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Derivation of a New One-Step Numerical Integrator for Solving First Order Ordinary Differential Equations

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Abstract

This article presents a new class of one-step second derivative hybrid block method to solve general first order initial value problems of ordinary differential equations. The method is derived by interpolating an approximated power series at one point and collocating its first and second derivatives at some carefully selected intra-step points. The method is implemented in a block form to overcome the setbacks of applying starting values and predictors. The basic properties of the method were analysed and found to be a method of order of accuracy 10 and A-stable. The efficacy of the method is tested on several first order ordinary differential equations which include autonomous and non-autonomous equations, stiff systems and oscillatory problems and results compared with exact solutions.

1. Introduction

This article presents a new one-step second derivative hybrid block method for solving general first-order IVPs of the form:

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$$x' = f(t, x),\ x(t_0) = x_0,\ \forall t_0 \le t \le t_N \tag{1}$$

with f assumed to be continuous on $[t_0, t_N]$and existence of a unique solution is satisfied. Many fields of application, notably spring and damping system, control system, chemical reaction, electrical circuits, diffusion and control theory, yield initial value problems of the form (1). However, many of these problems do not often have theoretical solution, hence seeking numerical approximation is of utmost importance. As a result, most famous numerical methods such as Euler forward and Euler backward methods, Adams Bashforth and Adams Moulton methods, linear multistep methods, Runge-Kutta, Heun methods and among other methods have been proposed by various experts in the field of Mathematics. See – [5, 6, 8, 11, 14, 15, 16, 17]. Also over the years, multi-derivatives of solution and application of off-step points have been considered by various notable researchers in order to develop effective numerical methods for (1), see [1, 3, 7, 9, 18].
The idea of this article is to derive a new continuous form of one-step second derivatives hybrid block method using the power series as basis function and implement the new method in block form to produce approximate solution to (1).

2. Derivation of the Method

The method is derived by assuming an approximate solution of the form:

$$X(t) = \sum_{j=0}^{r+2s-1} a_j t^j \tag{2}$$

where a_j's are unknown coefficients of the power series function in (2), r and s are numbers of interpolation and collocation points respectively. Interpolating at t_n and collocating the first and second derivatives of (2) at $t_n, t_{n+\frac{5}{34}}, t_{n+\frac{1}{2}}, t_{n+\frac{29}{34}}, t_{n+1}$. These lead to

$$\begin{cases} X(t_n) = x_n \\ X'(t_{n+j}) = f_{n+j} \\ X''(t_{n+j}) = g_{n+j} \\ j = 0, \frac{5}{34}, \frac{1}{2}, \frac{29}{34}, 1; \end{cases} \tag{3}$$

and the coefficient matrix is given as:

(4)

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 & 9x_n^8 & 10x_n^9 \\ 0 & 1 & 2A & 3A^2 & 4A^3 & 5A^4 & 6A^5 & 7A^6 & 8A^7 & 9A^8 & 10A^9 \\ 0 & 1 & 2B & 3B^2 & 4B^3 & 5B^4 & 6B^5 & 7B^6 & 8B^7 & 9B^8 & 10B^9 \\ 0 & 1 & 2C & 3C^2 & 4C^3 & 5C^4 & 6C^5 & 7C^6 & 8C^7 & 9C^8 & 10C^9 \\ 0 & 1 & 2D & 3D^2 & 4D^3 & 5D^4 & 6D^5 & 7D^6 & 8D^7 & 9D^8 & 10D^9 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 & 72x_n^7 & 90x_n^8 \\ 0 & 0 & 2 & 6A & 12A^2 & 20A^3 & 30A^4 & 42A^5 & 56A^6 & 72A^7 & 90A^8 \\ 0 & 0 & 2 & 6B & 12B^2 & 20B^3 & 30B^4 & 42B^5 & 56B^6 & 72B^7 & 90B^8 \\ 0 & 0 & 2 & 6C & 12C^2 & 20C^3 & 30C^4 & 42C^5 & 56C^6 & 72C^7 & 90C^8 \\ 0 & 0 & 2 & 6D & 12D^2 & 20D^3 & 30D^4 & 42D^5 & 56D^6 & 72D^7 & 90D^8 \end{pmatrix}$$

where, $A = (x_{n+\frac{5}{34}})$; $B = (x_{n+\frac{1}{2}})$; $C = (x_{n+\frac{29}{34}})$ and $D = (x_{n+1})$ which is solved using matrix inversion method to obtain a_j's and then substituted into (2) to get the continuous scheme of the form:

(5)

$$\begin{aligned} X(t) = x_n + h\Big(&\beta_0(t) f_n + \beta_{\frac{5}{34}}(t) f_{n+\frac{5}{34}} + \beta_{\frac{1}{2}}(t) f_{n+\frac{1}{2}} + \beta_{\frac{29}{34}}(t) f_{n+\frac{29}{34}} + \beta_1(t) f_{n+1}\Big) \\ +h^2\Big(&\gamma_0(t) g_n + \gamma_{\frac{5}{34}}(t) g_{n+\frac{5}{34}} + \gamma_{\frac{1}{2}}(t) g_{n+\frac{1}{2}} + \gamma_{\frac{29}{34}}(t) g_{n+\frac{29}{34}} + \gamma_1(t) g_{n+1}\Big) \end{aligned}$$

Evaluating (5) at $t = t_{n+\frac{5}{34}}, t_{n+\frac{1}{2}}, t_{n+\frac{29}{34}}$ and t_{n+1}, the block algorithms are given as follows:

(6)

$$\left.\begin{aligned} x_{n+\frac{5}{34}} = x_n &+ \frac{48236526812245}{791200932938304} hf_n + \frac{5815453137955}{69329555914752} hf_{n+\frac{5}{34}} + \frac{575288125}{463713937344} hf_{n+\frac{1}{2}} \\ &- \frac{29906050595}{69329555914752} hf_{n+\frac{29}{34}} + \frac{1109311642315}{791200932938304} hf_{n+1} + \frac{181522121275}{163696744745856} h^2 g_n \\ &- \frac{23029130875}{6773577302016} h^2 g_{n+\frac{5}{34}} - \frac{12819468125}{24024798277632} h^2 g_{n+\frac{1}{2}} - \frac{4299025}{14757249024} h^2 g_{n+\frac{29}{34}} \\ &- \frac{9971114725}{163696744745856} h^2 g_{n+1} \end{aligned}\right\}$$

(7)

$$\left.\begin{aligned} x_{n+\frac{1}{2}} = x_n &- \frac{2229002917}{20486760000} hf_n + \frac{531827571502403}{2548880732160000} hf_{n+\frac{5}{34}} + \frac{269281}{1632960} hf_{n+\frac{1}{2}} - \\ &\frac{2082565315751}{509776146432000} hf_{n+\frac{29}{34}} + \frac{89027711}{4097352000} hf_{n+1} + \frac{2540591}{847728000} h^2 g_n + \\ &\frac{37985434321}{2929747968000} h^2 g_{n+\frac{5}{34}} - \frac{82801}{4976640} h^2 g_{n+\frac{1}{2}} - \frac{103649561}{21701836800} h^2 g_{n+\frac{29}{34}} - \\ &\frac{157309}{169545600} h^2 g_{n+1} \end{aligned}\right\}$$

$$\left.\begin{aligned} &x_{n+\frac{29}{34}} = x_n + \frac{2618145660737}{20275557960000}hf_n + \frac{364210994519}{1776660480000}hf_{n+\frac{5}{34}} + \frac{761804585009}{2318569686720}hf_{n+\frac{1}{2}} + \\ &\frac{214416025769}{1776660480000}hf_{n+\frac{29}{34}} + \frac{1410449179487}{20275557960000}hf_{n+1} + \frac{94010208599}{24330669552000}h^2g_n + \\ &\frac{1033435097}{59222016000}h^2g_{n+\frac{5}{34}} - \frac{12819468125}{24024798277632}h^2g_{n+\frac{1}{2}} - \frac{788317237}{37287936000}h^2g_{n+\frac{29}{34}} - \\ &\frac{68512139849}{24330669552000}h^2g_{n+1} \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} &x_{n+1} = x_n + \frac{83566921}{640211250}hf_n + \frac{1018388173679}{4978282680000}hf_{n+\frac{5}{34}} + \frac{269281}{816480}hf_{n+\frac{1}{2}} + \\ &\frac{1018388173679}{4978282680000}hf_{n+\frac{29}{34}} + \frac{83566921}{640211250}hf_{n+1} + \frac{103973}{26491500}h^2g_n \\ &+ \frac{203039551}{11444328000}h^2g_{n+\frac{5}{34}} - \frac{203039551}{11444328000}h^2g_{n+\frac{29}{34}} - \frac{103973}{26491500}h^2g_{n+1} \end{aligned}\right\} \tag{9}$$

Following [10] and [13], let the linear operator defined on the methods (6) to (9) be

$$L\left[y\left(x\right);h\right] = \sum_{j=0}^{k}\left(\alpha_j y_{n+j} - h\left\{\beta_j f_{n+j}\right\} - h^2\left\{\gamma_j g_{n+j}\right\}\right) \tag{10}$$

Expanding the terms in (10) we get

$$L\left[y\left(t\right);h\right] = E_0 y(x) + E_1 h y(t) + \cdots + E_q h^q y(t) + \cdots \tag{11}$$

where the constant E_q, $\quad q = 0, 1, 2\cdots$ are given below:

$$\left.\begin{aligned} &E_0 = \textstyle\sum_{j=0}^{k}\alpha_j \\ &E_1 = \textstyle\sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k}\beta_j \\ &E_2 = \frac{1}{2!}\textstyle\sum_{j=1}^{k}(j)^2\alpha_j - \left(\sum_{j=1}^{k} j\beta_j + \sum_{j=0}^{k}\gamma_j\right) \\ &\vdots \\ &E_q = \frac{1}{q!}\textstyle\sum_{j=1}^{k}(j)^q\alpha_j - \frac{1}{(q-1)!}\sum_{j=1}^{k} j^{q-1}\beta_j - \frac{1}{(q-2)!}\sum_{j=1}^{k} j^{q-2}\gamma_j \end{aligned}\right\} \tag{12}$$

$q = 3, 4, \cdots$

According to [12], we say the method is of order p if $E_0 = E_1 = ...E_q = 0$, $E_{q+1} \neq 0$. E_{q+1} is the error constant.

From our analysis, the methods (6) to (9) have the following order and error constants summarized in table 1.

Table 1. Order and Error Constants for the method

Equation	Order	Error constants, E_{q+1}
(6)	10	$\frac{54355055225}{882539920625142485680128}$
(7)	10	$\frac{186341}{268844974497792000}$
(8)	10	$\frac{146131455427861}{110317490078142810710016000}$
(9)	10	$\frac{186341}{134422487248896000}$

In what follows, the methods (6) to (9) can generally be written as a matrix difference equation as follows:

$$P^{(1)}Y_w = P^{(0)}Y_{w-1} + h\left[Q^{(0)}F_{w-1} + Q^{(1)}F_w\right] + h^2\left[R^{(0)}G_{w-1} + R^{(1)}G_w\right] \tag{13}$$

where

$$Y_w = \left(y_{n+\frac{5}{34}}, y_{n+\frac{1}{2}}, y_{n+\frac{29}{34}}, y_{n+1}\right)^T; \quad Y_{w-1} = \left(y_{n-\frac{29}{34}}, y_{n-\frac{1}{2}}, y_{n-\frac{29}{34}}, y_n\right)^T$$

$$F_w = \left(f_{n+\frac{5}{34}}, f_{n+\frac{1}{2}}, f_{n+\frac{29}{34}}, f_{n+1}\right)^T; \quad F_{w-1} = \left(f_{n-\frac{69}{74}}, f_{n-\frac{3}{4}}, f_{n-\frac{1}{2}}, f_{n-\frac{1}{4}}, f_{n-\frac{5}{74}}, f_n\right)^T$$

$$G_w = \left(g_{n+\frac{5}{34}}, g_{n+\frac{1}{2}}, g_{n+\frac{29}{34}}, g_{n+1}\right)^T; \quad G_{w-1} = \left(g_{n-\frac{29}{34}}, g_{n-\frac{1}{2}}, g_{n-\frac{5}{34}}, g_n\right)^T$$

and $P^{(1)}, P^{(0)}, Q^{(1)}, Q^{(0)}, R^{(1)}, R^{(0)}$ are matrices whose elements are the coefficients of the block method.

3.2. Zero stability: Let

$$P^{(1)}Y_w - P^{(0)}Y_{w-1} = 0 \tag{14}$$

given by the characteristic polynomial $\rho(\lambda)$

$$\rho(\lambda) = |\lambda P^{(1)} - P^{(0)}| = \lambda^3(\lambda - 1) \tag{15}$$

Following [10] the block method (14) is zero-stable, since from (15) $\rho(\lambda) = 0$ satisfies $|\lambda_j| \leq 1, \quad j = 1, 2, 3, \cdots$ and for the roots whose $|\lambda_j| = 1$, the multiplicity does not exceed one. With reference to [12], we can say that the block method is convergent.

3.3. Region of Absolute Stability**:** Following [4], we apply the usual test equations

$$y' = \lambda y, \qquad y'' = \lambda^2 y$$

to yield

$$Y_w = \sigma(z) Y_{w-1}, \quad z = \lambda h,$$

where the matrix $\sigma(z)$ is given by

$$\sigma(z) = \left(P^{(1)} - zQ^{(1)} - z^2R^{(1)}\right)^{-1}\left(P^{(0)} - Q^{(0)} - R^{(0)}\right) \tag{16}$$

The Matrix $\sigma(z)$has eigenvalues $\{0, 0, 0, \lambda_4\}$. And the dominant eigenvalue $\lambda_4 : \mathrm{C} \rightarrow \mathrm{C}$ is a rational function given by

$$A(z) = \frac{-B(z)}{C(z)} \tag{17}$$

where
$$B(z) = \left(\begin{array}{l} 33627190836857898777375z^8 + 23423037210385101716384 00\mathrm{z}^7 + \\ 78104482440874985839524120z^6 + 173743819107801095342166 0672z^5 \\ +25791922532600978895713002752z^4 + 26933408665644355460945215 4880z^3 \\ +18814265677057630051940322816 00z^2 + 806680383707240225936518348 8000z \\ +1613360767414480451873036697 6000 \end{array} \right)$$

$$C(z) = 1663499961502784160 \left(\begin{array}{l} 21025z^8 - 1384170z^7 + 47476092z^6 - 1038752880z^5 + 15555540960z^4 \\ -161689812480z^3 + 1131595637760z^2 - 4849296076800z \\ +9698592153600 \end{array} \right)$$

It is clear from the stability function (16) that for $Re(z) < 0$, $|\lambda_4| \leq 1$. Thus the block method is A-stable since its region of absolute stability contains the left half-plane C.

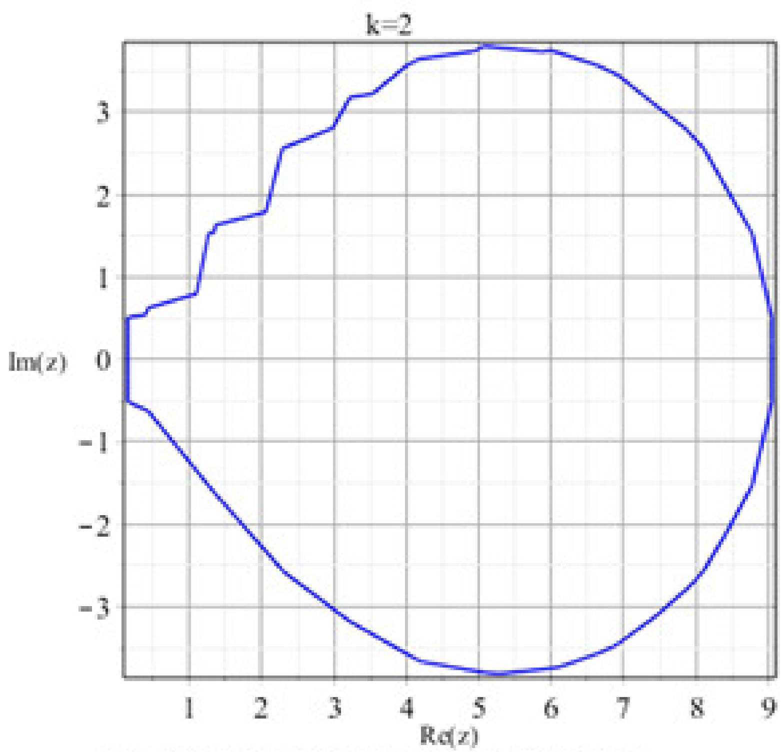

Fig. 1: Stability Region of the Block Method

2. Implementation

Our method is implemented as a block to solve some first order ODEs which requires no starting values and predictors. Initial conditions were obtained over sub-intervals $[x_0, x_1], \cdots, [x_{N-1}, x_N]$. For instance when $n = 0, (y_{\frac{5}{34}}, y_{\frac{1}{2}}, y_{\frac{29}{34}}, y_1)$ are obtained simultaneously over the sub-interval $[x_0, x_1]$, as y_0 is known from the IVP, for $n = 1, (y_{\frac{39}{34}}, y_{\frac{3}{2}}, y_{\frac{63}{34}}, y_2)$ are also obtained simultaneously over the sub-interval $[x_1, x_2]$, as y_1 is now known from the previous block, and so on. Therefore, the sub-interval $[x_n, x_{n+1}]$ do not overlap.

2.1. **Numerical experiment.** Problem 1

$x' = -x(t); \quad x(0) = 1, \quad 0 \leq t \leq 1, \quad x(t) = e^{-t}$

Table 1. Numerical results for problem 1 (h=0.1)

t	$x(t)$	$x_n(t)$	$\lvert x(t) - x_n(t)\rvert$
0.1	0.904837418035959573164249 1	0.9048374180359595731642616	$1.25*10^{-23}$
0.2	0.8187307530779818586699355	0.8187307530779818586699582	$2.27*10^{-23}$
0.3	0.7408182206817178660668738	0.7408182206817178660669046	$3.08*10^{-23}$
0.4	0.6703200460356393007444329	0.6703200460356393007444702	$3.73*10^{-23}$
0.5	0.6065306597126334236037995	0.6065306597126334236038417	$4.22*10^{-23}$
0.6	0.5488116360940264326284589	0.5488116360940264326285048	$4.59*10^{-23}$
0.7	0.4965853037914095147048001	0.4965853037914095147048485	$4.84*10^{-23}$
0.8	0.4493289641172215914301024	0.4493289641172215914301525	$5.01*10^{-23}$
0.9	0.4065696597405991118834542	0.4065696597405991118835053	$5.11*10^{-23}$
1.0	0.3678794411714423215955238	0.3678794411714423215955751	$5.13*10^{-23}$

Problem 2

$$x'(t) = tx(t); \quad x(0) = 1, \quad 0 \le t \le 1, \quad x(t) = e^{\frac{t^2}{2}}$$

Table 2. Numerical result for problem 2 (h=0.1)

t	$x(t)$	$x_n(t)$	$\lvert x(t) - x_n(t) \rvert$
0.1	1.005012520859401063383566	1.005012520859401063376291	$7.28*10^{-21}$
0.2	1.020201340026755810160144	1.020201340026755810129859	$3.03*10^{-20}$
0.3	1.046027859908716942713440	1.046027859908716942640655	$7.28*10^{-20}$
0.4	1.083287067674958554435988	1.083287067674958554294220	$1.42*10^{-19}$
0.5	1.133148453066826316829007	1.133148453066826316580299	$2.49*10^{-19}$
0.6	1.197217363121810164876824	1.197217363121810164465180	$4.12*10^{-19}$
0.7	1.277621313204886610669917	1.277621313204886610011406	$6.59*10^{-19}$
0.8	1.377127764335957084526877	1.377127764335957083494453	$1.03*10^{-18}$
0.9	1.499302500056766869653131	1.499302500056766868053147	$1.60*10^{-18}$
1.0	1.648721270700128146848651	1.648721270700128144384192	$2.46*10^{-19}$

Problem 3

$$y' = -10\,(y-1)^2\,; \qquad y\,(0) = 2, \quad 0 \le x \le 1, \ y\,(x) = 1 + \frac{1}{1+10x}$$

Table 3. Numerical result for problem 3 (h=0.1)

t	$x(t)$	$x_n(t)$	$\lvert x(t) - x_n(t) \rvert$
0.1	1.500000000000000000000000	1.500000373909081537963499	$3.746*10^{-7}$
0.2	1.333333333333333333333333	1.333333500267550197933720	$1.67*10^{-7}$
0.3	1.250000000000000000000000	1.250000093914041485721624	$9.39*10^{-8}$
0.4	1.200000000000000000000000	1.200000060105663416866361	$6.01*10^{-8}$
0.5	1.166666666666666666666667	1.166666708406771238941410	$4.17*10^{-8}$
0.6	1.142857142857142857142857	1.142857173523349616707315	$3.07*10^{-8}$
0.7	1.125000000000000000000000	1.125000023478815481783436	$2.35*10^{-8}$
0.8	1.111111111111111111111111	1.111111129662273925337306	$1.86*10^{-8}$
0.9	1.100000000000000000000000	1.100000015026441722037157	$1.50*10^{-8}$
1.0	1.090909090909090909090909	1.090909103327637645482818	$1.24*10^{-8}$

Problem 4

$x' = -10tx; \quad x(0) = 1, \quad 0 \le t \le 1, \; x(t) = e^{-5t^2}$

TABLE 4. Numerical result for problem 4 (h=0.1)

t	$x(t)$	$x_n(t)$	$\lvert x(t) - x_n(t)\rvert$
0.1	0.9512294245007140090914253	0.9512294245007074774117708	$6.53*10^{-15}$
0.2	0.8187307530779818586699355	0.8187307530779646574437491	$1.72*10^{-14}$
0.3	0.6376281516217732931437434	0.6376281516217553996559747	$1.79*10^{-14}$
0.4	0.4493289641172215914301024	0.4493289641172136055300991	$7.99*10^{-15}$
0.5	0.2865047968601901003248854	0.2865047968601914481193225	$1.35*10^{-15}$
0.6	0.1652988882215865382968047	0.1652988882215894103752785	$2.87*10^{-15}$
0.7	0.08629358649937051097207352	0.08629358649937012170943457	$3.89*10^{-16}$
0.8	0.04076220397836621516607926	0.04076220397836369630668346	$2.52*10^{-15}$
0.9	0.01742237463949351138695445	0.01742237463949164358479555	$1.87*10^{-15}$
1.0	0.006737946999085467096636048	0.006737946999085128172400253	$3.39*10^{-16}$

Problem 5

$x_1^{'} = -21x_1 + 19x_2 - 20x_3; \qquad x_1(0) = 1;$
$x_2^{'} = 19x_1 - 21x_2 + 20x_3; \qquad x_2(0) = 0;$
$x_3^{'} = 40x_1 - 40x_2 - 40x_3; \qquad x_3(0) = -1;$

$0 \leq t \leq 3$

$x_1(t) = \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-40t}\sin(40t) + \frac{1}{2}e^{-40t}\cos(40t);$
$x_2(t) = \frac{1}{2}e^{-2} - \frac{1}{2}e^{-40t}\sin(40t) - \frac{1}{2}e^{-40t}\cos(40t);$
$x_3(t) = e^{-40t}\sin(40t) - e^{-40t}\cos(40t).$

Table 5. Numerical result for problem 5a (h=0.2)

t	$x_1(t)$	$x_1(t_n)$	$\|x_1(t) - x_1(t_n)\|$
0.2	0.33530156446464362999698836	0.33530162840318170718097330	$5.27*10^{-5}$
0.4	0.22466441197379730426290920	0.22466441195286501517723620	$2.80*10^{-7}$
0.6	0.15059710594701431165545390	0.15059710593611725635108770	$9.77*10^{-11}$
0.8	0.10094825899733647829335050	0.10094825898761510605507270	$1.18*10^{-13}$
1.0	0.0676676416183063461130575	0.0676676416101608036584710	$9.12*10^{-17}$
1.2	0.0453589766447062516865825	0.0453589766381541072123331	$1.59*10^{-17}$
1.4	0.0304050313126089824978107	0.0304050313074849430813272	$1.24*10^{-17}$
1.6	0.0203811019891831075830396	0.0203811019852576831973204	$9.53*10^{-18}$
1.8	0.0136618612236462804007815	0.0136618612206860784140044	$7.19*10^{-18}$
2.0	0.00915781944436709014685901	0.00915781944216233155571750	$5.35*10^{-18}$
2.2	0.00613866995153422058946969	0.00613866994990853732114670	$3.95*10^{-18}$
2.4	0.00411487352451001442068101	0.00411487352332122014804627	$2.89*10^{-18}$
2.6	0.00275828221038038620899687	0.00275828220951710752481790	$2.10*10^{-18}$
2.8	0.00184893185824146541034631	0.00184893185761827909477796	$1.51*10^{-18}$
3.0	0.00123937608833317921152258	0.00123937608788560676894880	$1.09*10^{-18}$

TABLE 6. Numerical result for problem 5b (h=0.2)

t	$x_2(t)$	$x_2(t_n)$	$\lvert x_2(t) - x_2(t_n)\rvert$
0.2	0.3353503742883449718086651	0.3350184176163195391996491	$3.85*10^{-4}$
0.4	0.2246645197441742440098859	0.2246645521427212535649763	$3.13*10^{-7}$
0.6	0.1505971059310009834198948	0.1505971059543309545439383	$6.35*10^{-11}$
0.8	0.1009482589973259135506081	0.1009482589875975481671902	$1.25*10^{-13}$
1.0	0.06766764161830634894644666	0.06766764161016080332000290	$1.28*10^{-16}$
1.2	0.04535897664470625168749482	0.04535897663815410721434232	$1.59*10^{-17}$
1.4	0.03040503131260898249781037	0.03040503130748494308132687	$1.24*10^{-17}$
1.6	0.02038110198918310758303963	0.02038110198525768319732027	$9.53*10^{-18}$
1.8	0.01366186122364628040078153	0.01366186122068607841400428	$7.19*10^{-18}$
2.0	0.009157819444367090146859010	0.009157819442162331555717491	$5.35*10^{-18}$
2.2	0.006138669951534220589469695	0.006138669949908537321146664	$3.95*10^{-18}$
2.4	0.004114873524510014420681014	0.004114873523321220148046246	$2.89*10^{-18}$
2.6	0.002758282210380386208996878	0.002758282209517107524817879	$2.10*10^{-18}$
2.8	0.001848931858241465410346318	0.001848931857618279094777953	$1.51*10^{-18}$
3.0	0.001239376088333179211522584	0.001239376087885606768948792	$1.09*10^{-18}$

Problem 6.

$x_1' = -1002x_1 + 100x_2^2; \quad x_1(0) = 1;$
$x_2' = x_1 - x_2(1 + x_2); \quad x_2(0) = 1;$
$x_1(t) = e^{-2t}; \quad x_2(t) = e^{-t};$
$0 \le t \le 1$

Table 7. Numerical result for problem 6a (h=0.02)

t	$x_1(t)$	$x_1(t_n)$	$\|x_1(t) - x_1(t_n)\|$
0.1	0.81873075307798185866993550861903	0.81873075307798185866993550864216	2.31*10^{-29}
0.2	0.67032004603563930074443292514782	0.67032004603563930074443292517201	2.42*10^{-29}
0.3	0.54881163609402643262845891723256	0.54881163609402643262845891725640	2.38*10^{-29}
0.4	0.44932896411722159143010238501556	0.44932896411722159143010238503819	2.26*10^{-29}
0.5	0.36787944117144232159552377016146	0.36787944117144232159552377018239	2.09*10^{-29}
0.6	0.30119421191220209664497760708322	0.30119421191220209664497760710222	1.90*10^{-29}
0.7	0.24659696394160647693986123983376	0.24659696394160647693986123985077	1.70*10^{-29}
0.8	0.20189651799465540848517926764334	0.20189651799465540848517926765840	1.51*10^{-29}
0.9	0.16529888822158653829680472043221	0.16529888822158653829680472044541	1.32*10^{-29}
1.0	0.13533528323661269189399949497248	0.13533528323661269189399949498398	1.15*10^{-29}

Table 8. Numerical result for problem 6b (h=0.02)

t	$x_2(t)$	$x_2(t_n)$	$\|x_2(t) - x_2(t_n)\|$
0.1	0.90483741803595957316424905944643	0.90483741803595957316424905945019	3.76*10^{-30}
0.2	0.81873075307798185866993550861903	0.81873075307798185866993550862564	6.61*10^{-30}
0.3	0.74081822068171786606687377931781	0.74081822068171786606687377932652	8.71*10^{-30}
0.4	0.67032004603563930074443292514782	0.67032004603563930074443292515802	1.02*10^{-29}
0.5	0.60653065971263342360379953499118	0.60653065971263342360379953500239	1.12*10^{-29}
0.6	0.54881163609402643262845891723256	0.54881163609402643262845891724440	1.18*10^{-29}
0.7	0.49658530379140951470480009339752	0.49658530379140951470480009340970	1.22*10^{-29}
0.8	0.44932896411722159143010238501556	0.44932896411722159143010238502783	1.23*10^{-29}
0.9	0.40656965974059911188345423964562	0.40656965974059911188345423965780	1.22*10^{-29}
1.0	0.36787944117144232159552377016146	0.36787944117144232159552377017341	1.20*10^{-29}

Problem 7

$x' = -2\pi \sin(2\pi t) - \frac{1}{10^{-3}}(x - \cos(2\pi t));$ $x(t) = \cos(2\pi t);$
$x(0) = 1, 0 \le t \le 1$

Table 9. Numerical result for problem 7 (h=0.1)

t	$x(t)$	$x_n(t)$	$\lvert x(t) - x_n(t)\rvert$
0.1	0.80901699437494742241022935	0.80901699437494740446196635	1.96*10^{-17}
0.2	0.30901699437494742241022938	0.30901699437494736774798 71	5.64*10^{-17}
0.3	-0.30901699437494742241022934	-0.30901699437494750228633 66	7.82*10^{-17}
0.4	-0.80901699437494742241022930	-0.80901699437494749604711 13	7.19*10^{-17}
0.5	-1.00000000000000000000000000	-1.00000000000000003871011 0	3.87*10^{-17}
0.6	-0.80901699437494742241022935	-0.80901699437494741492281 14	9.18*10^{-18}
0.7	-0.30901699437494742241022941	-0.30901699437494737057487 96	5.35*10^{-17}
0.8	0.30901699437494742241022921	0.30901699437494750152240 91	7.42*10^{-17}
0.9	0.80901699437494742241022934	0.80901699437494749584067 64	7.17*10^{-17}
1.0	1.00000000000000000000000000	1.00000000000000003865431 6	3.87*10^{-17}

Problem 8

$x^{'} = 5e^{5t}(x-t)^2+1; \quad x(0)=-1;$

$x(t) = t - e^{-5t}; \quad 0 \le t \le 1$

TABLE 10. Numerical result for problem 8 (h=0.1)

t	$x(t)$	$x_n(t)$	$\|x(t)-x_n(t)\|$
0.1	-0.5065306597126334236037995	-0.5065306597126337531401969	$3.30*10^{-16}$
0.2	-0.1678794411714423215955238	-0.1678794411714426426991180	$3.21*10^{-16}$
0.3	0.0768698398515701710667195	0.07686983985156993170964293	$2.39*10^{-16}$
0.4	0.2646647167633873081060005	0.2646647167633871465219439	$1.62*10^{-16}$
0.5	0.4179150013761012048304713	0.4179150013761011007891173	$1.04*10^{-16}$
0.6	0.5502129316321360570206576	0.5502129316321359916959877	$6.53*10^{-17}$
0.7	0.6698026165776814992602137	0.6698026165776814588219596	$4.04*10^{-17}$
0.8	0.7816843611112658197062820	0.7816843611112657948787427	$2.48*10^{-17}$
0.9	0.8888910034617576935038569	0.8888910034617576783346459	$1.52*10^{-17}$
1.0	0.9932620530009145329033640	0.9932620530009145236621044	$9.24*10^{-17}$

Problem 9

$x_1' = -x_1 + 95x_2; \quad x(0) = 1;$
$x_2 = -x_1 - 97x_2; \quad x_2(0) = 1;$
$x_1(t) = \frac{95}{47}e^{-2t} - \frac{48}{47}e^{-96t};$
$x_2(t) = \frac{48}{47}e^{-96t} - \frac{1}{47}e^{-2t};$

TABLE 11. Numerical result for problem 9a (h=0.0625)

t	$x_1(t)$	$x_1(t_n)$	$\lvert x_1(t) - x_1(t_n)\rvert$
0.0625	1.781238843426735703571460	1.781235081308101235617369	$3.76*10^{-6}$
0.1250	1.574165520629585164451548	1.574165501965006985049470	$1.87*10^{-8}$
0.1875	1.389201718172411300331660	1.389201718102962425280066	$6.94*10^{-11}$
0.2500	1.225966227040172567257595	1.225966227039942867885903	$2.30*10^{-13}$
0.3125	1.081911398070203858018172	1.081911398070203145780032	$7.12*10^{-16}$
0.3750	0.9547834576680082137107413	0.9547834576680082115915321	$2.12*10^{-18}$
0.4375	0.8425934440310276216596920	0.8425934440310276216545069	$5.19*10^{-21}$
0.5000	0.7435861044954685223724756	0.7435861044954685223734167	$9.41*10^{-22}$
0.5625	0.6562124340221962623557724	0.6562124340221962623567239	$9.52*10^{-22}$
0.6250	0.5791054404620863729971088	0.5791054404620863729980418	$9.33*10^{-22}$
0.6875	0.5110587574776790508945492	0.5110587574776790508954549	$9.06*10^{-22}$
0.7500	0.4510077705127836967800351	0.4510077705127836967809070	$8.72*10^{-22}$
0.8125	0.3980129605191156331026058	0.3980129605191156331034394	$8.34*10^{-22}$
0.8750	0.3512452048466444049929392	0.3512452048466444049937312	$7.92*10^{-22}$
0.9375	0.3099728053248554045335878	0.3099728053248554045343368	$7.49*10^{-22}$
1.0000	0.2735500405846426751048926	0.2735500405846426751055977	$7.05*10^{-22}$

TABLE 12. Numerical result for problem 9b (h=0.0625)

t	$x_2(t)$	$x_2(t_n)$	$\|x_2(t) - x_2(t_n)\|$
0.0625	-0.01624503825754489784167498	-0.01624127613891042988730028	3.76*10^{-6}
0.1250	-0.01656395448677542796120752	-0.01656393582219724855862798	1.87*10^{-8}
0.1875	-0.01462316059046690324125611	-0.01462316052101802818899741	6.94*10^{-11}
0.2500	-0.01290490761490572004999644	-0.01290490761467602067752201	2.30*10^{-13}
0.3125	-0.01138854103222337410492722	-0.01138854103222266186592367	7.12*10^{-16}
0.3750	-0.01005035218597879943464812	-0.01005035218597879731452526	2.12*10^{-18}
0.4375	-0.008869404674010816489097666	-0.00886940467401081648297194	6.13*10^{-21}
0.5000	-0.007827222152583879181427998	-0.00782722215258387918142069	7.30*10^{-24}
0.5625	-0.006907499305496802761636139	-0.00690749930549680276164610	9.97*10^{-24}
0.6250	-0.006095846741706172347337978	-0.00609584674170617234734780	9.82*10^{-24}
0.6875	-0.005379565868186095272574202	-0.00537956586818609527258373	9.53*10^{-24}
0.7500	-0.004747450215924038913474053	-0.00474745021592403891348323	9.18*10^{-24}
0.8125	-0.004189610110727532980027430	-0.00418961011072753298003620	8.78*10^{-24}
0.8750	-0.003697317945754151631504623	-0.00369731794575415163151296	8.34*10^{-24}
0.9375	-0.003262871634998477942458819	-0.00326287163499847794246670	7.88*10^{-24}
1.0000	-0.002879474111417291316893606	-0.00287947411141729131690102	7.42*10^{-24}

Problem 10

$x^{'} = -\frac{x^3}{2}; \quad x(0) = 1;$

$x(t) = \frac{1}{\sqrt{1+t}}; \quad ;$

$0 \le t \le 4;$

TABLE 13. Numerical result for problem 10 (h=0.1)

t	x(t)	$x_n(t)$	$\|x(t) - x_n(t)\|$
0.4	0.8451542547285165775096181	0.8451542547285166351186796	5.76*10^{-17}
0.8	0.7453559924999298988030580	0.7453559924999299401460599	4.13*10^{-17}
1.2	0.6741998624632420862464909	0.6741998624632421169755264	3.07*10^{-17}
1.6	0.6201736729460422809512778	0.6201736729460423048849520	2.39*10^{-17}
2.0	0.5773502691896257645091489	0.5773502691896257838220228	1.93*10^{-17}
2.4	0.5423261445466404300001337	0.5423261445466404460077153	1.60*10^{-17}
2.8	0.5129891760425770477283374	0.5129891760425770612762802	1.35*10^{-17}
3.2	0.4879500364742665896771923	0.4879500364742666013365981	1.17*10^{-17}
3.6	0.4662524041201568828180069	0.4662524041201568929901913	1.02*10^{-17}
4.0	0.4472135954999579392818347	0.4472135954999579482581112	8.98*10^{-18}

Conclusions: One-Step Second Derivative Block intra-point Method for integrating general first order initial value problems of ordinary differential equations is proposed in this paper. The method which is found to be a uniform order of 10 and A-stable is implemented in block form to simultaneously solve IVPs over a partitioned interval of integration. We have demonstrated the efficiency of the method on ten numerical examples (which include linear and non-linear IVPs) and it competes favourably with the exact solution.

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