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Dissipative Energy Estimates for Non-Autonomous Stoke Equation

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Abstract

This paper derives the weighted energy equality for the linear nonautonomous stokes problem. This is a part of the process of solving Navier-Stokes equations to exclude the pressure term before it is determined in which function space solution (u, p) lies. A correction term v_ϕ is introduced when the pressure term is excluded in the Navier-Stokes system and the energy estimate is computed; because the parameter v_ϕ is only an artificial creation meant to exclude the pressure, thereby gauge the impact of the parameter on the final estimates. It is discovered that the introduction of a correction parameter v_ϕ did not essentially change the estimates as it is only found to be $\epsilon - small$. Hence, the energy estimate for the Navier-Stokes system remains valid.

1. Introduction

The efforts being made to solve the initial boundary value problems for Navier-Stokes equations has received enormous attention in recent times and has generated very important questions in mathematical hydrodynamics. There are lot of studies in papers and monographs (e. g. [5], [4], [3].). The existence theory, there developed, handles mainly flows of viscous fluids in bounded and exterior domains. Although some of these results do not depend on the shape of the

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boundary, however, there exist problems of scientific and practical interest bordering on kinds of flows of viscous incompressible fluid in unbounded domains or in domains with noncompact boundaries; there remain a lot of unsolved problems that is attracting attention in the scientific community. It is not surprising that during the last few years attention is being focused on research related to Navier-Stokes in unbounded domain. The global, in time, estimate for Navier Stokes equations have always relied on some kind of compactness of the semigroup generated by such equations. This compactness is gotten via the regularization property of the equations together with the compact embedding of the relevant Sobolev spaces (see [6] for instance). This is only possible when the domain is bounded because for unbounded domain, like in our case to be considered here, sobolev embeddings are not compact. To resolve the issue of noncompactness of Sobolev embedding for the case of unbounded domains, the notion of weighted energy spaces already used in Anthony[1] is adopted. The main idea behind the use of weighted spaces needed to be first captured.
Consider the Navier-Stokes System:

$$\begin{cases} \partial_t u + (u, \nabla)u - \nu\Delta u + \nabla p = g \\ div\ u = 0, u|_{\partial\Omega} = 0, u|_{t=0} = u_0; \end{cases} \tag{1}$$

in unbounded domain $\Omega = \mathbb{R} \times (-1, 1)$.
The usual energy estimate for (1) should require multiplication of the equation by a test function u and integrating over the domain Ω. However, due to the nature of the domain prescribed, we are unable to do this because the integral will not make a mathematical sense (it will be infinite). To get an infinite energy solution for (1), it will require interfacing the solution u with a weight function to be explained as given in Zelik [8] as follows: Let us define $B(x_0, 1)$-a unit rectangle centred at $(x_0, 0)$ given as:

$$B(x_0, 1) = \Big(-\frac{1}{2} < x_0 < \frac{1}{2} \Big) \times (-1, 1), x_0 \in \mathbb{R} \tag{2}$$

The following definitions are important in what follows:
Let us briefly state the definition of weight functions as presented by Zelik [7] which will be systematically used in the sequel (see also [2] for more details). We start with the class of admissible weight functions.

Definition 1.1. A function $\phi \in C_{loc}(\mathbb{R})$ is weight function of exponential growth rate $\mu > 0$ if the following inequalities hold:

$$\phi(x + y) \le C_\phi \phi(x) e^{\mu|y|}, \quad \phi(x) > 0, \tag{3}$$

for all $x, y \in \mathbb{R}$.

This paper seeks to understand the impact of introducing a correction parameter v_ϕ, for the purpose of excluding the pressure term of Navier-Stokes equation, on its energy solution in a Poincare-like domain. This artificial creation of a correction term v_ϕ is justified by our results in this article in that it is estimated to be only $\epsilon-small$; this presupposes that the final energy estimates is not altered by the introduction of v_ϕ which, for the moment, only served to exclude pressure. From the onset, we seek to use the weighted energy spaces to control the energy solutions of the Navier-Stokes equation; however, the emergence of cubic nonlinearity and nasty pressure terms led to the introduction of the corrector v_ϕ; this later became the solution of the auxiliary linear stokes problem consequent of the use of adjoint equation to simplify the entire Navier-Stokes system with constant flux conditions. This article justifies the adjoint simplification by linear theory to obtain the following estimates for the corrector v_ϕ thus:

$$\|v_\phi\|_{L^2_{\phi^{-1}}} + \|v_\phi\|_{L^3_{\phi^{-2}}} \leq C\|u\|_{C(0,T;L^2_\phi)}.$$

2. Weighted Energy Equality

Let us begin by deriving the weighted energy equality for the linear non-autonomous Stokes problem (1) under the assumptions that

$$g \in L_b^{4/3}(\mathbb{R}_+ \times \Omega) \cap L_b^1(\mathbb{R}_+, L_b^{3/2}(\Omega)), \quad u_0 \in H_b \tag{4}$$

this is used for the study of the nonlinear Navier-Stokes equation; Zelik [8] and Anthony [1].

Definition 2.1. A function $u(t,x)$ is a weak (energy) solution of (1) if

$$u \in L^\infty(0,T;H_b) \cap C(0,T;H_\phi), \quad \nabla_x u \in L_b^2([0,T]\times\Omega), \tag{5}$$

where ϕ is any weight of exponential growth rate such that $\phi \in L^2(\mathbb{R})$ and u solves (1) in the sense of distributions, namely, for any $\varphi \in C_0^\infty([0,T]\times\Omega)$ satisfying $div\ v\varphi = 0$,

$$-\int_{\mathbb{R}_+}(u,\partial_t\varphi)dt - \int_{\mathbb{R}_+}(u,\Delta_x\varphi)dt = \int_{\mathbb{R}_+}(g,\varphi)dt.$$

Here and below (u,v) stands for the standard inner product in $[L^2(\Omega)]^2$.

Consider equation (1) with all its multiplying terms with a view to factoring out the nonstationary linear stokes problem through adjoint simplification. Here, we multiply (1) by $(\theta^2 u - v_\theta)$ to obtain the following:

$$\begin{aligned}(\partial_t u,(\theta^2 u - v_\theta)) + ((u,\nabla_x)u,(\theta^2 u - v_\theta)) + (-\Delta_x u,(\theta^2 u - v_\theta)) \\ + (\nabla_x p,(\theta^2 u - v_\theta)) = (g,(\theta^2 u - v_\theta))\end{aligned} \tag{6}$$

where $div\ v(\theta^2 u - v_\theta) = 0$.

We explicitly simplify (6) by collecting similar terms and taking into account that the pressure term is excluded; the following is obtained:

$$(7)\quad (\partial_t u, \theta^2 u) + ((u,\nabla_x)u, \theta^2 u) + (-\Delta_x u, \theta^2 u) = (\partial_t u, v_\theta) + \\ + ((u,\nabla_x)u, v_\theta) - (\Delta_x u, v_\theta) + (g, \theta^2 u - v_\theta).$$

$$(8)\qquad (\partial_t u, \phi^2 u) + ((u,\nabla)u, \phi^2 u) - (\nu\Delta u, \phi^2 u) + (\nabla p, \phi\phi^2 u) = (f, \phi^2 u)$$

It is observed that the LHS of (7) is exactly equation (8) without the pressure term. Let's focus on other terms in the RHS. It is not immediately clear how to control the third term on the RHS of (7); therefore, introducing the auxiliary linear stokes problem is helpful:

$$(9)\quad -\partial_t v_\theta + \Delta_x v_\theta + \nabla_x q = 0, \quad div\ v_\theta = 2\theta\theta' u_1, \quad v_\theta\big|_{\partial\Omega} = 0.$$

The next Lemma derives (9):

Lemma 2.2. *Let u be a sufficiently smooth solution of (1) satisfying zero flux assumption $\mathbb{S}u_1(t;x_1) = \frac{1}{2}\int_{-1}^{1} u_1(t,x_1,x_2)dx_2 = 0$ and v_θ be a solution of the auxiliary problem (9) whose existence is established in Theorem 3.1. Then,*

$$(10)\qquad \frac{d}{dt}(u, v_\theta) = -((u,\nabla)u, v_\theta) + (f, v_\theta) + 2(p, \theta'\theta u_1)$$

Proof: By energy estimates the following is true:

$$(11)\qquad -(\partial_t u, v_\theta) - ((u\nabla)u, v_\theta) + (\Delta u, v_\theta) - (\nabla p, v_\theta) = -(f, v_\theta).$$

By Leibniz formula the first term of (11) gives

$$(12)\qquad -(\partial_t u, v_\theta) = -\frac{d}{dt}(u, v_\theta) + (u, \partial_t v_\theta).$$

Since the Laplacian is a positive self-adjoint operator, the third term of LHS (11) could be written as $(u, \Delta v_\theta)$. Simplify (11) and get

$$(13)\qquad \frac{d}{dt}(u, v_\theta) = (u, -\partial_t v_\theta + \Delta v_\theta) + 2(p, \theta'\theta u_1) + (f, v_\theta) - ((u\nabla)u, v_\theta)$$

$$(14)\qquad \frac{d}{dt}(u, v_\theta) = (u, \nabla q) + 2(p, \theta'\theta u_1) + (f, v_\theta) - ((u\nabla)u, v_\theta).$$

Assuming that $(u, \nabla q) = -(\text{div } u, q) = 0$. Hence, we obtain (10). This ends the proof of the Lemma. The assumption on (13) that

$$(15)\qquad -\partial_t v_\theta + \Delta_x v_\theta = \nabla_x q$$

will form the basis for the justification of the weighted energy equality. Formally, introduce the corrector v_θ as the solution of the following adjoint equation:

$$(16)\quad -\partial_t v_\theta + \Delta_x v_\theta + \nabla_x q = 0, \quad div\ v_\theta = 2\theta\theta' u_1, \quad v_\theta\big|_{\partial\Omega} = 0,$$
$$v_\theta\big|_{t=T} = (0, 2\theta\theta'\Psi(u(T))),$$

where $T > 0$ is a parameter and $\psi = \Psi(u)$ is a stream function of the vector field u. Then, $div\ (\theta^2 v - v_\theta) = 0$ and we may at least formally multiply equation (1) by $\theta^2 u - v_\theta$ without taking a special care on the pressure term $\nabla_x p$. Note also that due to Lemma 2.2, (16) is an adjoint equation; therefore, we need to solve it backward in time (for $t \le T$) and the unusual initial data at $t = T$ is chosen in order to satisfy the necessary compatibility condition

$$div\ u\big|_{t=T} = div\ u(T) = 2\theta\theta' u_1(T).$$

Furthermore, if we put (7) with the result of Lemma (2.2) into perspective and integrate over the domain, we get the following weighted energy equality:

$$(17)\quad \frac{d}{dt}(\frac{1}{2}\|u\|^2_{L^2_\theta} - (u, v_\theta)) + (u, -\partial_t v_\theta + \Delta_x v_\phi) + (\nabla_x u, \nabla_x(\theta^2 u)) = (g, \theta^2 u - v_\theta)$$

where $div\ u = div\ (\theta^2 u - v_\theta) = 0$ is used. Thus, obtain the key weighted energy identity

$$(18)\qquad \frac{d}{dt}\Big(\frac{1}{2}\|u\|^2_{L^2_\theta} - (u, v_\theta)\Big) + (\nabla_x u, \nabla_x(\theta^2 u)) = (g, \theta^2 u - v_\theta).$$

Since our aim is to justify the weighted energy equality (18). To this end, it is important to study the solutions of the auxiliary problem (16). For simplicity, let's switch back to forward in time solutions by the change of time variable $t \to T - t$ and consider slightly more general problem

$$(19)\quad \partial_t v - \Delta_x v + \nabla_x q = 0, \quad div\ v = \partial_{x_2}\psi(t), \quad v\big|_{\partial\Omega} = 0, \quad v\big|_{t=0} = (0, \psi(0))^t,$$

where the function $\psi \in C(0, T; W^{1,2}_\theta(\Omega))$ for some weight ϕ of exponential growth rate which satisfies the additional assumption

$$(20)\qquad |\theta'(x)| + |\theta''(x)| + |\theta'''(x)| \le C\epsilon\theta(x),$$

and the parameter $\epsilon > 0$ is small enough. Then, the choice

$$(21)\qquad \psi(t) = 2\theta\theta'\Psi(u_1(t))$$

corresponds to the considered auxiliary problem (16). $\square$.
The next section gives the estimate for the solution v in terms of the function ψ.

3. Linear Stokes Problem

The following non-stationary linear Stokes problem is obtained earlier:

$$(22)\qquad \begin{cases} \partial_t v - \Delta v = \nabla p \\ \text{div } v = \partial_{x_2}\psi, \mathbb{S}u_1 \equiv 0, \\ v|_{\partial\Omega} = 0, v\big|_{t=T} = (0, 2\theta\theta'\Psi(u(T))) \end{cases}$$

assume that

$$(23)\qquad \psi \in (L^2(0,T;H^1_\theta)), \psi|_{\partial\Omega} = 0$$

where $\theta = \theta(x)$ is a weight function of small exponential growth rate.
The following theorem gives a priori estimates and the solvability result for the problem (22). Moreover, the corrector is not local in space and in time.

$$(24)\qquad \phi_\epsilon(x) = (1 + \epsilon^2|x|^2)^{-\frac{1}{2}}$$

Theorem 3.1. *Let θ be a weight of exponential growth rate which satisfies (24) for sufficiently small $\epsilon \le \epsilon_0$ and let $\psi \in C(0,T;H^1_\theta(\Omega))$. Then, there exists a unique solution $v(t)$ of problem (22) such that*

$$(25)\qquad v \in C(0,T;H^{1/3}(\Omega)), \ \int_0^t v(t)\,dt \in L^2(0,T;H^{2+1/3}_\theta(\Omega)), \ \ \mathbb{S}v_1 \equiv 0$$

and the following estimate holds:

$$(26)\qquad \|v\|_{C(0,T;H^{1/3}_\theta(\Omega))} \le C\|\psi\|_{C(0,T;H^1_\theta(\Omega))},$$

where the constant C is independent of ψ, T, ϵ and the choice of the weight θ.

Proof: Lets verify (26) by transforming the equation; this is to reduce (22) in the following way: $\bar{v} = v - w$ and write $w(t) = (0, \psi(t))$. Then, $\bar{v}(t)$ solves:

$$(27)\qquad \partial_t\bar{v} - \Delta\bar{v} - \nabla p = -\partial_t w + \Delta w, \ \text{div } \bar{v} = 0 \text{ and} v|_{t=0} = 0.$$

From the definition of $w(t)$, we have from (27) that:

$$(28)\qquad \Delta w(t) \in C(0,T;H^{-1});$$

and from (23) the following estimates hold:

$$(29)\qquad \|\Delta w(t)\|_{C(0,T;H^{-1}_\theta)} \le C\|w(t)\|_{C(0,T;H^1_\theta)} \le C\|\psi(t)\|_{C(0,T;H^1_\theta)}.$$

By yet another transformation on $\bar{v}$, we write: $\tilde{w} = \bar{v} + \tilde{v}$ to obtain:

$$(30)\qquad \partial_t\tilde{v} - \Delta\tilde{v} - \nabla p = -\partial_t(w + \tilde{w}) + \Delta w + \Delta\tilde{w}, \tilde{w}|_{\partial\Omega} = 0, \text{div } \tilde{w} = 0$$

Since, as yet,(30) have terms in w which are not divergent free, the following reformulation is important:

$$(31)\qquad \Delta w + \Delta\tilde{w} = \nabla q, \ \tilde{w}|_{\partial\Omega} = 0 \text{ and div } \tilde{w} = 0$$

Substituting the above into (30), we obtain the following:

$$\partial_t \tilde{v} - \Delta \tilde{v} - \nabla(p+q) = -\partial_t(w+\tilde{w}). \tag{32}$$

Therefore we have:

$$\partial_t \tilde{v} - \Delta \tilde{v} - \nabla \tilde{p} = -\partial_t(w(t)+\tilde{w}(t)), \text{div } \tilde{v} = 0, \ \tilde{v}|_{t=0} = -\tilde{w}|_{t=0} = 0. \tag{33}$$

From (29) and (31):

$$C\|\tilde{w}(t)\|_{C(0,T;H^1_\theta)} \leq \|\Delta w\|_{C(0,T;H^{-1}_\theta)} \leq C\|\psi(t)\|_{C(0,T;H^1_\theta)}. \tag{34}$$

To get rid of the time derivative which, up to the moment, still has the divergent term w, we fix some $\alpha > 0$ and introduce the new function $V(t)$ via:

$$V(t) = \int_0^t e^{-\alpha(t-s)}\tilde{w}(s)ds. \tag{35}$$

Obviously, (35) solves:

$$V'(t) + \alpha V(t) = \tilde{w}(t), V|_{t=0} = 0 \tag{36}$$

Similarly, from (36) we may reformulate equation (33) as follows:

$$V'(t) + \alpha V(t) = \partial_t \tilde{v}(t) - \Delta \tilde{v} - \nabla p; \tag{37}$$

hence,

$$V(t) = \int_0^t e^{-\alpha(t-s)} \left(\partial_t \tilde{v}(s) - \Delta \tilde{v}(s) - \nabla p(s)\right) ds. \tag{38}$$

For simplicity, we will take the integrals on the RHS of (38) in turn and perform some basic integration by parts as follows:

$$\int_0^t e^{-\alpha(t-s)}\partial_s \tilde{v}(s)ds = \tilde{v}(t) - \tilde{v}(0)e^{-\alpha t} - \alpha \int_0^t e^{-\alpha(t-s)}\tilde{v}(s)ds. \tag{39}$$

The second and the third integrals of (38) yield:

$$-\left(\int_0^t e^{-\alpha(t-s)}(\Delta \tilde{v}(s) + \nabla P(s))ds\right) =$$

$$= -\left(\Delta \int_0^t \tilde{v}(s)e^{-\alpha(t-s)}ds + \nabla \int_0^t P(s)e^{-\alpha(t-s)}ds\right). \tag{40}$$

Next, we posit that $\tilde{v}(t) = V'(t) + \alpha V(t)$ and write (38) as:

$$V(t) = \partial_t V + \alpha V - \tilde{v}(0)e^{-\alpha t} - \alpha V - \Delta V - \nabla P; \tag{41}$$

this reduces to:

$$V(t) = \partial_t V - \tilde{v}(0)e^{-\alpha t} - \Delta V - \nabla P. \tag{42}$$

Similarly, we consider the RHS of (32) and write:

$$\partial_t V + \alpha V = -\partial_t(w(t) + \tilde{w}(t)) \tag{43}$$

which results to:

$$V(t) = -\int_0^t e^{-\alpha(t-s)}\partial_s(w(s)+\tilde{w}(s))ds = -(\tilde{w}(t)+w(t))+$$

$$(\tilde{w}(0)+w(0))e^{-\alpha t} + \alpha\int_0^t \tilde{(}w(s)+w(s))e^{-\alpha(t-s)}ds. \tag{44}$$

Hence, $V(t)$ solves:

$$\partial_t V(t) - \Delta V(t) - \nabla P = H, V|_{t=0} = 0, \text{div } V(t) = 0. \tag{45}$$

We take

$$H = -(\tilde{w}(t)+w(t)) + (\tilde{w}(0)+w(0))e^{-\alpha t} + \alpha\int_0^t \tilde{(}w(s)+w(s))e^{-\alpha(t-s)}ds.$$

Therefore,

$$\sup_{t\in[0,T]}\|H\|_{H^1_\theta} \le \sup_{t\in[0,T]}\|\tilde{w}(t)+w(t)\|_{H^1_\theta} + \sup_{t\in[0,T]}\|\tilde{w}(0)+w(0)\|_{H^1_\theta}$$

$$\alpha\sup_{t\in[0,T]}\|(w(t)+\tilde{w}(t))\|_{H^1_\theta}\int_0^t e^{-\alpha(t-s)}ds. \tag{46}$$

And from (34) we obtain:

$$\|H\|_{C(0,T;H^1_\theta} \le \|w(t)+\tilde{w}(t)\|_{C(0,T;H^1_\theta)} \le C\|\psi\|_{C(0,t;H^1_\theta)}. \tag{47}$$

We take into account:

$$\|\tilde{w}\|_{L^3_\theta} \le \|\partial_t V(t)\|_{L^3_\theta} + \|\Delta V(t)\|_{L^3_\theta}. \tag{48}$$

And by the Sobolev embedding $H^{\frac{1}{3}} \hookrightarrow L^3$, we see that to prove the estimate (26) we only need to prove the following lemma:

Lemma 3.2. *Let $V(t)$ be the solution of problem (45), then the following estimate holds:*

$$\|\partial_t V(t)\|_{C(0,T;H^{\frac{1}{3}}_\theta)} + \|\Delta V(t)\|_{C(0,T;H^{\frac{1}{3}}_\theta)} \le \|H\|_{C(0,T;H^1_\theta)} \tag{49}$$

where C is independent of T, H and θ

We introduce the Stokes operator $A = -\Pi\Delta$; applying the Leray projector on both sides of (45) excludes the pressure. We obtain:

$$\partial_t V(t) + AV(t) = \Pi H(t) = h(t), V|_{t=0} = 0. \tag{50}$$

Since Π maps $H^1_\theta(\Omega)$ to $H^1_\theta(\Omega)$ see [8]. We have from (47) that:

$$\|h\|_{C(0,T;H^1_\theta)} \le C\|\psi\|_{C(0,T;H^1_\theta)}. \tag{51}$$

Furthermore, we remind ourselves that due to [8] the following are true:

$$\|AV\|_{H_\theta^{\frac{1}{3}}} \sim \|\Delta V\|_{H_\theta^{\frac{1}{3}}} \sim \|V\|_{H_\theta^{2+\frac{1}{3}}}. \tag{52}$$

Therefore to verify (49), it is enough to check that:

$$\|AV(t)\|_{C(0,T;H_\theta^{\frac{1}{3}})} \leq \|h(t)\|_{C(0,T;H_\theta^{\frac{1}{3}})} \tag{53}$$

for the solution $V(t)$ of problem (50). $\square$.

4. Energy Solution of the non-autonomous Stokes Equation

The following corollary is crucial:

Corollary 4.1. *Let the weight exponential growth rate $\phi \in L^{4/3}(\mathbb{R})$ and satisfy (20) with sufficiently small $\epsilon > 0$. Let also $u_0 \in H_b$ and g satisfy (4). Then, there exists a unique energy solution $u(t)$ of the Stokes problem and this solution satisfies the estimate*

$$\|u\|_{C(0,T;L^2_\phi)} + \|u\|_{L^2(0,T;W^{1,2}_\phi)} \leq \tag{54}$$
$$\leq C_T\Big(\|u_0\|_{H_\phi} + \|g\|_{L^{4/3}(0,T;L^{4/3}_\phi)\cap L^1(0,T;L^{3/2}_\phi)}\Big),$$

where the constant C_T may depend on T, but is independent of the concrete choice of the weight. Moreover, the function $t \to \frac{1}{2}\|u(t)\|^2_{L^2_\phi} - (u(t), v_\phi(t))$ is absolutely continuous and the energy identity (18) holds for almost all $t \in (0,T)$.

Proof: We first derive estimate (54) assuming that the validity of (18) is already verified. Then, for sufficiently small $\epsilon > 0$,

$$(\nabla_x u, \nabla_x(\phi^2 u)) = \|\nabla_x u\|^2_{L^2_\phi} + 2(\nabla_x u, \phi\phi' u) \geq \tag{55}$$
$$\geq \|\nabla_x u\|^2_{L^2_\phi} - C\epsilon(\phi^2|\nabla_x u|, |u|) \geq \frac{1}{2}\|\nabla_x u\|^2_{L^2_\phi} - C\epsilon^2\|u\|^2_{L^2_\phi} \geq \frac{1}{4}\|u\|^2_{W^{1,2}_\phi},$$

where we have implicitly used the weighted version of the Poincare inequality

$$\|u\|_{L^2_\phi} \leq C\|\nabla_x u\|_{L^2_\phi}.$$

$$\|v_\theta\|_{C(0,T;L^2_{\theta-1})} + \|v_\theta\|_{C(0,T;L^3_{\theta-1})} \leq C\epsilon\|u\|_{C(0,T;L^2_\theta)}. \tag{56}$$

Moreover, due to (56) and the weighted Ladyzhenskaya inequality

$$\|u\|^2_{L^4_\phi} \leq C\|u\|_{L^2_\phi}\|u\|_{W^{1,2}_\phi}, \tag{57}$$

together with the Hölder inequality,

(58)
$$|(g,\phi^2 u - v_\phi)| \le C\|g\|_{L^{4/3}_\phi}\|u\|_{L^4_\phi} + \|g\|_{L^{3/2}_\phi}\|v_\phi\|_{L^3_\phi} \le C\|g\|_{L^{\frac{4}{3}}_\phi}\|u\|^{1/2}_{L^2_\phi}\|u\|^{1/2}_{W^{1,2}_\phi} +$$
$$+ C\|g\|_{L^{3/2}_\phi}\|u\|_{L^2_\phi} \le C\|g\|^{4/3}_{L^{4/3}_\phi}\|u\|^{2/3}_{L^2_\phi} + C\|g\|_{L^{3/2}_\phi}\|u\|^{1/2}_{L^2_\phi} + 1/8\|u\|^2_{W^{1,2}_\phi}.$$

Integrating now the energy identity (18) in time and using (58), (55) and the obvious estimate

$$|(u,v_\phi)| \le \|u\|_{L^2_\phi}\|v_\phi\|_{L^2_{\phi^{-1}}} \le C\epsilon\|u\|^2_{L^2_\phi},$$

we arrive at

$$(59)\quad (1-C\epsilon)\|u\|^2_{C(0,T;L^2_\phi)} + \|u\|^2_{L^2(0,T;W^{1,2}_\phi)} \le C\|g\|^{4/3}_{L^{4/3}(0,T;L^{4/3}_\phi)}\|u\|^{2/3}_{C(0,T;L^2_\phi)} +$$
$$+ C\|g\|_{L^1(0,T;L^2_\phi)}\|u\|_{C(0,T;L^2_\phi)} + (1-C\epsilon)\|u_0\|^2_{C(0,T;L^2_\phi)}$$

and estimate (54) is an immediate corollary of this estimate if $\epsilon > 0$ is small enough.

Proposition 4.2. *Let ϕ be the weight function of sufficiently small exponential growth rate. Then, for every $g \in L^2(0,T;L^2_\phi(\Omega))$ and every $u_0 \in V_\phi$, there is a unique solution $u(t)$ of the problem*

$$(60)\qquad \partial_t u - \Delta_x u + \nabla_x p = g(t), div\ u = 0,\quad u\big|_{\partial\Omega} = 0,\ u\big|_{t=0} = u_0$$

which satisfies $\partial_t u, \Delta_x u \in L^2(0,T;L^2_\phi)$ and the following estimate holds:

$$(61)\quad \|u(t)\|^2_{L^2_\phi} + \|\partial_t u\|^2_{L^2(\max\{0,t-1\},t;L^2_\phi)} + \|\Delta_x u\|^2_{L^2(\max\{0,t-1\},t;L^2_\phi)} \le$$
$$\le C\|u_0\|^2_{L^2_\phi}e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)}\|g(s)\|^2_{L^2_\phi}\,ds,$$

where C and $\alpha > 0$ are independent of $t \ge 0$, u_0, g and the concrete choice of the weight ϕ.

Note that the uniqueness of a solution u can be done exactly as in Theorem 3.1. To verify the existence, again similar to the proof of Theorem 3.1, we approximate the initial data u_0 by the sequence $u_0^n \in H_\phi$ of smooth initial data which is convergent to u_0 in that space and, analogously, we approximate the external force g by the sequence g_n of smooth ones which is convergent in $L^{4/3}(0,T;L^{4/3}_\phi)\cap L^1(0,T;L^{3/2}_\phi)$. We note that, since the weight $\phi \in L^{4/3}(\mathbb{R})$ then it is not difficult to check, $\phi \in L^{3/2}(\mathbb{R})$ as well and, thanks to (62)

$$u_0 \in H_\phi,\quad g \in L^{4/3}(0,T;L^{4/3}_\phi) \cap L^1(0,T;L^{3/2}_\phi)$$

and, therefore, such approximations exist. Let u_n be the corresponding solutions of (1) which exist due to Proposition 4.2. Then, applying the proved estimate (54) to the differences $u_n - u_m$ of to approximation solutions (since they are smooth, the energy identity hold for them), we have

$$\|u_n - u_m\|_{C(0,T;L^2_\phi)} + \|u_n - u_m\|_{L^2(0,T;W^{1,2}_\phi)} \le$$
$$\le C_T\Big(\|u_0^n - u_0^m\|_{H_\phi} + \|g_n - g_m\|_{L^{4/3}(0,T;L^{4/3}_\phi)\cap L^1(0,T;L^{3/2}_\phi)}\Big).$$

Lemma 4.3. *Let ϕ_ϵ be a weight function defined by (24). Then, for all l and $1 \le p \le \infty$, the map T_{ϕ_ϵ} is an isomorphism between spaces $W^{l,p}(\Omega)$ and $W^{l,p}_\phi(\Omega)$ and the following inequalities hold:*

$$C_1\|\phi u\|^2_{W^{l,p}} \le \|u\|^2_{W^{l,p}_\phi} \le C_2\|\phi u\|^2_{W^{l,p}} \tag{62}$$

where C_1 and C_2 are independent of ϵ but may depend on l and p

Thus, $u_n - u_m$ is a Cauchy sequence in $C(0,T;L^2_\phi)\cap L^2(0,T;W^{1,2}_\phi)$ and, passing to the limit $n\to\infty$, we construct a solution u of problem (1) belonging to this space and justify estimate (54). Moreover, applying this estimate with the shifted weights $\phi(x_1 - s)$, taking the supremum over $s\in\mathbb{R}$ and using (62), we check that u belongs to the uniformly local spaces (5). Thus, the existence of an energy solution is also verified. It only remains to prove the energy identity. To this end, we write the energy identity for u_n in the equivalent integral form:

$$\frac{1}{2}\|u_n(s)\|^2_{L^2_\phi} - (u_n(s), v^n_\phi(s)) - \frac{1}{2}\|u_n(\tau)\|^2_{L^2_\phi} + (u_n(\tau), v^n_\phi(\tau)) =$$
$$= \int_\tau^s (g_n(t), \phi^2 u_n(t) - v^n_\phi(t)) - (\nabla_x u_n(t), \nabla_x(\phi^2 u_n(t)))\,dt, \tag{63}$$

where v^n_ϕ are the solutions of the auxiliary problem (16) which correspond to the solutions u_n. Note that, due to estimate (56) applied to $v^n_\phi - v^m_\phi$, we know that v^n_ϕ converges strongly to v_ϕ in the spaces $C(0,T;L^2_{\phi^{-1}})$ and $C(0,T;L^3_{\phi^{-1}})$. This allows us to pass to the limit $n\to\infty$ in (63) and verify that the limit function u also satisfies this integral identity. Since the integral form (63) of the energy identity is equivalent to the differential form (18), the energy equality is proved and the corollary is also proved. □.

Conclusions: It noteworthy that Theorem 3.1 does not give us control over the $L^2(0,T;W^{1,2}_{\phi^{-1}})$-norm of the corrector v_ϕ, hence, multiplying equation (1) by $\phi^2 u - v_\phi$ directly is not allowed (the term $(\Delta_x u, v_\phi)$ a priori may have no sense). By this reason, we have to justified this multiplication in a different way based on the approximations and the fact that all bad terms are canceled out since v_ϕ solves the *adjoint* equation.

It worth mentioning that the validity of the energy identity (18) remains true if we replace the weight function ϕ by the proper *cut-off* function φ with finite support (the proof just repeats word by word the one given in Corollary 4.1).

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