeria
[6]Department of Mathematics and Statistics, Federal Polytechnic Bida, Nigiria
E-mails: isaacadekunle1234@gmail.com; krauf@unilorin.edu.ng
Corresponding author Phone Number: +2348062146309
ORCID of the corresponding author: https://orcid.org/0000-0000-0000-0000

## 1. Introduction

Transformation semigroup is one of the most fundamental mathematical facts. Transformation semigroups are also of utmost important for semigroup theory as every semigroup is isomorphic to a transformation semigroup. Transformation semigroup is a pair $(X, S)$ where $S$ is a semigroup of transformations on $X$ and a transformation $X$ is just a function from $X$ to itself not necessarily invertible and therefore $S$ is simply a set of transformation on $X$ which is closed under composition of functions.

Let $X_n = \{1, 2, 3, \cdots, n\}$, then a (partial) transformation $\alpha : Dom\alpha \subseteq X_n \to Im\alpha$ is said to be full or total if $Dom\alpha = X_n$ otherwise called strictly partial $SP_n$. The set of all partial transformations on $n$-objects forms a semigroup under the usual composition of transformation, it is denoted by $P_n$ when it is partial, $T_n$ when it is full or total and $I_n$ when it is one to one partial transformation semigroups. The elements of $I_n$, which we sometimes called the symmetric inverse semigroup are usually called charts.

Thus, these are the three main fundamentals semigroups. Also, the subsemigroup of transformation include the following:

- **(i):** the Subsemigroup of partial contractions $-CP_n$;
- **(ii):** the Semigroup of Order-Preserving or Reversing Partial Contractions $ORCP_n$;
- **(iii):** the Subsemigroup of Order-Decreasing Partial Contractions $DCP_n$;
- **(iv):** the Subsemigroup of Order-Preserving Partial Contractions $OCP_n$;
- **(v):** the Subsemigroup of Order-Decreasing and Order-Reserving Partial Contraction $DRCP_n$;
- **(vi):** the Subsemigroup of Order-Preserving and Order-Decreasing Partial contractions $ODCP_n$.

Algebraic and combinatorial properties of transformation semigroup have been studied by some researchers with interesting results. Among them are [4], [3], [8] and [7]. The fact that contraction mapping is a new class of transformation semigroups, [11], [1] and [9] worked on this new class of transformation semigroups. Quite a good number of Algebraic Scientists have done researches on many interesting results. However, [7] studied the semigroups of order-preserving and order-decreasing of a finite set $X_n = \{1, 2, 3, \cdots, x_n\}$ and showed that the order and number of idempotent of $O_n$ (the semigroup of order-preserving full transformation on $X_n$) are:

$$|O_n| = \begin{pmatrix} 2n-1 \\ n-1 \end{pmatrix}$$

and $E = |O_n| = F_n$ respectively. He also showed that the Fibonacci number $F_n$ defined by $F_1 = F_2 = 1$ and $F_0 = 0$ where

$$F_n = F_{n-1} + F_{n-2}, \ (n \geq 3)$$

and $F_{2n}$ is alternate Fibonacci numbers.
[2] derived the formula for the number of semigroup $|S|$ in partial one-one transformation to be $C_n = \sum_{r=0}^{n} \binom{n}{r}^2 r!, C_0 = 1, C_1 = 2$. He also derived the formula for the number of semigroup of the order-decreasing partial one-one transformation to be the bell number $B_n$ where $B_{n+1} = \sum_{r=0}^{n} \binom{n}{r} B_n, B_0 = 1 = B_1$. He derived further the formula for the number of idempotent $|E(S)|$ of the order-decreasing partial one - to - one transformation to be $2^n$ for $n \geq 1$. In addition, [5, 6] obtained some delightful results concerning $N_n$ where $N_n = O_n \cap D_n$, the semigroup of all maps that are both order-preserving and order-decreasing and showed that $|O_n \cap D_n| = |C_n|$ the catalon number defined as:

$$C_n = \frac{1}{n+1} \begin{pmatrix} 2n \\ n \end{pmatrix}$$

and number of idempotent elements are

$$|E(C_n)| = 2^{n-1}.$$

He stated further that the nature of the mapping with the property that $|x\alpha - y\alpha| \leq |x - y|$ for all $x$ and $y$ to be called a contraction mapping. The sets of all partial contraction maps and of all order-preserving contraction maps in $P_n$ (Partial transformation Semigroup) are respectively denoted by $CP_n$ and $OCP_n$ respectively and constitute the subsemigroup of $P_n$.
[3] also derived the formula for the number of idempotent $|E(s)|$ for the order-preserving partial one-to-one transformation to be $2^n$ for $n \geq 1$. Hence, he derived the formula for the number of semigroup of the order-preserving partial one-to-one transformation to be

$$CO_n = \begin{pmatrix} 2n \\ n \end{pmatrix}$$

The work of [1] centered on full contraction mapping and its subsemigroup of order-preserving ($OCT_n$) order-preserving or order-reserving ($ORCT_n$) and order-decreasing ($ODCT_n$) of full contracting mapping. His results are summarized below:

| $S$ | $\|S\|$ | $E(S)$ |
|---|---|---|
| $OCT_n$ | $(n+1)2^{n-2}$ | $\begin{pmatrix} n+1 \\ 2 \end{pmatrix}$ |
| $ORCT_n$ | $(n+1)2^{n-1} - n$ | $\begin{pmatrix} n+1 \\ 2 \end{pmatrix}$ |
| $ODCT_n$ | $2^{n-1}$ | $\begin{pmatrix} n+1 \\ p-1 \end{pmatrix}$ |

However, [10], compute the number of all idempotent in $PC_n$ ( the semigroup of order- preserving and order- decreasing partial transformation) and showed that $e(n,r,k) = |\{\alpha \in PC_n : \alpha^2 = \alpha, |Dom\alpha| = rmax?(Im\alpha) = k\}|$ to have

$$e(n,r,0) = \begin{cases} 1 \ (r = 0); \\ \\ 0 \ (r > 0); \end{cases} \qquad e(n,0,k) = \begin{cases} 1 \ (k = 0); \\ \\ 0 \ (k > 0); \end{cases}$$

and $e(n,r,1) = \binom{n-1}{r-1}$ of width $r$ and $Im\alpha = \{1\}$
i.e the number of the subsets of $X_n$ each containing the element 1 of size $r$.

## 2. The Techniques

This section presents the results on semigroup of order-preserving and order-decreasing partial contraction transformation. The method employed in this work was the results generated by Gap structure and studying the elements that are order-preserving, order-decreasing contraction transformation for $n = 1, 2, 3, 4, 5$.and their idempotents. Also, the combinatorial results were investigated.

**Examples**

The elements of $ODCP_1$ has 1 element when $n = 1$
$|Im\alpha| = 1$

| $Ker\alpha \| Im\alpha$ | $\{1\}$ |
|---|---|
| 1 | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ |

When $n = 2$, $ODCP_2$ has 5 elements

$|Im\alpha| = 1$

| $Ker\alpha \mid Im\alpha$ | $\{1\}$ | $\{2\}$ |
|---|---|---|
| 1 2 | $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ | |
| 1 | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | |
| 2 | $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ |

$|Im\alpha| = 2$

| $Ker\alpha \mid Im\alpha$ | $\{1, 2\}$ |
|---|---|
| 1/2 | $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ |

When $n = 3$, $ODCP_3$ has 19 elements.
$|Im\alpha| = 1$

| $Ker\alpha \mid Im\alpha$ | $\{1\}$ | $\{2\}$ | $\{3\}$ |
|---|---|---|---|
| 1 2 3 | $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ | | |
| 1 2 | $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ | | |
| 1 3 | $\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ | | |
| 2 3 | $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ | $\begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}$ | |
| 1 | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | | |
| 2 | $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ | |
| 3 | $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ | $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ |

$|Im\alpha| = 2$

| $Ker\alpha \vert Im\alpha$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ |
|---|---|---|---|
| 1 2/3 | $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$ | | |
| 1/2 3 | $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}$ | | |
| 1/2 | $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ | | |
| 1/3 | $\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$ | $\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$ | |
| 2/3 | $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ | | $\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$ |

$|Im\alpha| = 3$

| $Ker\alpha \vert Im\alpha$ | $\{1,2,3\}$ |
|---|---|
| 1/2/3 | $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ |

When $n = 4$, $ODCP_4$ has 67 elements

## 3. Combinatorial Results

In this section, the tables for each subsemigroups are drawn. However, for given natural number

$$n \geq P \geq M \geq 0,$$

we define the following:

- **(i):** $F(n;p) = |\{\alpha \in S : h(\alpha) = |Im\alpha|| = p\}$
- **(ii):** $F(n;m) = |\{\alpha \in S : f(\alpha) = m\}|$
- **(iii):** $F(n;k) = |\{\alpha \in S : w^+(\alpha) = max(Im\alpha) = K\}|$
- **(iv):** $F(n;r) = |\{\alpha \in S : h(\alpha) = p \wedge w^+(\alpha) = r\}|$

$\square$.

The tables below show these subgroups:

TABLE 1. Height of Image of $ODCP_n$

| $n/p$ | 1 | 2 | 3 | 4 | 5 | $\sum f(n;p) = \lvert ODCP_n \rvert$ |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 1 |
| 2 | 4 | 1 | | | | 5 |
| 3 | 11 | 7 | 1 | | | 19 |
| 4 | 26 | 30 | 10 | 1 | | 67 |
| 5 | 57 | 102 | 58 | 13 | 1 | 231 |

TABLE 2. Size of the Domain of $ODCP_n$

| $n/r$ | 1 | 2 | 3 | 4 | 5 | $\sum f(n;r) = \lvert ODCP_n \rvert$ |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 1 |
| 2 | 3 | 2 | | | | 5 |
| 3 | 6 | 9 | 4 | | | 19 |
| 4 | 10 | 25 | 24 | 8 | | 67 |
| 5 | 15 | 55 | 85 | 64 | 12 | 231 |

TABLE 3. Maximum Element of $ODCP_n$

| $n/k$ | 1 | 2 | 3 | 4 | 5 | $\sum f(n;k) = \lvert ODCP_n \rvert$ |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 1 |
| 2 | 3 | 2 | | | | 5 |
| 3 | 7 | 8 | 4 | | | 19 |
| 4 | 15 | 24 | 20 | 8 | | 67 |
| 5 | 31 | 64 | 72 | 48 | 16 | 231 |

TABLE 4. Fixed Element of $ODCP_n$

| $n/m$ | 0 | 1 | 2 | 3 | 4 | 5 | $\sum f(n;m) = \lvert ODCP_n \rvert$ |
|---|---|---|---|---|---|---|---|
| 1 | - | 1 | | | | | 1 |
| 2 | 1 | 3 | 1 | | | | 5 |
| 3 | 5 | 9 | 4 | 1 | | | 19 |
| 4 | 19 | 29 | 13 | 5 | 1 | | 67 |
| 5 | 68 | 96 | 42 | 18 | 6 | 1 | 231 |

Furthermore, the tables for idempotent of order-preserving and order-decreasing partial contraction transformations are constructed and briefly studied.

Hence, the following are results of elements generated for $n = 6$ to $n = 20$ respectively as: 791, 2703, 9231, 31519, 107615, 367423, 1254463, 4283007, 14623103, 49926399, 170459391, 581984767, 1987020287, 6784111615 and 23162405887.

TABLE 5. Heights of the Image of Idempotent Elements in $ODCP_n$

| $n/p$ | 1 | 2 | 3 | 4 | 5 | $\sum f(n;p) = \lvert E(ODCP_n)\rvert$ |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 1 |
| 2 | 3 | 1 | | | | 4 |
| 3 | 7 | 4 | 1 | | | 12 |
| 4 | 15 | 11 | 5 | 1 | | 32 |
| 5 | 31 | 26 | 16 | 6 | 1 | 80 |

TABLE 6. Size of the Domain of Idempotent Elements in $ODCP_n$

| $n/r$ | 1 | 2 | 3 | 4 | 5 | $\sum f(n;r) = \lvert E(ODCP_n)\rvert$ |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 1 |
| 2 | 2 | 2 | | | | 4 |
| 3 | 3 | 6 | 3 | | | 12 |
| 4 | 4 | 12 | 12 | 4 | | 32 |
| 5 | 5 | 20 | 30 | 20 | 5 | 80 |

TABLE 7. Maximum Elements in the Image of Idempotent Elements in $ODCP_n$

| $n/k$ | 1 | 2 | 3 | 4 | 5 | $\sum f(n;k) = \lvert E(ODCP_n)\rvert$ |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 1 |
| 2 | 2 | 2 | | | | 4 |
| 3 | 4 | 4 | 4 | | | 12 |
| 4 | 8 | 8 | 8 | 8 | | 32 |
| 5 | 16 | 16 | 16 | 16 | 16 | 80 |

TABLE 8. Fixed Elements in the Image of Idempotent Elements in $ODCP_n$

| $n/m$ | | 2 | 3 | 4 | 5 | $\sum f(n;m) = \|E(ODCP_n)\|$ |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 1 |
| 2 | 3 | 1 | | | | 4 |
| 3 | 7 | 4 | 1 | | | 12 |
| 4 | 15 | 11 | 5 | 1 | | 32 |
| 5 | 30 | 27 | 16 | 6 | 1 | 80 |

Based on above tables the following Lemma is generated.

**Lemma 3.1.** *Let $S = ODCP_n$ and $E(ODCP_n)$ be the set of idempotent elements in $S$, then for every $n$, $F(n,k) = 2^{n-1}\ \forall k$.*

*Proof.* For each $n$, the number of elements is the same and it forms a sequence that gives a formula $2^{n-1}$. That is $F(n,k) = 2^{n-1}$ for all $k$ □

**Theorem 3.2.** *When $S = OCP_n \cap DCP_n$. The total number of idempotents*

$$|E(S)| = \sum_{r=0}^{n} r \begin{pmatrix} n \\ r \end{pmatrix}$$

*Proof.* For each $n$ there is an $(n-1)$ element set on which we can find identity elements for an idempotent, so there are $2^{n-1}$ ways, hence the total number of ways is $n \times 2^{n-1}$. Hence, for overall idempotents elements, $|E(S)| = \sum_{r=0}^{n} r \begin{pmatrix} n \\ r \end{pmatrix}$ □

## 4. Semigroup of Order-preserving or Order-reversing partial Contraction Transformation

A transformation $\alpha \in P_n$ is said to be order-preserving (order-reversing) if (for all $x, y \in Dom\alpha)x \leq y \implies \alpha \leq y\alpha(x\alpha \geq y\alpha)$.
We used similar method as in section two to obtain delightful results as follows:

**Examples**
When n=1, by investigating the elements of $ORCP_1$, there exists only 1 element $|Im\alpha| = 1$

TABLE 9.

| $Ker\alpha \| Im\alpha$ | $\{1\}$ |
|---|---|
| 1 | $\binom{1}{1}$ |

$$|P_1| = 2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ - \end{pmatrix}.$$

When n=2, the elements of $P_2$ has 8 elements.

$$|P_2| = 9 :$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ - & - \end{pmatrix}$$

$|Im\alpha| = 1$

TABLE 10.

| $Ker\alpha \vert Im\alpha$ | $\{1\}$ | $\{2\}$ |
|---|---|---|
| 1/2 | $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$ |
| 1 | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ |
| 2 | $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ |

$|Im\alpha| = 2$

TABLE 11.

| $Ker\alpha \vert Im\alpha$ | $\{1,2\}$ |
|---|---|
| 1\|2 | $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ |

When n=3, $ORCP_3$ has 45 elements.
$|Im\alpha| = 1$

Table 12.

| $Ker\alpha \vert Im\alpha$ | {1} | {2} | {3} |
|---|---|---|---|
| 1/2/3 | $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}$ | $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}$ |
| 1/2 | $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$ | $\begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}$ |
| 1/3 | $\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$ | $\begin{pmatrix} 1 & 3 \\ 3 & 3 \end{pmatrix}$ |
| 2/3 | $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ | $\begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}$ | $\begin{pmatrix} 2 & 3 \\ 3 & 3 \end{pmatrix}$ |
| 1 | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ |
| 2 | $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ | $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ |
| 3 | $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ | $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ |

$|Im\alpha| = 2$

Table 13.

| $Ker\alpha \vert Im\alpha$ | {1,2} | {2,3} | {1,3} |
|---|---|---|---|
| 1 2/3 | $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}$ | |
| 1/2 3 | $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}$ | |
| 1/2 | $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ | |
| 1/3 | $\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$ | $\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ |
| 2/3 | $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$ | $\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$ $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ | |

$|Im\alpha| = 3$

TABLE 14.

| $Ker\alpha \| Im\alpha$ | {1,2,3} |
|---|---|
| 1/2/3 | $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ |

Similarly, when n=4, $ORCP_4$ has 218 elements and when $n = 5$, $ORCP_5$ has 963 elements, hence, tables are constructed in the same way.

## 5. Combinatorial Results

In this section, tables for subsemigroups, the idempotent and nilpotent are given. Also, from the tables, sequence of pattern was formed and study. Thus, the height $h(\alpha) = p$,breath $b(\alpha) = r, w^+(\alpha) = k$ and fix of $\alpha f(\alpha) = m$ are defined as following:

(i) $F(n;p) = |\{\alpha \in S : h(\alpha) = |Im\alpha|p\}|$
(ii) $F(n;r) = |\alpha \in S : b|(\alpha) = Pw^+(\alpha) = r|$
(iii) $F(n;k) = |\alpha \in S : w^+(\alpha) = max?(Im\alpha) = k|$
(iv) $F(n;m) = |\alpha \in S : f(\alpha) = m|$

The following lemmas and theorems are required in this section:

**Lemma 5.1.** *Table 15*
*If* $S \supseteq ORCP_n$, *then* $F(n, 1) = n^2 \forall n \in \mathbb{N}$ *whenever* $P = 1$.

**Lemma 5.2.** *Table 16*
*If* $S \supseteq ORCP_n, then F(n, 1) = n^2 \forall n \in \mathbb{N}$ *whenever* $r = 1$.

**Lemma 5.3.** *Table 17*
*If* $S \supseteq ORCP_n, then F(n, 1) = n(2^n - 1) \forall n \in \mathbb{N}$ *whenever* $k = 1$.

**Theorem 5.4.** *Let* $S \supseteq ORCP_n$, *then* $|E(S)| = n(n + 3)2^{n-3}$

$$for\ n \geq 1.$$

*Proof.* Let $X_n = \{1, 2, 3, \dots, n\}$ such that $\alpha : dom(\alpha) \subseteq X_n \to Im(\alpha) \subset X_n$ and each $i \to j$, then the mapping is one-to-one i.e each element from $X_n$ taken as the domain would occur n times $(n + 3)$ and $2^{n-3}$ ways. By multiplying and summing over n, gives the result. Hence, $|E(S)| = n(n + 3)2^{n-3}$. □

**Theorem 5.5.** *Let* $S \supseteq ORCP_n$ *and* $E(ORCP_n)$ *be the idempotent in S, then*

$$F(n, 1) = n(2^{n-1}) \forall n \in N whenever p = 1.$$

*Proof.* Let $\alpha \in E|(ORCP)|$ such that $F(\alpha)\{n, 1\}$. By the contradiction property and Order-Preserving (reversing) properties for all $x \in Dom(\alpha)$, we must have $x\alpha = x$ which implies that $\begin{pmatrix} 1 & n \\ 1 & n \end{pmatrix}$ is the unique idempotent contraction mapping satisfying $F(\alpha) = \{n, 1\}$. Hence, $F(n, 1) = n(2^{n-1})\forall n \in N$ whenever p = 1. □

**Theorem 5.6.** *Let* $S \supseteq ORCP_n$ *and* $E|(ORCP_n)$ *be the set of idempotent elements in S. Then,* $F(n, 1) = 2^{n-1}\forall n \in N$ *whenever* $k = 1$ *and* $n \geq 1$.

*Proof.* Let $\alpha \in E|(ORCP_n)|$ be such that $\{1\} = F(\alpha) = Im\alpha. Then, forall x \in$ $Dom(\alpha)$, we must have $x\alpha = 1$ satisfying $F(n, 1) = n(2^{n-1})$ ways. □

**Theorem 5.7.** *Let* $S \supseteq ORCP_n$ *for all* $n = p = k = r = 1$, *then,* $|N(ORCP_n)|$ *is the set of nilpotent elements.*

*Proof.* Let $\alpha \in |N(ORCP_n)|$, it is clear that the empty map in $ORCP_n$ for all $n = p = k = r = 1$ is nilpotent. Thus, by nilpotent property and for $x \in Dom(\alpha)$, we must have $a^n = 0$. Hence the prove. □

In the tables below, the subsemigroups are shown:

TABLE 15. Height of the Image of $ORCP_n$.

| n/p | 1 | 2 | 3 | 4 | 5 | $\sum F(n;p) = \|ORCP_n$ |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 1 |
| 2 | 6 | 2 | | | | 8 |
| 3 | 21 | 22 | 2 | | | 45 |
| 4 | 60 | 124 | 32 | 2 | | 218 |
| 5 | 155 | 516 | 248 | 42 | 2 | 963 |

TABLE 16. Size of the Domain of $ORCP_n$.

| n/r | 1 | 2 | 3 | 4 | 5 | $\sum F(n;r) = \|ORCP_n$ |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 1 |
| 2 | 4 | 4 | | | | 8 |
| 3 | 9 | 23 | 13 | | | 45 |
| 4 | 16 | 74 | 92 | 36 | | 218 |
| 5 | 25 | 180 | 364 | 303 | 91 | 963 |

TABLE 17. Maximum elements in the Image of $ORCP_n$.

| n/k | 1 | 2 | 3 | 4 | 5 | $\sum F(n;k) = \|ORCP_n$ |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 1 |
| 2 | 3 | 5 | | | | 8 |
| 3 | 7 | 17 | 21 | | | 45 |
| 4 | 15 | 49 | 73 | 81 | | 218 |
| 5 | 31 | 129 | 225 | 281 | 297 | 963 |

TABLE 18. Fixed elements in the Image of $ORCP_n$.

| n/m | 1 | 0 | 2 | 3 | 4 | 5 | $\sum F(n;m) = \|ORCP_n$ |
|---|---|---|---|---|---|---|---|
| 1 | - | 1 | | | | | 1 |
| 2 | 3 | 4 | 1 | | | | 8 |
| 3 | 20 | 19 | 5 | 1 | | | 45 |
| 4 | 98 | 92 | 21 | 6 | 1 | | 218 |
| 5 | 426 | 417 | 85 | 27 | 7 | 1 | 963 |

Furthermore, the tables for Idempotent and Nilpotent elements of Order-preserving or Order-reserving partial contraction transformation are also constructed.

TABLE 19. Heights of the Image of Idempotent elements in $ORCP_n$.

| n/p | 1 | 2 | 3 | 4 | 5 | $\sum F(n;p) = \|E(ORCP_n)$ |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 1 |
| 2 | 4 | 1 | | | | 5 |
| 3 | 12 | 5 | 1 | | | 18 |
| 4 | 32 | 17 | 6 | 1 | | 56 |
| 5 | 80 | 49 | 23 | 7 | 1 | 160 |

TABLE 20. Size of the domain of Idempotent elements in $ORCP_n$.

| n/r | 1 | 2 | 3 | 4 | 5 | $\sum F(n;r) = \|E(ORCP_n)$ |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 1 |
| 2 | 2 | 3 | | | | 5 |
| 3 | 3 | 9 | 6 | | | 18 |
| 4 | 4 | 18 | 24 | 10 | | 56 |
| 5 | 5 | 30 | 60 | 50 | 15 | 160 |

Table 21. Maximum elements in the Image of domain of Idempotent in $ORCP_n$.

| n/k | 1 | 2 | 3 | 4 | 5 | $\sum F(n;k) = \|E(ORCP_n)\|$ |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 1 |
| 2 | 2 | 3 | | | | 5 |
| 3 | 4 | 6 | 8 | | | 18 |
| 4 | 8 | 12 | 16 | 20 | | 56 |
| 5 | 16 | 24 | 32 | 40 | 48 | 160 |

Table 22. Fixed elements in the Image of Idempotent in $ORCP_n$.

| n/m | 1 | 2 | 3 | 4 | 5 | $\sum F(n;m) = \|E(ORCP_n)\|$ |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 1 |
| 2 | 4 | 1 | | | | 5 |
| 3 | 12 | 5 | 1 | | | 18 |
| 4 | 32 | 17 | 6 | 1 | | 56 |
| 5 | 80 | 49 | 23 | 7 | 1 | 160 |

Hence, the following are results of elements further generated for $n = 6$ to $n = 20$ respectively as:
$432, 1120, 2816, 6912, 16640, 39424, 92160, 212992, 487424, 1105920, 2222490368,$
$5570560, 12386304, 2794048$ and $60293120$

Table 23. Heights of the Image of Nilpotent elements in $ORCP_n$.

| n/p | 1 | 2 | 3 | 4 | 5 | $\sum F(n;p) = \|N(ORCP_n)\|$ |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | 2 | | | | | 2 |
| 3 | 9 | 4 | | | | 13 |
| 4 | 28 | 38 | 5 | | | 71 |
| 5 | 75 | 190 | 54 | 4 | | 323 |

Table 24. Size of the domain of Nilpotent elements in $ORCP_n$.

| n/r | 1 | 2 | 3 | 4 | 5 | $\sum F(n;r) = \|N(ORCP_n)\|$ |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | 2 | | | | | 2 |
| 3 | 6 | 7 | | | | 13 |
| 4 | 12 | 36 | 23 | | | 71 |
| 5 | 20 | 106 | 140 | 57 | | 323 |

Table 25. Maximum elements in the Image of Nilpotent elements in $ORCP_n$.

| n/k | 1 | 2 | 3 | 4 | 5 | $\sum F(n;k) = \|N(ORCP_n)\|$ |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | 1 | 1 | | | | 2 |
| 3 | 3 | 5 | 5 | | | 13 |
| 4 | 7 | 18 | 23 | | | 71 |
| 5 | 15 | 49 | 77 | 93 | 89 | 323 |

## Conclusion

The combinatorial properties of subsemigroup of Order-Preserving and Order-Decreasing and Order-Preserving or Order-Reversing partial contraction transformations $ODCP_n$ and $ORCP_n$ were studied together with their Idempotent and Nilpotent. Thus, the functions such as $F(n,p), F(n,k), F(n,m)$ and $F(n,r)$were used to carry out the order of the semigroup. Based on the results obtained, the work can be extended to other Subsemigroups of Order-Decreasing and Oder-Reversing Partial Contraction Transformation ($DRCP_n$) for further study. $\square$.

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