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Third Derivative Block Hybrid Obrechkoff Methods

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Abstract

This paper presents a family of Third Derivative Block Hybrid Obrechkoff Methods of Adams-type for solving stiff Initial Value Problems (IVPs). Multistep collocation techniques, with power series as basis function, are employed for the derivation of the continuous form of the hybrid methods. The continuous Hybrid Obrechkoff Methods are used to generate several implicit Hybrid Integrators that are expressed in block form and are applied as block integrators at both grid and off-grid points. The Block Methods are of order $2k+3$ and their performance on some problems considered shows the accuracy and efficiency of the methods. The basic properties of the developed methods are also discussed.

1. Introduction

Consider the Initial Value Problem (IVP) of the form:

$$y' = f(t, y), t \in [t_0, t_n]. \tag{1}$$

Stiff problems are IVPs of the form (1) with Jacobian whose eigenvalues have negative real parts. The function satisfies the Lipschitz condition as given in [13]. This type of equations arise frequently in several engineering, applied sciences and economics problems. An essential property of the majority of computational

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methods for the solution of (1) is that of discretization, that is: seeking an approximate solution, not on the continuous interval $t_0 \leq t \leq t_n$ but on the discrete point set t_n. Several authors have proposed various methods including Multiderivative linear multistep methods (MLMM) for the solution of equation (1) such as in [2], [3], [4], [5], [7], [8], [9], [10], [15], [18], [20] to mention but a few. Multi-derivative linear multistep method was first proposed by Obrechkoff (see [17]) and the k-step Obrechkoff method using the first l derivatives of y is given in the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = \sum_{i=0}^{l} h^i \sum_{j=0}^{k} \beta_{ij} y_{n+j}^{(i)} \tag{2}$$

[17] first advocated the use of Obrechkoff method, in which the method depends on higher derivatives of y for the solution of (1). The Obrechkoff method has been modified to hybrid Obrechkoff method to further circumvent Dahlquist barrier theorem (see [6]), where v are the off-step points, in the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = \sum_{i=0}^{l} h^i \left[\sum_{j=0}^{k} \beta_{ij} y_{n+j}^{(i)} + \sum_{v=0}^{k} \beta_v y_{n+v}^{(i)} \right] \tag{3}$$

Incorporating these off-step points into the derivation process (3) converts the $k-$ step Obrechkoff Multistep Method to the $k-$ step Hybrid Obrechkoff method. This method was motivated by the desire to increase the order without increasing the step number k (see [16]). [10] proposed a family of third derivative methods which is an extension of [8] second derivative LMM. Recently, a family of high order third derivative hybrid Obrechkoff method for the numerical integration of stiff IVPs in ODEs was presented by [20]. These methods were A-stable for $k = 1, 2, 3$ and A(α) stable for $k = 4$ up to 18 and unstable for $k \geq 19$. Inspite of these advantages, these methods are implemented in predictor-corrector mode, and Taylor series expansion are adopted to supply starting values. The setback of the predictor-corrector methods are that they are very costly to implement, longer computer time and greater human effort with reduced order of accuracy which affects the accuracy of the method. In this paper, a family of third derivative block hybrid Obrechkoff method of Adams type of the form:

$$y_{n+k} = y_{n+k-1} + \sum_{i=1}^{3} h^i \left[\sum_{j=0}^{k} \beta_{ij} y_{n+j}^{(i)} + \sum_{j=1}^{k} \beta_{v_j} y_{n+v_j}^{(i)} \right] \tag{4}$$

are constructed and applied to stiff Initial Value Problems (IVPs). It should be noted that block methods has the advantage of being self-starting and is capable of giving evaluations at different grids points without overlapping as in the predictor-corrector method, hence does not require the development of separate predictors, or starting values (see [1], [3] and [11, 12]).

2. Derivation of TDBHOM

This section considers the Multistep collocation approach of [21] for the derivation of the k-step Hybrid Obrechkoff Methods and extends to third derivative of the form

$$y(t) = y_{n+k-1} + h\left[\sum_{j=0}^{k}\beta_j f_{n+j} + \sum_{j=1}^{k}\beta_{v_j} f_{n+v_j}\right] + h^2\phi_k g_{n+k} + h^3\rho_k r_{n+k} \tag{5}$$

where $\alpha_j(t)$, $\beta_j(t)$, $\phi_k(t)$ and $\rho_k(t)$ are parameters of the method which are to be determined uniquely. The exact solution is approximated by seeking a continuous solution of the form

$$y(t) = \sum_{j=0}^{2k+3} b_j t^j \tag{6}$$

where b_j represent the unknown coefficients to be determined and $v = \frac{1}{2}$. The class of method is constructed by imposing the following conditions:

$$y(t_{n+i}) = y_{n+i}, \quad i = k-1 \tag{7}$$

$$y'(t_{n+i}) = f_{n+i}, \quad i = 0\left(\frac{1}{2}\right)k \tag{8}$$

$$y''(t_{n+i}) = g_{n+i}, \quad i = k \tag{9}$$

$$y'''(t_{n+i}) = r_{n+i}, \quad i = k \tag{10}$$

The resulting system of $2k+4$ equations from equations (7)-(10) are solved using Gauss Elimination Method to obtain b_j and the values of b_j are substituted into (6) to form the Continuous Hybrid Obrechkoff Formula (CHOF) expressed in the form:

$$y(t) = \sum_{j=0}^{k-1}\alpha_j(t) y_{n+j} + h\sum_{j=0}^{k}\beta_j(t) f_{n+j} + h^2\phi_k(t) g_{n+k} + h^3\rho_k(t) r_{n+k} \tag{11}$$

where k is the step length, $\alpha_j(t)$, $\beta_j(t)$, $\phi_k(t)$ and $\rho_k(t)$ are the determined continuous coefficients. The main scheme is generated by evaluating CHOF (11) at t_{n+k} while the additional schemes are generated at the off-step points t_{n+v} and t_n . The developed main and additional schemes are combined and implemented simultaneously as a single block hybrid Obrechkoff method for the numerical integration of stiff IVPs (1). Setting $k = 1$ in equations (7)-(10), we obtained 6 equations which are solved simultaneously to obtain $b_j, j = 0(1)5$. Substituting the values of $b_j, j = 0(1)5$ into equation (6) and simplified, it will be expressed in the form of equation (11) where

$$\alpha_0(t) = 1$$
$$\beta_0(t) = (t - t_n) - \tfrac{5}{2}\tfrac{(t-t_n)^2}{h} + 3\tfrac{(t-t_n)^3}{h^2} - \tfrac{7}{4}\tfrac{(t-t_n)^4}{h^3} + \tfrac{2}{5}\tfrac{(t-t_n)^5}{h^4}$$
$$\beta_{\frac{1}{2}}(t) = 8\tfrac{(t-t_n)^2}{h} - 16\tfrac{(t-t_n)^3}{h^2} + 12\tfrac{(t-t_n)^4}{h^3} - \tfrac{16}{5}\tfrac{(t-t_n)^5}{h^4}$$
$$\beta_1(t) = -\tfrac{11}{2}\tfrac{(t-t_n)^2}{h} + 13\tfrac{(t-t_n)^3}{h^2} - \tfrac{41}{4}\tfrac{(t-t_n)^4}{h^3} + \tfrac{14}{5}\tfrac{(t-t_n)^5}{h^4}$$
$$\phi_1(t) = 2(t - t_n)^2 - 5\tfrac{(t-t_n)^3}{h} + \tfrac{17}{4}\tfrac{(t-t_n)^4}{h^2} - \tfrac{6}{5}\tfrac{(t-t_n)^5}{h^3}$$
$$\rho_1(t) = -\tfrac{1}{4}h(t - t_n)^2 + \tfrac{2}{3}(t - t_n)^3 - \tfrac{5}{8}\tfrac{(t-t_n)^4}{h} + \tfrac{1}{5}\tfrac{(t-t_n)^5}{h^2}$$

Interpolating equation (11) at t_{n+1} to generate the main method, we obtained

$$y_{n+1} = y_n + h\left[\frac{3}{20}f_n + \frac{4}{5}f_{n+\frac{1}{2}} + \frac{1}{20}f_{n+1}\right] + \frac{1}{20}h^2 g_{n+1} - \frac{1}{120}h^3 r_{n+1} \tag{12}$$

Also interpolating equation (11) at $t_{n+\frac{1}{2}}$ to generate the additional method, we obtained

$$y_{n+\frac{1}{2}} = y_n + h\left[\frac{49}{320}f_n + \frac{13}{20}f_{n+\frac{1}{2}} - \frac{97}{320}f_{n+1}\right] + \frac{33}{320}h^2 g_{n+1} - \frac{23}{1920}h^3 r_{n+1} \tag{13}$$

Equations (12) and (13) are combined and implemented simultaneously as a single method known as the one-step Third Derivative Block Hybrid Obrechkoff Method (1TDBHOM) for the numerical integration of stiff IVPs. Tables (5-7) in the appendix section shows the discrete coefficients of $k-$ step TDBHOM for $k = 1(1)4$. From equation (5), the coefficients $\alpha_j, \beta_j, j = 0(\frac{1}{2})k, \phi_k$ and ρ_k are defined. The 1TDBHOM can be represented in a matrix block form as

$$A_{(1)}Y_\varpi = A_{(0)}Y_{\varpi-1} + hB_{(1)}F_\varpi + hB_{(0)}F_{\varpi-1} + h^2C_{(1)}G_\varpi + h^3D_{(1)}R_\varpi \tag{14}$$

where

$$Y_\varpi = \begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \end{pmatrix}; Y_{\varpi-1} = \begin{pmatrix} y_{n-\frac{1}{2}} \\ y_n \end{pmatrix}; F_\varpi = \begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \end{pmatrix}; F_{\varpi-1} = \begin{pmatrix} f_{n-\frac{1}{2}} \\ f_n \end{pmatrix}$$

$$G_{\varpi} = \begin{pmatrix} g_{n+\frac{1}{2}} \\ g_{n+1} \end{pmatrix}; R_{\varpi} = \begin{pmatrix} r_{n+\frac{1}{2}} \\ r_{n+1} \end{pmatrix}$$

The 2 by 2 matrices $A_{(0)}, A_{(1)}, B_{(0)}, B_{(1)}, C_{(1)}, D_{(1)}$ of the 1TDBHOM (12) and (13) are defined as follows:

$$A_{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A_{(0)} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$B_{(1)} = \begin{pmatrix} \frac{13}{20} & -\frac{97}{320} \\ \frac{4}{5} & \frac{1}{20} \end{pmatrix}$$

$$B_{(0)} = \begin{pmatrix} 0 & -\frac{49}{320} \\ 0 & \frac{3}{20} \end{pmatrix}$$

$$C_{(1)} = \begin{pmatrix} 0 & \frac{33}{320} \\ 0 & \frac{1}{20} \end{pmatrix}$$

$$D_{(1)} = \begin{pmatrix} 0 & -\frac{23}{1920} \\ 0 & -\frac{1}{120} \end{pmatrix}$$

3. Analysis of TDBHOM

3.1. **Order and Error Constant of the Method.** Following [12] and [15], a method was proposed for finding the order η and error constant $W_{\eta+1}$ of the block method (14) by first expanding $y-, f-, g-$ and $r-$ functions by Taylors series expansion about t and then comparing the coefficients of h . It is established from the computations that the derived k-step Block Hybrid Obrechkoff Method have order η and error constant $W_{\eta+1}$ [see appendix Table 8].

3.2. **Zero Stability.** A numerical method is said to be zero-stable if the roots R_j , $j = 1, 2, \ldots, N$ of the first characteristic polynomial $\rho(R)$ satisfies $|R_j| \leq 1, j = 1, 2, \ldots, N$ and those roots with $|R_j| = 1$ is simple ([15]). Applying the above conditions to the derived block method, the first characteristic polynomial $\rho(R) = 0$ is calculated as

$$\rho(R) = \det(RA_{(1)} - A_{(0)}) = R(R-1)$$

The 1TDBHOM is found to be zero-stable since $\rho(R) = 0$ satisfies $|R_j| \leq 1$ for $j = 1, 2$.

3.3. **Convergence.** According to [13], a numerical method converges if it is consistent and zero-stable. Since TDBHOM for $k = 1$ is of order $5 > 1$, then it is consistent and we have established earlier that the method satisfies the conditions of zero-stability. Therefore, the block method (14) converges and the TDBHOM for $k = 1(1)4$ are convergent methods.

3.4. **Stability of TDBHOM.** Applying the derived TDBHOM for $k = 1(1)4$ to the test equations

$$y' = \lambda y, \quad y'' = \lambda^2 y, \quad y''' = \lambda^3 y, \lambda < 0$$

yields

$$Y_{\varpi} = Q(z) Y_{\varpi-1}, \quad z = \lambda h$$

where $Q(z)$ is the amplification matrix given by

$$Q(z) = \frac{A_{(0)} + zB_{(0)}}{A_{(1)} + zB_{(1)} + z^2 C_{(1)} + z^3 D_{(1)}}$$

The matrix $Q(z)$ has eigenvalues $\zeta_1, \zeta_2, \cdots, \zeta_k$ where the dominant eigenvalue ζ_k is the stability function with real coefficient and denoted as $S(z)$ given in Table 9. The one-step and 4-step TDBHOM are L-stable methods while the 2-step and 3-step TDBHOM are L(0)-stable in nature. They are L-stable methods since the stability region covers the entire left plane of the graph (A-stability) and the limit of the stability function is zero as $z \to \infty$. Some of the derived methods are L(0)-stable since the stability region do not cover the entire left plane of the graph at an angle α and the limit of the stability function is zero as $z \to \infty$. The stability region for $k-$ step TDBHOM for $k = 1(1)4$ are given in Figures $(1-4)$.

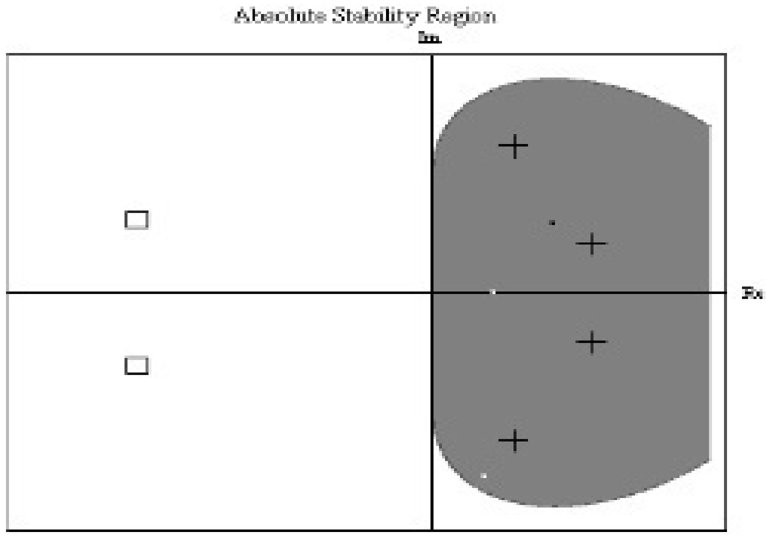

Figure 1: Stability Region of one-step TDBHOM

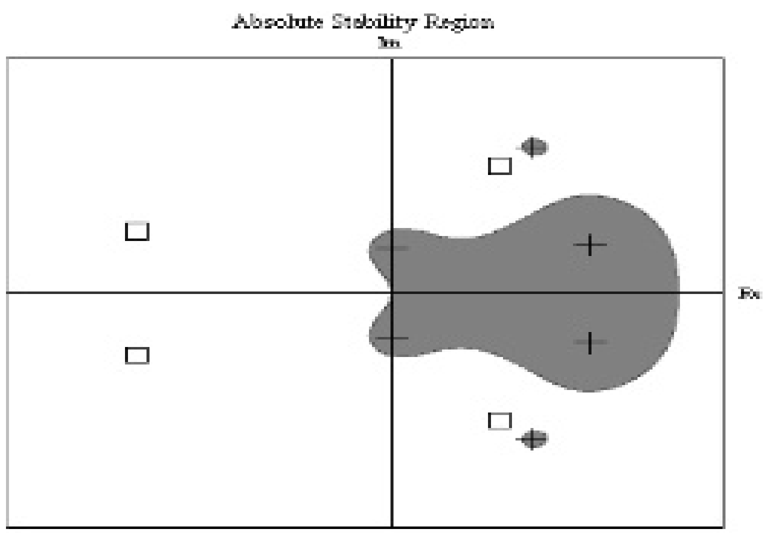

Figure 2: Stability Region of two-step TDBHOM

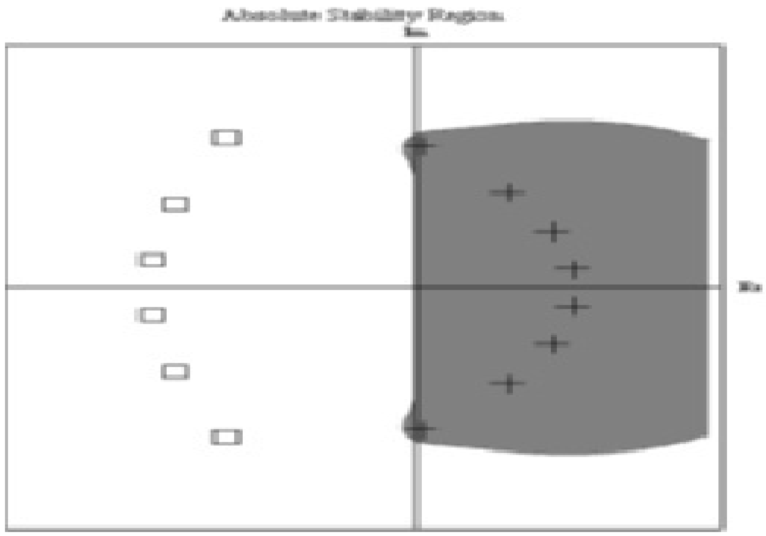

Figure 3: Stability Region of three-step TDBHOM

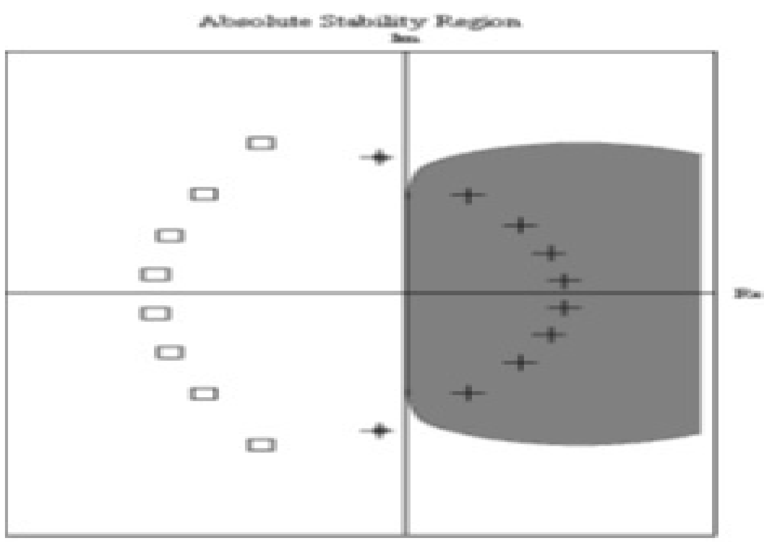

Figure 4: Stability Region of four-step TDBHOM

4. Numerical Results

This section deals with some numerical experiments to evaluate the accuracy of the proposed methods TDBHOM executed in Mathematica 9.0. We solved some systems of IVPs and compared the obtained results with some existing methods in literature. Existing methods considered for comparison are Third Derivative Hybrid Linear Multistep Methods (TDHLMM) of order 6 in [20], Third Derivative Backward Differentiation Formula (TDBDF) of order 6 in [19] and Third Derivative Multistep Methods (TDMM) in [10] used to solve same set of IVPs. TDHLMM implemented in predictor-corrector mode:

$$y_{n+\frac{1}{2}} = \frac{1}{16}\left(y_n + 15y_{n+1}\right) - \frac{7}{16}hf_{n+\frac{1}{2}} + \frac{3}{32}h^2g_{n+1} - \frac{1}{96}h^3r_{n+1}, p = 4$$

$$y_{n+1} = y_n + h\left(\frac{1}{10}f_n + \frac{4}{5}f_{n+\frac{1}{2}} + \frac{1}{10}f_{n+1}\right) + \frac{1}{60}h^3r_{n+1}, p = 6$$

TDBDF:

$$y_{n+4} = \frac{1}{5845}\left(-27y_n + 256y_{n+1} - 1296y_{n+2} + 6912y_{n+3}\right) + \frac{996}{1169}hf_{n+4} - \frac{360}{16}h^2g_{n+4} + \frac{288}{5845}h^3r_{n+4}$$

TDMM:

$$y_{n+3} = y_{n+2} + h\left(\frac{1}{810}f_n - \frac{7}{480}f_{n+1} + \frac{1}{3}f_{n+2} + \frac{8813}{12960}f_{n+3}\right) + \frac{83}{432}h^2g_{n+3} + \frac{17}{720}h^3r_{n+3}$$

Tables 1-4 shows the numerical results for the stiff Initial Values Problems.

Example 1: A stiff system of equations

$$y_1' = y_2$$
$$y_2' = -100y_1 - 101y_2$$

with initial conditions

$$y_1(0) = 1.01, \quad y_2(0) = -2, \quad t \in [0, 15]$$

whose exact solution is given as

$$y_1(t) = 0.01e^{-100t} + e^{-t}, \quad y_2(t) = e^{-100t} - e^{-10t}$$

Example 2: A stiff system of equations

$$y_1' = -8y_1 + 7y_2$$

$$y_2' = 42y_1 - 43y_2$$

with initial conditions

$$y_1(0) = 1, \quad y_2(0) = 8, \quad t \in [0, 15]$$

whose exact solution is given as

$$y_1(t) = 2e^{-t} - e^{-50t}, \quad y_2(t) = 2e^{-t} + 6e^{-50t}$$

Examples 1 and 2 were integrated using step size $h = 10^{-4}$ to aid in comparing with other methods in literature shown in Tables 1 and 2. It was discovered that ITDBHOM was better in accuracy than the three existing methods compared with.

Example 3
Consider the non-linear system of first order Ordinary Differential Equations, compared with Second Derivative Multistep Method (SDMM) of [14] and Multi-derivative Hybrid Implicit Runge Kutta metheod (MHIRK) of [3]

$$y_1' = \lambda y_1 + y_2^2$$
$$y_2' = -y_2$$

with initial conditions

$$y_1(0) = -1/(\lambda + 2), \quad y_2(0) = 1$$

where $\lambda = 10000$,the exact solution is given as

$$y_1(t) = e^{-2t}/(\lambda + 2), \quad y_2(t) = e^{-t}$$

From Table 3, the result obtained with the derived one-step Third Derivative Block Hybrid Obrechkoff Method (1TDBHOM) using step size $h = 10^{-1}$ is found superior to MHIRK in [3] that used same step length and same order of 5. 1TDBHOM is also better than SDMM in [14] that used $h = 10^{-4}$.

Example 4: A stiff system of equations

$$y_1' = -1002y_1 + 1000y_2$$
$$y_2' = y_1 - y_2 - y_2^2$$

with initial conditions

$$y_1(0) = 1, \quad y_2(0) = 1$$

where $\lambda = 10000$,the exact solution is given as

$$y_1(t) = e^{-2t}, \quad y_2(t) = e^{-t}$$

In Table 4, the numerical result of the method was shown using step size $h = 10^{-3}$. The numerical result obtained is quite close to exact solution, thus the derived method can integrate stiff initial value problems.

Table 1: A Comparison of absolute errors of methods for Example 1

t	Methods	Error $\lvert y(t_n) - y_n \rvert$
5	1TDBHOM	4.9622×10^{-15}
	[20]	1.4321×10^{-10}
	[19]	6.7386×10^{-3}
	[10]	6.7383×10^{-3}
10	1TDBHOM	4.4427×10^{-15}
	[20]	1.9299×10^{-12}
	[19]	4.5405×10^{-5}
	[10]	4.5405×10^{-5}
15	1TDBHOM	3.9133×10^{-15}
	[20]	1.9506×10^{-14}
	[19]	3.0593×10^{-7}
	[10]	3.0593×10^{-7}

Table 2: A Comparison of absolute errors of methods for Example 2

t	Methods	Error $\lvert y(t_n) - y_n \rvert$
5	1TDBHOM	5.3724×10^{-15}
	[19]	1.5472×10^{-2}
	[10]	1.5476×10^{-2}
10	1TDBHOM	7.9971×10^{-15}
	[19]	9.0808×10^{-5}
	[10]	9.0808×10^{-5}
15	1TDBHOM	8.7149×10^{-15}
	[19]	6.1186×10^{-7}
	[10]	6.1186×10^{-7}

Table 3: Numerical Results for Example 3

t	y_i	Error $\|y(t_n) - y_n\|$ [14]	Error $\|y(t_n) - y_n\|$ [3]	Error $\|y(t_n) - y_n\|$ 1TDBHOM
3	y_1	$2.478147E-11$	$3.06450E-15$	$9.76732E-16$
	y_2	$2.471093E-06$	$3.07825E-10$	$9.81115E-11$
5	y_1	$3.450271E-14$	$9.35475E-17$	$2.9807E-17$
	y_2	$2.304573E-08$	$6.94326E-11$	$2.21299E-11$
10	y_1	$3.456372E-18$	$8.49412E-21$	$6.46422E-20$
	y_2	$3.150734E-10$	$9.35666E-10$	$2.98221E-13$

Table 4: Numerical Results for Example 4

t	y_i	Exact solution	Error $\|y(t_n) - y_n\|$ 1TDBHOM
3	y_1	2.47875×10^{-3}	1.06425×10^{-15}
	y_2	4.97871×10^{-2}	1.06859×10^{-14}
5	y_1	4.53999×10^{-5}	6.09864×10^{-19}
	y_2	6.73795×10^{-3}	4.51028×10^{-17}
7	y_1	8.31529×10^{-7}	1.11967×10^{-18}
	y_2	9.11882×10^{-4}	6.17128×10^{-16}
10	y_1	2.06115×10^{-9}	6.44953×10^{-21}
	y_2	4.53999×10^{-5}	4.35036×10^{-18}

Conclusions: A class of stable high order Third Derivative Block Hybrid Obrechkoff Methods (TDBHOM) has been developed for solving stiff system of Initial Value Problems (IVPs) numerically. The TDBHOM has the advantage of being self-starting and has a wider stability region than A-stable methods since they are L-stable. The plot of the boundary locus of the roots of the stability polynomial in Table 9 and Figures (1)-(4) shows that the methods are either L-stable or L(0)-stable in nature. The $k-$ step TDBHOM for $k = 1(1)4$ are zero-stable, consistent, convergent and are of order $2k + 3$. To attest the accuracy of the methods, we compared their errors (see Tables (1)-(4)) and found that the methods are highly competitive with existing methods and is efficient for solving stiff IVPs.

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Appendix

Table 5: The Discrete Coefficients of $k-$ step TDBHOM for $k = 1(1)4$

k	1	1	2	2	2	2	3	3
Method	y_{n+1}	$y_{n+\frac{1}{2}}$	y_{n+2}	$y_{n+\frac{3}{2}}$	$y_{n+\frac{1}{2}}$	y_n	y_{n+3}	$y_{n+\frac{5}{2}}$
α_{k-1}	1	1	1	1	1	1	1	1
β_k	$\frac{1}{20}$	$-\frac{97}{320}$	$\frac{17791}{90720}$	$-\frac{65059}{483840}$	$-\frac{270113}{1451520}$	$\frac{12293}{30240}$	$\frac{23446261}{102060000}$	$-\frac{1142206241}{13063680000}$
$\beta_{k-\frac{1}{2}}$	$\frac{4}{5}$	$\frac{13}{20}$	$\frac{64}{105}$	$\frac{361}{840}$	$\frac{269}{840}$	$-\frac{64}{105}$	$\frac{2636}{4725}$	$\frac{54041}{151200}$
β_{k-1}	$\frac{3}{20}$	$\frac{49}{320}$	$\frac{43}{210}$	$\frac{243}{1120}$	$-\frac{1499}{3360}$	$\frac{9}{70}$	$\frac{881}{3780}$	$\frac{248533}{967680}$
$\beta_{k-\frac{3}{2}}$	0	0	$-\frac{32}{2835}$	$-\frac{101}{7560}$	$-\frac{4387}{22680}$	$-\frac{736}{945}$	$-\frac{656}{25515}$	$-\frac{27611}{816480}$
β_{k-2}	0	0	$\frac{1}{1120}$	$\frac{59}{53760}$	$\frac{97}{17920}$	$-\frac{493}{3360}$	$\frac{181}{30240}$	$\frac{32479}{3870720}$
$\beta_{k-\frac{5}{2}}$	0	0	0	0	0	0	$-\frac{124}{118125}$	$-\frac{1441}{945000}$
β_{k-3}	0	0	0	0	0	0	$\frac{47}{510300}$	$\frac{17881}{130636800}$
$\beta_{k-\frac{7}{2}}$	0	0	0	0	0	0	0	0
β_{k-4}	0	0	0	0	0	0	0	0
ϕ_k	$\frac{1}{20}$	$\frac{33}{320}$	$-\frac{17}{3024}$	$\frac{629}{16128}$	$\frac{2887}{48384}$	$-\frac{139}{1008}$	$-\frac{2743}{162000}$	$\frac{480983}{20736000}$
ρ_k	$-\frac{1}{120}$	$-\frac{23}{1920}$	$-\frac{1}{1008}$	$-\frac{19}{5376}$	$-\frac{97}{16128}$	$\frac{5}{336}$	$\frac{7}{32400}$	$-\frac{7667}{4147200}$

Table 6: The continuation of Table 5
The Discrete Coefficients of $k-$ step TDBHOM for $k = 1(1)4$

k	3	3	3	3	4	4
Method	$y_{n+\frac{3}{2}}$	y_{n+1}	$y_{n+\frac{1}{2}}$	y_n	y_{n+4}	$y_{n+\frac{7}{2}}$
α_{k-1}	1	1	1	1	1	1
β_k	$-\frac{878728129}{13063680000}$	$\frac{2104229}{102060000}$	$-\frac{28936657}{161280000}$	$\frac{2416754}{3189375}$	$\frac{35610575353}{146694240000}$	$-\frac{121822988417}{18776862720000}$
$\beta_{k-\frac{1}{2}}$	$\frac{20929}{151200}$	$-\frac{116}{4725}$	$\frac{1851}{5600}$	$-\frac{6112}{4725}$	$\frac{83459}{155925}$	$\frac{12699877}{39916800}$
β_{k-1}	$-\frac{336523}{967680}$	$-\frac{671}{3780}$	$-\frac{17457}{35840}$	$\frac{767}{945}$	$\frac{156677}{623700}$	$\frac{23026391}{79833600}$
$\beta_{k-\frac{3}{2}}$	$-\frac{198139}{816480}$	$-\frac{16144}{25515}$	$-\frac{3307}{10080}$	$-\frac{35648}{25515}$	$-\frac{755}{18711}$	$-\frac{7119853}{119750400}$
β_{k-2}	$\frac{86591}{3870720}$	$-\frac{5851}{30240}$	$-\frac{95091}{143360}$	$\frac{74}{945}$	$\frac{7603}{498960}$	$\frac{788687}{31933440}$
$\beta_{k-\frac{5}{2}}$	$-\frac{3179}{945000}$	$-\frac{964}{118125}$	$-\frac{6171}{35000}$	$-\frac{96352}{118125}$	$-\frac{20479}{3898125}$	$-\frac{1794901}{199584000}$
β_{k-3}	$\frac{36089}{130636800}$	$-\frac{257}{510300}$	$\frac{5897}{1612800}$	$-\frac{18031}{127575}$	$\frac{2557}{1871100}$	$\frac{580603}{239500800}$
$\beta_{k-\frac{7}{2}}$	0	0	0	0	$-\frac{1747}{7640325}$	$-\frac{813557}{1955923200}$
β_{k-4}	0	0	0	0	$\frac{29}{1596072}$	$\frac{95999}{2554675200}$
ϕ_k	$\frac{406327}{20736000}$	$-\frac{1127}{162000}$	$\frac{13991}{256000}$	$-\frac{2404}{10125}$	$-\frac{2442707}{116424000}$	$\frac{24126283}{1490227200}$
ρ_k	$-\frac{7123}{4147200}$	$\frac{23}{32400}$	$-\frac{259}{51200}$	$\frac{46}{2025}$	$\frac{997}{1663200}$	$-\frac{25253}{21288960}$

Table 7: The continuation of Table 5
The Discrete Coefficients of $k-$ step TDBHOM for $k = 1(1)4$

k	4	4	4	4	4	4
Method	$y_{n+\frac{5}{2}}$	y_{n+2}	$y_{n+\frac{3}{2}}$	y_{n+1}	$y_{n+\frac{1}{2}}$	y_n
α_{k-1}	1	1	1	1	1	1
β_k	$-\frac{326270824373}{93884313600000}$	$-\frac{883174711}{146694240000}$	$-\frac{1611061031}{38635520000}$	$\frac{95810809}{3667356000}$	$-\frac{2784872197}{15021490176}$	$\frac{675242851}{603680000}$
$\beta_{k-\frac{1}{2}}$	$\frac{3272453}{39916800}$	$\frac{331}{155925}$	$\frac{45909}{492800}$	$-\frac{6208}{155925}$	$\frac{587725}{1596672}$	$-\frac{4077}{1925}$
β_{k-1}	$-\frac{24164297}{79833600}$	$-\frac{26623}{124740}$	$-\frac{297049}{985600}$	$-\frac{23707}{155925}$	$-\frac{1875505}{3193344}$	$\frac{15221}{7700}$
$\beta_{k-\frac{3}{2}}$	$-\frac{33400781}{119750400}$	$-\frac{282811}{467775}$	$-\frac{9335}{19712}$	$-\frac{308288}{467775}$	$-\frac{781525}{4790016}$	$-\frac{5657}{1925}$
β_{k-2}	$\frac{1442311}{31933440}$	$-\frac{21017}{99792}$	$-\frac{226593}{394240}$	$-\frac{44573}{124740}$	$-\frac{5385125}{6386688}$	$\frac{10179}{6160}$
$\beta_{k-\frac{5}{2}}$	$-\frac{13629769}{9979200000}$	$\frac{60673}{3898125}$	$-\frac{2643929}{12320000}$	$-\frac{495296}{779625}$	$-\frac{356441}{1596672}$	$-\frac{96079}{48125}$
β_{k-3}	$\frac{807707}{239500800}$	$-\frac{4867}{1871100}$	$\frac{14297}{985600}$	$-\frac{88481}{467775}$	$-\frac{6711725}{9580032}$	$\frac{2383}{7700}$
$\beta_{k-\frac{7}{2}}$	$-\frac{1074901}{1955923200}$	$\frac{521}{1528065}$	$-\frac{42597}{24147200}$	$\frac{52928}{7640325}$	$-\frac{13145885}{78236928}$	$-\frac{81891}{94325}$
β_{k-4}	$\frac{15689}{364953600}$	$-\frac{923}{39916800}$	$\frac{151}{1261568}$	$-\frac{1823}{4989600}$	$\frac{287575}{102187008}$	$-\frac{67003}{492800}$
ϕ_k	$\frac{70639687}{7451136000}$	$\frac{158909}{116424000}$	$\frac{3209301}{275968000}$	$-\frac{23591}{2910600}$	$\frac{3213715}{59609088}$	$-\frac{1425321}{4312000}$
ρ_k	$-\frac{80777}{106444800}$	$-\frac{139}{1663200}$	$-\frac{3771}{3942400}$	$\frac{31}{41580}$	$-\frac{19825}{4257792}$	$\frac{1791}{61600}$

Table 8: The Order and Error Constant of $k-$ step TDBHOM for $k = 1(1)4$

k	Order (p)	Error Constant C_{p+1}
1	$(5,5)^T$	$\left(\frac{1}{12800}, \frac{1}{14400}\right)^T$
2	$(7,7,7,7)^T$	$\left(-\frac{1}{56448}, \frac{197}{43352064}, \frac{23}{14450688}, \frac{1}{846720}\right)^T$
3	$(9,9,9,9,9,9)^T$	$\left(-\frac{1}{241920}, \frac{161}{235929600}, -\frac{23}{174182400}, \frac{661}{495454600}, \frac{761}{8918138880}, \frac{1}{19353600}\right)^T$
4	$(11,11,11,11,11,11,11,11)^T$	$\left(-\frac{3107}{3469312000}, \frac{7445}{63946358784}, -\frac{31}{1317254400}, \frac{12153}{888143872000}\right)^T$ $\left(-\frac{59}{31223808000}, \frac{229517}{30831280128000}, \frac{11021}{15986589696000}, \frac{2739}{843042816000}\right)^T$

Table 9: The Stability Functions of $k-$ step TDBHOM for $k = 1(1)4$

k	$S(z)$
1	$\frac{1+0.3z+0.025z^2}{1-0.7z+0.225z^2-0.0416667z^3+0.00416667z^4}$
2	$\frac{1+0.10476z+0.005159z^2+0.010714z^3+0.001786z^4}{1-0.67619z+0.77976z^2-0.4123z^3+0.1113z^4-0.0165z^5+0.0011906z^6}$
3	$\frac{1+1.166667z+0.6076z^2+0.18229z^3+0.03356z^4+0.00365z^5+0.000186z^6}{1-1.833z+1.6076z^2-0.89z^3+0.346z^4-0.0986z^5+0.0207z^6-0.0031z^7+0.000279z^8}$
4	$\frac{1+1.636z+1.2409z^2+0.5727z^3+0.177z^4+0.0379z^5+0.0055z^6+0.0005z^7+0.000024z^8}{1-2.364z+2.695z^2-1.967z^3+1.0256z^4-0.404z^5+0.1236z^6-0.0296z^7+0.0055z^8-0.000749z^9+0.000063z^{10}}$