



On the Nilpotent Fuzzy Subgroups of the Abelian Type:

$$\mathbb{Z}_{32} \times \mathbb{Z}_{2^n}, n \geq 5$$

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ABSTRACT

The extension of the finite nilpotent groups is now being diversified. As such, results up to two dimensions are now obtainable. In this paper, the fuzzy subgroups of the nilpotent abelian structure given by: $\mathbb{Z}_{32} \times \mathbb{Z}_{2^n}$, the cartesian product of two abelian subgroups of orders 2^n and 32 respectively for every integer $n > 5$ have been carefully studied and the explicit formulae for its number distinctly given.

1. INTRODUCTION

Before this time, several methods, techniques and approaches were used for the classification of which some are obtainable in [5], [6] and [8]. For example, the natural equivalence relation was introduced in [10]. In this work, solution to the problems defined is given using the technique which is known as the normal Inclusion-Exclusion Principle. The aftermath of this method gives rise to some forms of common relations. This paves way to reasonable results and the

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problems are completely solved easily by using some simple computations as required. In the process of our computation, the use of GAP (Group Algorithm and Programming) was actually applied.

1.1. Basic Definitions and Terms. Let (H, \cdot, e) be a group which has an identity denoted by e . Collection of all fuzzy subsets of H , and denote this by $P(H)$.

Definition 1.1. [7]: An element $\tau \in P(H)$ would be known as a fuzzy subgroup of H provided that each of the conditions given below hold and are strictly maintained: $\tau(mn) \geq \min\{\tau(m), \tau(n)\}$ for all $m, n \in H$ and that $\tau(m^{-1}) \geq \tau(m)$ whenever $m \in H$. Now since $(m^{-1})^{-1} = m$, it implies that $\tau(m^{-1}) = \tau(m)$, as $m \in H$. Following similar definitions, $\tau(e) = \sup \tau(H)$.

We shall denote the subgroups of H which are maximal in their own right in H by M_1, M_2, \dots, M_t . Now let $h(H)$ be the number of chains of subgroups of H whose ends lie in H .

Theorem 1.2. [7]: *The lattice under the usual ordering of fuzzy set inclusion is hereby represented by the set denoted by $G(H)$.*

This contains every subgroup of H which are fuzzified and referred to as the fuzzy subgroup lattice of H . In the same vein define as follows:

Definition 1.3. [7]: The level subset: $\tau H_\delta = \{m \in H | \tau(m) \geq \delta\}$ for each $\delta \in [0, 1]$.

By this definition, the fuzzy subgroups of a finite p -group H are thus, characterized, according to the subsets. Note that, τ is a fuzzy subgroup of H provided the level subsets are contained in H .

In practice however, certain relations on $P(H)$ can also be used naturally, when dealing with the concept of classifying the subgroups which are fuzzified for every finite q -group H (see [9] and [10]). One of them is defined by: $\tau \sim \gamma$ iff $(\tau(m) > \gamma(n) \iff \tau(m) > \gamma(n), \forall m, n \in H)$. Also, two fuzzy subgroups τ, γ of H are said to be distinct if $\tau \not\sim \gamma$.

As a result of this development, let H be a finite p -group. Now, suppose that $\tau : H \rightarrow [0, 1]$ is fuzzy for H . Set $\tau(H) = \{\beta_1, \beta_2, \dots, \beta_k\}$ with the assumption that $\beta_1 > \beta_2 > \dots > \beta_k$. Then, H is determined by τ following the chain given below:

$$\tau H_{\beta_1} \subset \tau H_{\beta_2} \subset \dots \subset \tau H_{\beta_k} = H \quad (a)$$

Also, we have that:

$$\tau H(a) = \beta_t \iff t = \max\{r/a \in \tau H_{\beta_r}\} \iff a \in \tau H_{\beta_t} \setminus \tau H_{\beta_{t-1}},$$

for any $a \in H$ and $t = 1, \dots, k$. Here, we set $\tau H_{\beta_0} = \phi$, conventionally as required.

2. THE TECHNIQUES

Here, we clarify the technique commonly applied for the process of getting the chains of fuzzy subgroups of an arbitrary finite p -group H . Let V_1, V_2, \dots, V_k be the subgroups which are maximal for H . Denote by $h(H)$, the number of chains of subgroups of H whose ends lie in H . What we do is to apply the method which computes $h(H)$, by simply using the technique of the Inclusion-Exclusion Principle. It follows from here that:

$$(1) \quad h(H) = 2 \left(\sum_{b=1}^k h(V_b) - \sum_{1 \leq b_1 < b_2 \leq k} h(V_{b_1} \cap V_{b_2}) + \dots + (-1)^{k-1} h \left(\bigcap_{b=1}^k V_b \right) \right)$$

In [8], (1) was used to obtain the explicit formulas for some positive integers n .

Theorem 2.1. [10]: *The number of every distinct fuzzy subgroups of a p -group which is finite and possesses the order that is equal to p^n and having maximal subgroups whose one is cyclic is given by: (i) $h(\mathbb{Z}_{p^n}) = 2^n$
(ii) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}(2 + (n-1)p)$*

3. THE FUZZY SUBGROUPS FOR $\mathbb{Z}_{16} \times \mathbb{Z}_{16}$

The abelian structure possesses three similar maximal subgroups $A \sim \mathbb{Z}_8 \times \mathbb{Z}_{16}$

Lemma 3.1. [2]: *Let G be abelian such that $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}, n \leq m$ then, there exists $p+1$ maximal subgroups in G .*

Lemma 3.2. [1]: *Let $Q = \mathbb{Z}_8 \times \mathbb{Z}_{2^n}$. Then, for $n > 2, h(Q) = \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24)$*

Proposition 3.3. [3]: *Let $G = \mathbb{Z}_{16} \times \mathbb{Z}_{16}, h(G) = 6400$*

Proof: Using Lemma 3.2 and by carefully applying equation (1), we have that: $\frac{1}{2}h(G) = 3h(\mathbb{Z}_8 \times \mathbb{Z}_{16}) - 3h(\mathbb{Z}_8 \times \mathbb{Z}_{16}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{16})$. Thus $h(G) = 2h(\mathbb{Z}_8 \times \mathbb{Z}_{2^4}) = 6400$ \square .

Proposition 3.4. : *Suppose that $G = \mathbb{Z}_{2^5} \times \mathbb{Z}_{16}$. Then, $h(G) = 27136$*

Proof: By (1), $\frac{1}{2}h(G) = h(\mathbb{Z}_{16} \times \mathbb{Z}_{16}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^5}) - h(\mathbb{Z}_8 \times \mathbb{Z}_{2^4}), h(G) = 27136$ \square .

Lemma 3.5. : *If $G = \mathbb{Z}_{2^6} \times \mathbb{Z}_{16}$, then $h(G) = 95488$*

Proof: Using (1), $\frac{1}{2}h(G) = h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^5}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^6}) - h(\mathbb{Z}_8 \times \mathbb{Z}_{2^5}) \therefore h(G) = 95488$ \square .

Theorem 3.6. *Suppose that $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_{16}$.*

Then $h(G) = \frac{1}{3}(2^{n+2})(n^3 + 12n^2 + 17n - 24) + 2^n(200)$

$$+ \frac{1}{3}(2^{n+1}) \sum_{k=1}^{n-5} [(n-k)^3 + 12(n-k)^2 + 17(n-k) - 24]$$

Proof: The three maximal subgroups, one is isomorphic to $\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}}$, while two are isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_{2^n}$ for $n > 4$.

We have: $h(G) = 2[h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^n}) - h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}})]$
 $= \frac{1}{3}(2^{n+2})(n^3 + 12n^2 + 17n - 24) + 2^n(200)$

$$+ \frac{1}{3}(2^{n+1}) \sum_{k=1}^{n-5} [(n-k)^3 + 12(n-k)^2 + 17(n-k) - 24]$$

□

Proposition 3.7. (i)[2]: *Suppose that $G = \mathbb{Z}_4 \times \mathbb{Z}_{2^n}$, $n \geq 2$. Then, $h(G) = 2^n[n^2 + 5n2]$*

Proposition 3.8. (ii)[3]: *Let $G = \mathbb{Z}_8 \times \mathbb{Z}_{2^n}$. Then $h(G) = \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24)$*

Proposition 3.9. (iii)[4]: *Suppose that $G = \mathbb{Z}_{16} \times \mathbb{Z}_{2^n}$. Then $h(G) = \frac{1}{3}(2^{n+2})(n^3 + 12n^2 + 17n - 24) + 2^n(200)$*

$$+ \frac{1}{3}(2^{n+1}) \sum_{k=1}^{n-5} [(n-k)^3 + 12(n-k)^2 + 17(n-k) - 24].$$

Lemma 3.10. *Let $G = \mathbb{Z}_{32} \times \mathbb{Z}_{2^5}$, then $h(G) = 13124$*

Proof: $\frac{1}{2}h(G) = 3h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^4})3h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^4}) + h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^4}) = h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^5}) = 6562$,
 by (iii) above. $\therefore h(G) = 2 \times 6562 = 13124$ □

Lemma 3.11. [3]: *Let $G = \mathbb{Z}_{32} \times \mathbb{Z}_{2^6}$, then, $\frac{1}{2}h(G) = h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^5}) + 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^6})$
 $- h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^6}) - 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^5}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^5}) = h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^5})$
 $+ h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^6}) - h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^5})$.
 $\therefore h(G) = 2[h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^5}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^6}) - h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^5})]$*

Lemma 3.12. [3]: *Let $G = \mathbb{Z}_{32} \times \mathbb{Z}_{2^7}$, then, $\frac{1}{2}h(G) = h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^6}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^7})$
 $- 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^6})$
 $\therefore h(G) = 2[h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^6}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^7}) - 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^6})]$*

Lemma 3.13. [4]: *Let $G = \mathbb{Z}_{32} \times \mathbb{Z}_{2^8}$, then, $\frac{1}{2}h(G) = h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^7}) + 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^8})$
 $- h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^8}) - 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^7}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^7}) = h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^7})$
 $+ h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^8}) - h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^7})$.*

Lemma 3.14. *Let $G = \mathbb{Z}_{32} \times \mathbb{Z}_{2^n}$ Then*

$$h(G) = 2[h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}})] + \sum_{k=1}^{n-6} 2^{k+1} h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1-k}}).$$

Proof: For any integer $n > 5$, we have three maximal subgroups which are isomorphic to $\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1}}$, $\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}$ and $\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}}$.

Then, $\frac{1}{2}h(G) = h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1}}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) - h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}})$.

$$\begin{aligned} \therefore h(G) &= 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) - 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}}) \\ &= 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) + 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}}) - 4h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-2}}) + 4h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-2}}) \\ &= 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) + 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}}) + 4h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-2}}) - 8h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-3}}) \\ &\quad + 8h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-3}}). \end{aligned}$$

hence $h(G) = 2[h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}})]$

$$\begin{aligned} &\quad + \sum_{k=1}^{n-6} 2^{k+1} h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1-k}}) 2^{n-5} h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^5}) + 2^{n-5} h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^5}) \\ &= 2[h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}})] \\ &\quad + \sum_{k=1}^{n-6} 2^{k+1} h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1-k}}) - 13124 + 13124 \\ &= 2[h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}})] + \sum_{k=1}^{n-6} 2^{k+1} h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1-k}}). \end{aligned}$$

as required □.

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REFERENCES

- [1] ADEBISI S. A., OGIUGO M. AND ENIOLUWAFE M. (2020). Computing the number of distinct fuzzy subgroups for the nilpotent p-Group of $D_{2^n} \times C_4$. *International J. Math. Combin.* **1**(2020), 86-89.
- [2] ADEBISI S. A., OGIUGO M. AND ENIOLUWAFE M. (2020). Determining the number of distinct fuzzy subgroups for the abelian structure: $\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}$, $n > 2$. *Trans. NAMP.* **11**(Jan.-June, 2020 Issue), 5-6.

- [3] ADEBISI S. A., OGIUGO M. AND ENIOLUWAFE M. (2020). The fuzzy subgroups for the abelian structure: $\mathbb{Z}_8 \times \mathbb{Z}_{2^n}; n > 2$. *Journal of the Nigerian Mathematical Society*. **39**(2), 167-171.
- [4] ADEBISI S. A., OGIUGO M. AND ENIOLUWAFE M. (2020). Fuzzy subgroups for the cartesian product of the abelian structure $\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}; n > 3$ (Submitted).
- [5] MASHINCHI M., MUKAIDONO M. (1992). A classification of fuzzy subgroups. *Ninth Fuzzy System Symposium, Sapporo, Japan*. 649-652.
- [6] TARNAUCEANU MARIUS (2009). The number of fuzzy subgroups of finite cyclic groups and Delannoy numbers. *European J. Combin.* **30**, 283-289.
- [7] TARNAUCEANU MARIUS (2013). Classifying fuzzy subgroups for a class of finite p -groups. *ALL-CUZA Univ. Iasi, Romania*, 30-39.
- [8] TARNAUCEANU MARIUS (2012). Classifying fuzzy subgroups of finite nonabelian groups. *Iran J. Fussy Systems*. **9**, 33-43.
- [9] TARNAUCEANU M. AND BENTEA L. (2008). A note on the number of fuzzy subgroups of finite groups. *Sci. An. Univ."ALL.Cuza" Iasi, Matt.* **54**, 209-220.
- [10] TARNAUCEANU M. AND BENTEA L. (2008). On the number of fuzzy subgroups of finite abelian groups. *Fuzzy Sets and Systems*. **159**, 1084-1096.