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On Jacobi Spectral Polynomials and Evaluation of Definite Integrals

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Abstract

In this paper, we construct a new class of spectral polynomials associated with the product $\mathsf{P}_{k,m}^{(\alpha,\beta;\gamma,\delta)} := \mathscr{P}_k^{(\alpha,\beta)} \times \mathscr{P}_m^{(\gamma,\delta)}$ $(k, m \geq 0;\ \alpha, \beta, \gamma, \delta > -1)$, where $\mathscr{P}_j^{(\nu,\mu)}$ are the normalised Jacobi polynomials with $j \geq 0; \nu, \mu > -1$. These spectral polynomials are then used to evaluate certain definite integrals involving the combination of logarithmic, exponential and trigonometric functions. These integrals, which are interesting in their own right, are expressed explicitly as polynomials in the eigenvalues $\lambda_k^{\alpha,\beta}$ and $\lambda_m^{\gamma,\delta}$, where $\lambda_j^{\nu,\mu} := j(j + \nu + \mu + 1), j \geq 0$.

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1. Introduction

Orthogonal polynomials have proved useful in spectral and pseudo-spectral approximations of solutions of differential equations and they also play important roles in mathematical analysis and its applications; for examples, in interpolation, Gaussian quadrature processes, least square approximation of functions, differential equations, difference equations, Fourier series and polynomial chaos expansions (see, e.g., [9], [12], [13], [14], [15], [26]). It is a well-known fact ([8], [30]) that the spherical functions (normalised eigenfunctions) of the Laplacian on

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compact symmetric spaces of rank one are explicitly described by the normalised Jacobi polynomials $\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta) := P_k^{(\alpha,\beta)}(\cos\theta)/P_k^{(\alpha,\beta)}(1)$ with α,β depending only on the dimension of these spaces. In spectral methods and approximations, the coefficients of classical orthogonal polynomials are used in the construction of filtered kernels such that the underlying kernel is exponentially localised (see [17], [21], [23]). Jacobi polynomials are therefore indispensable tools in the study of Sturm-Liouville problems which arise frequently from science and engineering applications. Jacobi polynomials are also key in the construction of Jacobi coefficients that describe the power series expansion of the heat kernels on rank one symmetric spaces of compact type (see [5], [6], [7]).

In applications, solutions of differential equations can be formulated in the series of Jacobi polynomials. To see this, a smooth function $f(t)$ defined on the closed interval $-1 \le t \le 1$ can be expressed in the series of Jacobi polynomials:

$$f(t) = \sum_{j=0}^{\infty} a_j P_j^{(\alpha,\beta)}(t), \tag{1}$$

where the expansion coefficients a_j are given by the integral formula

$$a_j = \left(\eta_j^{\alpha,\beta}\right)^{-1} \int_{-1}^{1} f(t)(1-t)^{\alpha}(1+t)^{\beta} P_j^{(\alpha,\beta)}(t)\,dt, \tag{2}$$

with $\eta_j^{\alpha,\beta}$ given by (10). In solving boundary value problems, one may assume a trial solution of the form (1). The coefficients a_j decay asymptotically faster than any algebraic power of j; this means that $a_j = O\left(j^{-m}\right)$ for any integer m as $j \nearrow \infty$ ([19]). In spectral methods, one is interested in the spectral approximation of eigenfunctions of a suitable Sturm-Liouville problem. Such expansions and their applications in solving boundary value problems have been considered in [18] and [19] in the special cases of Chebyshev polynomials of the first kind and second kind respectively ($\alpha = \beta = -1/2, \alpha = \beta = 1/2$). In spectral collocation and Galerkin methods, the coefficients appearing in the differented equation:

$$\frac{d^m}{dt^m} f(t) = \sum_{j=0}^{\infty} b_j^{(m)} P_j^{(\alpha,\beta)}(t), \qquad m \ge 1, \tag{3}$$

are often needed; such coefficients $b_j^{(m)}$ are called differentiated coefficients, with $b_j^{(0)} = a_j$. Explicit formulae for the coefficients $b_j^{(m)}$ in terms of the original coefficients a_j have been presented by notable authors, namely, in [22], explicit formulae for $b_j^{(m)}$ in the special case of Chebyshev polynomials were constructed, while [25] presented explicit formulae for the differentiated coefficient $b_j^{(m)}$ in the case of Legendre polynomials; note that in the case of Legendre polynomials, $\alpha = \beta = 0$. Following [18], instead of differentiating the trial solution (1) m

times to obtain a required solution of the associated sturm-Liouville problem, it is found convenient to reduce the differentiated equation (3) by integrating m times to recover the trial solution $f(t)$ defined by (1); the resulting equations, which are algebraic systems then contain a finite number of terms which must be solved. The authors in [26], in the case of ultraspherical polynomials or Gegenbauer polynomials ($\alpha = \beta$), presented a formula for the coefficients $b_j^{(m)}$ associated with the differentiated function $f(t)$ in terms of the original coefficients a_j of which the Chebyshev polynomials of the first, second, third & fourth kinds and Legendre polynomials are important special cases. It was shown in [12] (see also [13]) that the formula derived in [26] could be rederived in a simpler and compact form, when the underlying polynomials are ultraspherical or Gegenbauer polynomials. Explicit expressions for $b_j^{(m)}$ in the special case of Chebyshev polynomials of the third and fourth kinds were considered in [15]. In [14], solutions of ordinary differential equations with varying coefficients were obtained via the differentiated trial solution (3). It was shown in [24], a closed form formula for the coefficients of all classes of orthogonal polynomials whose recurrence relation admits a prescribed form; such orthogonal polynomials include Jacobi polynomials (of which Chebyshev, Legendre and Gegenbauer polynomials are notable special cases), Hermite and Laguerre polynomials. The expression of the expansion coefficients of Gegenbauer polynomials as Fourier coefficients of integral transform of the function $f(t)$ was discussed in [10].

A central result in the Maclaurin spectral analysis of the Laplacian on rank one compact symmetric spaces, as discussed in [5], is a spectral identity that expresses the higher (even (2ℓ)th, $\ell \geq 1$) derivatives of the normalised Jacobi polynomials $\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)$, $k \geq 0; \alpha, \beta > -1$ (evaluated at $\theta = 0$) as a linear combination of the mth derivatives of $\mathscr{P}_k^{(\alpha,\beta)}(t)$ (evaluated at $t = 1$):

$$\frac{d^{2\ell}}{d\theta^{2\ell}} P_k^{(\alpha,\beta)}(\cos\theta)\bigg|_{\theta=0} = \sum_{m=1}^{\ell} a_m^\ell \frac{d^m}{dt^m} P_k^{(\alpha,\beta)}(t)\bigg|_{t=1}, \tag{4}$$

where $\left(a_m^\ell : 1 \leq m \leq \ell\right)$ are integer coefficients. The explicit formula for these coefficients are given in (45). The derivatives in (4), upon further simplification tranforms into a ℓth-degree polynomial $\widetilde{\mathscr{R}}_\ell^{(\alpha,\beta)}(\lambda_k) := \widetilde{R}(\lambda_k^{\alpha,\beta})$ whose coefficients are the usual Jacobi coefficients $c_j^\ell(\alpha, \beta)$ $(1 \leq j \leq \ell;\ \alpha, \beta > -1)$; these coefficients depend only on α and β. Here the scalars $\lambda_k^{\alpha,\beta} = k(k + \alpha + \beta + 1)$ $(k \geq 0)$ are the eigenvalues of the Laplacian on rank one symmetric spaces of compact type. The introductory applications of the Jacobi coefficients in the description of the Maclaurin spectral functions on compact symmetric spaces of rank one appeared in ([5]). See [7] for the explicit calculation of $c_j^\ell(\alpha, \beta)$ $(1 \leq j \leq \ell \leq 4)$ using the derivative formula (11). In [4], we applied the Leibniz rule of derivatives to the Jacobi differential equation (7) in constructing a recursion formula that expressed $\frac{d^m}{dt^m} P_k^{(\alpha,\beta)}(t)|_{t=1}$ as a mth-degree polynomial in the eigenvalue $\lambda_k^{\alpha,\beta} = k(k+\alpha+\beta+1)$.

In this paper, we introduce and construct a new class of spectral polynomials associated with the products $\mathsf{P}_{k,m}^{(\alpha,\beta;\gamma,\delta)} := \mathscr{P}_k^{(\alpha,\beta)} \times \mathscr{P}_m^{(\gamma,\delta)}$ $(k, m \geq 0;\ \alpha, \beta, \gamma, \delta > -1)$ with the newly introduced *product Jacobi coefficients*. These spectral polynomials are then used to evaluate certain definite integrals involving logarithmic, exponential and trigonometric functions. These integrals which are interesting in their own right are expressed explicitly as polynomials in the eigenvalues $\lambda_k^{\alpha,\beta}$ and $\lambda_m^{\gamma,\delta}$. It is believed that the integral formulae presented in this paper will be useful to experts specialising in special functions. As discussed in [5], the Jacobi coefficients $c_j^\ell(\alpha,\beta)$ play a significant role in the description of the heat coefficients associated with rank one symmetric spaces of compact type and by extension, are useful in the expressions of power series coefficients of eigenfunctions of Sturm-Liouville problems. The product Gegenbauer polynomials $\mathsf{C}_{k,m}^{\nu,\rho} := \mathscr{C}_k^{\nu} \times \mathscr{C}_m^{\rho}$ $(k, m \geq 0;\ \nu, \rho > -1/2)$ are an important special case. For the evaluation of definite integrals via the spectral polynomials associated with the product $\mathsf{C}_{k,m}^{\nu}$, see [3].

2. The Jacobi, Gegenbauer and Laguerre Polynomials

The Jacobi polynomials $P_k^{(\alpha,\beta)} = P_k^{(\alpha,\beta)}(t)$ $(k \geq 0$, real $\alpha, \beta > -1)$ are defined by the generating function

$$2^{\alpha+\beta} R^{-1} (1-z+R)^{-\alpha} (1+z+R)^{-\beta} = \sum_{k=0}^{\infty} P_k^{(\alpha,\beta)}(t) z^k, \tag{5}$$

where $R = \sqrt{1-2tz+z^2}$, $|z| < 1$.

It is seen that $P_k^{(\alpha,\beta)}(t)$ is a degree k polynomial admitting the truncated series representation

$$P_k^{(\alpha,\beta)}(t) = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+\beta+1)} \sum_{m=0}^{k} \binom{k}{m} \frac{\Gamma(k+\alpha+\beta+m+1)}{2^m \Gamma(\alpha+m+1) k!} (t-1)^m. \tag{6}$$

The Jacobi polynomial $y = P_k^{(\alpha,\beta)}(t)$ satisfies the second-order differential equation

$$\left(1-t^2\right) \frac{d^2 y}{dt^2} - (\alpha - \beta + (\alpha+\beta+2)t) \frac{dy}{dt} + k(k+\alpha+\beta+1) y = 0, \tag{7}$$

which in turn constitutes a regular Sturm-Liouville system, where the associated Jacobi operator is a non-negative, self-adjoint, second-order, linear, differential operator in the Hilbert space $L^2[-1,1;(1-t)^\alpha(1+t)^\beta dt]$. The spectrum of the Jacobi operator is discrete and given by the sequence of eigenvalues and eigenfunctions

$$\lambda_k^{\alpha,\beta} = k(k+\alpha+\beta+1), \qquad y = P_k^{(\alpha,\beta)}(t), \qquad k \geq 0. \tag{8}$$

As an orthogonal polynomial the Jacobi polynomial satisfies the orthogonality relations

$$\int_{-1}^{1} P_k^{(\alpha,\beta)}(t) P_m^{(\alpha,\beta)}(t) \, (1-t)^{\alpha} \, (1+t)^{\beta} \; dt = \eta_k^{\alpha,\beta} \delta_{k,m}, \;\; k, m \geq 0, \tag{9}$$

where the scalars $\eta_k^{\alpha,\beta}$ on the right are given by

$$\eta_k^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}}{2k+\alpha+\beta+1} \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}. \tag{10}$$

The Jacobi polynomials satisfy the differential formula

$$\frac{d^m}{dt^m} P_k^{(\alpha,\beta)}(t) = \frac{1}{2^m} \frac{\Gamma(k+m+\alpha+\beta+1)}{\Gamma(k+\alpha+\beta+1)} P_{k-m}^{(\alpha+m,\beta+m)}(t), \qquad m \geq 1, \tag{11}$$

with the reflection symmetry and pointwise identities

$$P_k^{(\alpha,\beta)}(-t) = (-1)^k P_k^{(\beta,\alpha)}(t), \quad P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k+\alpha+1)}{k!\Gamma(\alpha+1)}. \tag{12}$$

We note that in the sequel it is often convenient to use the normalised form of the Jacobi polynomials

$$\mathscr{P}_k^{(\alpha,\beta)}(t) := \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\alpha,\beta)}(1)} = \frac{k!\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} P_k^{(\alpha,\beta)}(t), \tag{13}$$

and as a result of this normalisation, $\mathscr{P}_k^{(\alpha,\beta)}(1) = 1$. It is also worth noting that

$$\mathscr{P}_k^{(\alpha,\beta)}(-t) = \mathscr{P}_k^{(\beta,\alpha)}(t). \tag{14}$$

As a notable special case, the Gegenbauer polynomial $y = C_k^{\nu}(t), \; k \geq 0, \nu > -1/2$, satisfies the second-order homogenous differential equation

$$\left(1-t^2\right) \frac{d^2y}{dt^2} - (2\nu+1)t\frac{dy}{dt} + k(k+2\nu)y = 0, \tag{15}$$

with discrete spectrum given by the sequence of eigenvalues and eigenfunctions

$$\widetilde{\lambda}_k^{\nu} := \lambda_k^{\nu-1/2,\nu-1/2} = k(k+2\nu), \qquad y = C_k^{\nu}(t), \qquad k \geq 0. \tag{16}$$

The Jacobi, Gegenbauer and Legendre polynomials are related to one another for a suitable choice of (α, β) parameters:

$$P_k^{(0,0)}(t) = C_k^{1/2}(t) = P_k(t), \quad C_k^{\nu}(t) = \frac{\Gamma(k+2\nu)\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(2\nu)\Gamma\left(k+\nu+\frac{1}{2}\right)} P_k^{\left(\nu-\frac{1}{2},\nu-\frac{1}{2}\right)}(t). \tag{17}$$

The Gegenbauer polynomials $C_k^\nu(t)$ satisfy the derivative formula

$$\frac{d^m}{dt^m} C_k^\nu(t) = 2^m \frac{\Gamma(\nu+m)}{\Gamma(\nu)} C_{k-m}^{\nu+m}(t). \tag{18}$$

The normalised Gegenbauer polynomials $\mathscr{C}_k^\nu(t)$ are given in terms of the normalised Jacobi polynomials by

$$\mathscr{C}_k^\nu(t) = \mathscr{P}_k^{(\nu-1/2,\nu-1/2)}(t). \tag{19}$$

The Laguerre polynomial of order α and degree k in t, is defined by

$$L_k^\alpha(t) = \sum_{j=0}^{k} (-1)^j \binom{k+\alpha}{k-j} \frac{t^j}{j!}, \quad k = 0, 1, 2, \dots, \tag{20}$$

with the special values

$$L_0^\alpha(t) = 1, \qquad L_k^\alpha(0) = \binom{k+\alpha}{k}. \tag{21}$$

The Laguerre polynomial $y = L_k^\alpha(t)$ satisfies the second-order differential equation

$$t\frac{d^2y}{dt^2} + (\alpha - t + 1)\frac{dy}{dt} + ky = 0. \tag{22}$$

As an orthogonal polynomial the Laguerre polynomial satisfies the orthogonality relation

$$\int_0^\infty L_k^\alpha(t) L_m^\alpha(t) e^{-t} t^\alpha \, dt = \Gamma(\alpha+1)\binom{k+\alpha}{k}\delta_{k,m}, \ k, m \geq 0, \tag{23}$$

and the limit relation

$$L_k^\alpha(t) = \lim_{\beta \nearrow \infty} P_k^{(\alpha,\beta)}\left(1 - \frac{2t}{\beta}\right). \tag{24}$$

Similarly, the Laguerre polynomials satisfy the differential formula

$$\frac{d^m}{dt^m} L_k^\alpha(t) = (-1)^m L_{k-m}^{\alpha+m}(t), \qquad m \geq 1. \tag{25}$$

For more information and further reading on this scale of orthogonal polynomials, the interested reader is referred to [2], [20] and [29].

3. Product Jacobi Coefficients

In this section, we prove a spectral identity relating the (2ℓ)th differential action on the product $\mathsf{P}_{k,m}^{(\alpha,\beta;\gamma,\delta)}(\cos\theta) := \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\mathscr{P}_m^{(\gamma,\delta)}(\cos\theta)$ (evaluated at $\theta = 0$) to the spectral polynomials in the eigenvalues $\lambda_k^{\alpha,\beta}$ and $\lambda_k^{\gamma,\delta}$ whose non zero coefficients are the product Jacobi coefficients. Applications of the resulting spectral polynomials in the explicit evaluation of certain definite integrals will be discussed in the next section.

We start with the following known result.

Proposition 3.1 (Jacobi coefficients [6], [7]). *Consider the normalised Jacobi polynomials* $\mathscr{P}_k^{(\alpha,\beta)}$ *with* $k \geq 0; \alpha, \beta > -1$. *Then for any integer* $\ell \geq 1$ *we have*

$$\widetilde{\mathscr{R}}_\ell^{(\alpha,\beta)}(\lambda_k) := \frac{d^{2\ell}}{d\theta^{2\ell}} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0} = \sum_{j=1}^{\ell} c_j^\ell(\alpha,\beta)[\lambda_k^{\alpha,\beta}]^j. \tag{26}$$

The numbers $(c_j^\ell(\alpha,\beta) : 1 \leq j \leq \ell)$ *are constant coefficients, called the Jacobi coefficients,* $(\lambda_k^{\alpha,\beta} : k \geq 0)$ *are the eigenvalues of the Jacobi operator from* (7)-(8), *and* $\widetilde{\mathscr{R}}_\ell^{(\alpha,\beta)}(X)$ *are* ℓ*-degree polynomials in* X.

Note that all the Jacobi coefficients considered in this paper satisfy the property that $d_0^m = 1$ *for* $m = 0$ *and* $d_0^m = 0$ *for* $m \geq 1$, *i.e.,* $d_0^m = \delta_{0m}$.

Remark. The first Jacobi coefficients $c_j^\ell(\alpha,\beta)$ are given below.

(27)

$$c_1^1(\alpha,\beta) = -\frac{1}{2(\alpha+1)},\quad c_1^2(\alpha,\beta) = -\frac{\alpha+3\beta+2}{4(\alpha+1)(\alpha+2)},\quad c_2^2(\alpha,\beta) = \frac{3}{4(\alpha+1)(\alpha+2)}$$

$$c_1^3(\alpha,\beta) = -\frac{4\alpha^2+30\alpha\beta+30\beta^2+20\alpha+60\beta+24}{8(\alpha+1)(\alpha+2)(\alpha+3)}$$

$$c_2^3(\alpha,\beta) = \frac{15(\alpha+3\beta+2)}{8(\alpha+1)(\alpha+2)(\alpha+3)},\quad c_3^3(\alpha,\beta) = -\frac{15}{8(\alpha+1)(\alpha+2)(\alpha+3)}$$

$$c_1^4(\alpha,\beta) = -\frac{34\alpha^3+306\alpha^2+884\alpha+816}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} - \frac{630\beta^3+2310\beta^2+2604\beta+462\alpha^2\beta+1050\alpha\beta^2+2184\alpha\beta}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}$$

$$c_2^4(\alpha,\beta) = \frac{147\alpha^2+714\alpha+924+1155\beta^2+2310\beta+1050\alpha\beta}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}$$

$$c_3^4(\alpha,\beta) = -\frac{210\alpha+630\beta+420}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}$$

$$c_4^4(\alpha,\beta) = \frac{105}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}.$$

For the explicit calculations of these coefficients, see [7].

Now, let $\alpha, \beta, \gamma, \delta > -1$ and $k, m \geq 0$. Consider the differential action

$$\frac{d^{2\ell}}{d\theta^{2\ell}} \mathsf{P}_{k,m}^{(\alpha,\beta;\gamma,\delta)}(\cos\theta)\Big|_{\theta=0} := \frac{d^{2\ell}}{d\theta^{2\ell}}\left[\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\mathscr{P}_m^{(\gamma,\delta)}(\cos\theta)\right]\Big|_{\theta=0}. \tag{28}$$

Proposition 3.2 (Spectral polynomials)**.** *For any integer $\ell \geq 1$, the normalised product Jacobi polynomials $\mathsf{P}_{k,m}^{(\alpha,\beta;\gamma,\delta)}(\cos\theta)$ with $k, m \geq 0; \alpha, \beta, \gamma, \delta > -1$, satisfy the spectral relation*

$$\begin{aligned}\mathscr{R}_\ell^{(\alpha,\beta;\gamma,\delta)}(\lambda_k, \lambda_m) &:= \frac{d^{2\ell}}{d\theta^{2\ell}} \mathsf{P}_{k,m}^{(\alpha,\beta;\gamma,\delta)}(\cos\theta)\Big|_{\theta=0} \\ &= \sum_{p=0}^{\ell} \binom{2\ell}{2p} \sum_{i=0}^{\ell-p} \sum_{j=0}^{p} c_i^{\ell-p}(\alpha,\beta) c_j^p(\delta,\gamma) [\lambda_k^{\alpha,\beta}]^i [\lambda_m^{\gamma,\delta}]^j. \end{aligned} \tag{29}$$

The scalars $(c_j^\ell(\alpha,\beta) : 1 \leq j \leq \ell)$ are the usual Jacobi coefficients and $\mathscr{R}_\ell^{(\alpha,\beta;\gamma,\delta)}(X, Y)$ are ℓ-degree polynomials in X and Y.

Proof. By Leibniz rule of differentiation, we have

$$\begin{aligned}\frac{d^{2\ell}}{d\theta^{2\ell}} \mathsf{P}_{k,m}^{(\alpha,\beta;\gamma,\delta)}(\cos\theta)\Big|_{\theta=0} &= \frac{d^{2\ell}}{d\theta^{2\ell}} \left[\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta) \mathscr{P}_m^{(\gamma,\delta)}(\cos\theta)\right]\Big|_{\theta=0} \\ &= \sum_{r=0}^{2\ell} \binom{2\ell}{r} \frac{d^{2\ell-r}}{d\theta^{2\ell-r}} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta) \frac{d^r}{d\theta^r} \mathscr{P}_m^{(\gamma,\delta)}(\cos\theta)\Big|_{\theta=0} \\ &= \sum_{p=0}^{\ell} \binom{2\ell}{2p} \frac{d^{2\ell-2p}}{d\theta^{2\ell-2p}} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta) \frac{d^{2p}}{d\theta^{2p}} \mathscr{P}_m^{(\gamma,\delta)}(\cos\theta)\Big|_{\theta=0}. \end{aligned} \tag{30}$$

Upon applying Proposition 3.1 gives the result as required. □

The proof of Proposition 3.2 does not give the polynomial $\mathscr{R}_\ell = \mathscr{R}_\ell^{(\alpha,\beta;\gamma,\delta)}$ explicitly. Here we compute the first few polynomials $\mathscr{R}_\ell^{(\alpha,\beta;\gamma,\delta)}$ in explicit forms.

- ($\ell = 1$) Clearly, we have

$$\mathscr{R}_1^{(\alpha,\beta;\gamma,\delta)}(\lambda_k, \lambda_m) = c_1^1(\alpha,\beta)\lambda_k^{\alpha,\beta} + c_1^1(\gamma,\delta)\lambda_m^{\gamma,\delta}. \tag{31}$$

- ($\ell = 2$) In this case, we see that

$$\begin{aligned}\mathscr{R}_2^{(\alpha,\beta;\gamma,\delta)}(\lambda_k, \lambda_m) =& c_1^2(\alpha,\beta)\lambda_k^{\alpha,\beta} + c_1^2(\gamma,\delta)\lambda_m^{\gamma,\delta} + c_2^2(\alpha,\beta)\left[\lambda_k^{\alpha,\beta}\right]^2 + c_2^2(\gamma,\delta)\left[\lambda_m^{\gamma,\delta}\right]^2 \\ &+ 6c_1^1(\alpha,\beta)c_1^1(\gamma,\delta)\lambda_k^{\alpha,\beta}\lambda_m^{\gamma,\delta}.\end{aligned}$$

- ($\ell = 3$) Here we have

$$\begin{aligned}\mathscr{R}_3^{(\alpha,\beta;\gamma,\delta)}(\lambda_k,\lambda_m) =& c_1^3(\alpha,\beta)\lambda_k^{\alpha,\beta} + c_1^3(\gamma,\delta)\lambda_m^{\gamma,\delta} + c_2^3(\alpha,\beta)\left[\lambda_k^{\alpha,\beta}\right]^2 + c_2^3(\gamma,\delta)\left[\lambda_m^{\gamma,\delta}\right]^2 \\ &+ c_3^3(\alpha,\beta)\left[\lambda_k^{\alpha,\beta}\right]^3 + c_3^3(\gamma,\delta)\left[\lambda_m^{\gamma,\delta}\right]^3 \\ &+ 15\left[c_1^1(\alpha,\beta)c_1^2(\gamma,\delta) + c_1^1(\gamma,\delta)c_1^2(\alpha,\beta)\right]\lambda_k^{\alpha,\beta}\lambda_m^{\gamma,\delta} \\ &+ 15\left[c_1^1(\alpha,\beta)c_2^2(\gamma,\delta)\lambda_k^{\alpha,\beta}\left[\lambda_m^{\gamma,\delta}\right]^2 + c_1^1(\gamma,\delta)c_2^2(\alpha,\beta)\left[\lambda_k^{\alpha,\beta}\right]^2\lambda_m^{\gamma,\delta}\right].\end{aligned}$$

- ($\ell = 4$) Indeed,

$$\begin{aligned}\mathscr{R}_4^{(\alpha,\beta;\gamma,\delta)}(\lambda_k,\lambda_m) =& c_1^4(\alpha,\beta)\lambda_k^{\alpha,\beta} + c_1^4(\gamma,\delta)\lambda_m^{\gamma,\delta} + c_2^4(\alpha,\beta)\left[\lambda_k^{\alpha,\beta}\right]^2 + c_2^4(\gamma,\delta)\left[\lambda_m^{\gamma,\delta}\right]^2 \\ &+ c_3^4(\alpha,\beta)\left[\lambda_k^{\alpha,\beta}\right]^3 + c_3^4(\gamma,\delta)\left[\lambda_m^{\gamma,\delta}\right]^3 + c_4^4(\alpha,\beta)\left[\lambda_k^{\alpha,\beta}\right]^4 \\ &+ c_4^4(\gamma,\delta)\left[\lambda_m^{\gamma,\delta}\right]^4 + 2\left[28c_1^3(\alpha,\beta)c_1^1(\gamma,\delta) + 35c_1^2(\alpha,\beta)c_1^2(\gamma,\delta)\right]\lambda_k^{\alpha,\beta}\lambda_m^{\gamma,\delta} \\ &+ \left[28c_2^3(\alpha,\beta)c_1^1(\gamma,\delta) + 70c_2^2(\alpha,\beta)c_1^2(\gamma,\delta)\right]\left[\lambda_k^{\alpha,\beta}\right]^2\lambda_m^{\gamma,\delta} \\ &+ \left[28c_1^1(\alpha,\beta)c_2^3(\gamma,\delta) + 70c_1^2(\alpha,\beta)c_2^2(\gamma,\delta)\right]\lambda_k^{\alpha,\beta}\left[\lambda_m^{\gamma,\delta}\right]^2 \\ &+ 28c_1^1(\alpha,\beta)c_3^3(\gamma,\delta)\lambda_k^{\alpha,\beta}\left[\lambda_m^{\gamma,\delta}\right]^3 + 28c_3^3(\alpha,\beta)c_1^1(\gamma,\delta)\left[\lambda_k^{\alpha,\beta}\right]^3\lambda_m^{\gamma,\delta} \\ &+ 70c_2^2(\alpha,\beta)c_2^2(\gamma,\delta)\left[\lambda_k^{\alpha,\beta}\right]^2\left[\lambda_m^{\gamma,\delta}\right]^2.\end{aligned}$$

4. Evaluation of Definite Integrals

This section is concerned with the evaluation of definite integrals via the spectral polynomials $\mathscr{R}_\ell^{(\alpha,\beta;\gamma,\delta)}$. We first evaluate definite integrals involvng the combination of logarithmic, exponential and trigonometric functions and then reduce the integrals to those involving the combination of exponential and trigonometric functions.

Towards this end, the following remarkable product formula for the Jacobi polynomials due to Srivastava and Panda [28] is our starting point.

Proposition 4.1 ([28]). *Let $\alpha+\beta>-1, \alpha+\gamma>-1, \gamma+\delta>-1$ and k,m be nonnegative integers. Then*

$$
\begin{aligned}
&P_k^{(\alpha,\beta)}(\cos\theta)P_m^{(\gamma,\delta)}(\cos\vartheta)\\
=&h_{k,m}^{\alpha,\beta;\gamma,\delta}\int_0^1\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\log\frac{1}{s}\right)^{k+\alpha+\beta}\left(\log\frac{1}{t}\right)^{m+\gamma+\delta}e^{(k-m)\phi i}e^{(\alpha-\gamma)\varphi i}\cos^{k+m}\phi\\
&\times\cos^{\alpha+\gamma}\varphi\, L_{k+m}^{\alpha+\gamma}\left(\Phi\left(\sin^2(\theta/2)\log(1/s),\sin^2(\vartheta/2)\log(1/t);\phi,\varphi\right)\right)d\phi\, d\varphi\, dt\, ds,
\end{aligned}
\tag{32}
$$

where the constant $h_{k,m}^{\alpha,\beta;\gamma,\delta}$ is given by

$$
h_{k,m}^{\alpha,\beta;\gamma,\delta}=\frac{2^{k+m+\alpha+\gamma}\Gamma(k+\alpha+1)\Gamma(m+\gamma+1)}{\pi^2\Gamma(k+\alpha+\beta+1)\Gamma(m+\gamma+\delta+1)\Gamma(k+m+\alpha+\gamma+1)}
\tag{33}
$$

and

$$
\Phi\left(u,v;\phi,\varphi\right)=\frac{\cos\phi}{\cos\varphi}\left[ue^{(\phi-\varphi)i}+ve^{-(\phi-\varphi)i}\right].
\tag{34}
$$

See [1] for the product and linearisation formula for Jacobi polynomials in the special cases $\gamma=\beta, \delta=\alpha, k=m$; and [11] for the integral representation of the product of Jacobi polynomials in the special cases $\alpha=\gamma, \delta=\beta, k=m$.
We now state the main result of this paper.

Theorem 4.2 (Integral-spectral identity). *Let $\alpha+\beta>-1, \alpha+\gamma>-1, \gamma+\delta>-1$ and k,m be nonnegative integers. Then for any integer $\ell\geq 1$ we have the following integral formula:*

$$
\begin{aligned}
&\sum_{j=1}^{\ell}\sum_{p=0}^{j}\frac{a_j^\ell/2^j\binom{j}{p}}{(k+m-j)!\Gamma(\alpha+\gamma+j+1)}\int_0^1\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\log\frac{1}{s}\right)^{k+\alpha+\beta+p}\left(\log\frac{1}{t}\right)^{m+\gamma+\delta+j-p}\\
&\times e^{(k-m+2p-j)\phi i}e^{(\alpha-\gamma-2p+j)\varphi i}\cos^{k+m+j}\phi\cos^{\alpha+\gamma-j}\varphi\, d\phi\, d\varphi\, dt\, ds\\
&=H_{k,m}^{\alpha,\beta;\gamma,\delta}\mathscr{R}_\ell^{(\alpha,\beta;\gamma,\delta)}\left(\lambda_k,\lambda_m\right),
\end{aligned}
\tag{35}
$$

where the constant coefficients a_m^ℓ are as in (45), $\mathscr{R}_\ell^{(\alpha,\beta;\gamma,\delta)}\left(\lambda_k,\lambda_m\right)$ is the spectral polynomial in Proposition 3.2 and the constant $H_{k,m}^{\alpha,\beta;\gamma,\delta}$ is given by

$$
H_{k,m}^{\alpha,\beta;\gamma,\delta}=\frac{\pi^2\Gamma(k+\alpha+\beta+1)\Gamma(m+\gamma+\delta+1)}{2^{k+m+\alpha+\gamma}k!m!\Gamma(\alpha+1)\Gamma(\gamma+1)}.
\tag{36}
$$

Moreover,

$$\sum_{j=1}^{\ell}\sum_{p=0}^{j} a_j^{\ell}\frac{j!\Gamma(k+\alpha+\beta+p+1)\Gamma(m+\gamma+\delta+j-p+1)}{2^j p!(j-p)!(k+m-j)!\Gamma(\alpha+\gamma+j+1)}$$

$$\times\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(k-m+2p-j)\phi i}e^{(\alpha-\gamma-2p+j)\varphi i}\cos^{k+m+j}\phi\cos^{\alpha+\gamma-j}\varphi\, d\phi\, d\varphi$$

$$= H_{k,m}^{\alpha,\beta;\gamma,\delta}\mathscr{R}_{\ell}^{(\alpha,\beta;\gamma,\delta)}(\lambda_k,\lambda_m)\,. \tag{37}$$

Proof. Setting $\theta=\vartheta$, we see that equation (32) becomes

$$P_k^{(\alpha,\beta)}(\cos\theta)P_m^{(\gamma,\delta)}(\cos\theta)$$

$$=h_{k,m}^{\alpha,\beta;\gamma.\delta}\int_0^1\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\log\frac{1}{s}\right)^{k+\alpha+\beta}\left(\log\frac{1}{t}\right)^{m+\gamma+\delta}e^{(k-m)\phi i}e^{(\alpha-\gamma)\varphi i}\cos^{k+m}\phi\cos^{\alpha+\gamma}\varphi$$

$$\times L_{k+m}^{\alpha+\gamma}\left(\frac{\cos\phi}{\cos\varphi}\sin^2(\theta/2)\left[\log(1/s)e^{(\phi-\varphi)i}+\log(1/t)e^{-(\phi-\varphi)i}\right]\right)d\phi\, d\varphi\, dt\, ds. \tag{38}$$

Now, for any integer $\ell\geq 1$, consider the differential relation

$$\frac{d^{2\ell}}{d\theta^{2\ell}}\left\{P_k^{(\alpha,\beta)}(\cos\theta)P_m^{(\gamma,\delta)}(\cos\theta)\right\}\Big|_{\theta=0}$$

$$=h_{k,m}^{\alpha,\beta;\gamma.\delta}\int_0^1\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\log\frac{1}{s}\right)^{k+\alpha+\beta}\left(\log\frac{1}{t}\right)^{m+\gamma+\delta}e^{(k-m)\phi i}e^{(\alpha-\gamma)\varphi i}\cos^{k+m}\phi\cos^{\alpha+\gamma}\varphi$$

$$\times\frac{d^{2\ell}}{d\theta^{2\ell}}\left[L_{k+m}^{\alpha+\gamma}\left(\frac{\cos\phi}{\cos\varphi}\sin^2(\theta/2)\left[\log(1/s)e^{(\phi-\varphi)i}+\log(1/t)e^{-(\phi-\varphi)i}\right]\right)\right]\Big|_{\theta=0}d\phi\, d\varphi\, dt\, ds. \tag{39}$$

Note that the vanishing of the odd terms in the above identity is due to L_k^{ν} being even in the θ-variable. Indeed, from the recursion formula (25) we have

$$\frac{d^{2\ell}}{d\theta^{2\ell}}\left\{P_k^{(\alpha,\beta)}(\cos\theta)P_m^{(\gamma,\delta)}(\cos\theta)\right\}\Big|_{\theta=0}$$

$$=h_{k,m}^{\alpha,\beta;\gamma.\delta}\sum_{j=1}^{\ell}\frac{a_j^{\ell}}{2^j}L_{k+m-j}^{\alpha+\gamma+j}(0)\int_0^1\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\log\frac{1}{s}\right)^{k+\alpha+\beta}\left(\log\frac{1}{t}\right)^{m+\gamma+\delta}\frac{\cos^j\phi}{\cos^j\varphi}$$

$$\times\left(\log(1/s)e^{(\phi-\varphi)i}+\log(1/t)e^{-(\phi-\varphi)i}\right)^j e^{(k-m)\phi i}e^{(\alpha-\gamma)\varphi i}\cos^{k+m}\phi\cos^{\alpha+\gamma}\varphi\, d\phi\, d\varphi\, dt\, ds, \tag{40}$$

where a_j^ℓ are constant coefficients given in (45). Upon using the generalised binomial expansion and the second identity on the right-hand of (21) one obtains

$$\frac{d^{2\ell}}{d\theta^{2\ell}}\left\{P_k^{(\alpha,\beta)}(\cos\theta)P_m^{(\gamma,\delta)}(\cos\theta)\right\}\Big|_{\theta=0}$$
$$=h_{k,m}^{\alpha,\beta;\gamma,\delta}\sum_{j=1}^{\ell}\sum_{p=0}^{j}\frac{(a_j^\ell/2^j)\binom{j}{p}\Gamma(k+m+\alpha+\gamma+1)}{(k+m-j)!\Gamma(\alpha+\gamma+j+1)}\int_0^1\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\log\frac{1}{s}\right)^{k+\alpha+\beta+p}$$

(41)

$$\times\left(\log\frac{1}{t}\right)^{m+\gamma+\delta+j-p}e^{(k-m)\phi i}e^{(\alpha-\gamma)\varphi i}\cos^{k+m+j}\phi\cos^{\alpha+\gamma-j}\varphi e^{(\phi-\varphi)pi}e^{-(\phi-\varphi)(j-p)i}\,d\phi\,d\varphi\,dt\,ds.$$

In view of Proposition 3.2 we obtain the spectral relation

$$h_{k,m}^{\alpha,\beta;\gamma,\delta}\sum_{j=1}^{\ell}\sum_{p=0}^{j}\frac{(a_j^\ell/2^j)\binom{j}{p}\Gamma(k+m+\alpha+\gamma+1)}{(k+m-j)!\Gamma(\alpha+\gamma+j+1)}\int_0^1\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\log\frac{1}{s}\right)^{k+\alpha+\beta+p}$$
$$\times\left(\log\frac{1}{t}\right)^{m+\gamma+\delta+j-p}e^{(k-m)\phi i}e^{(\alpha-\gamma)\varphi i}\cos^{k+m+j}\phi\cos^{\alpha+\gamma-j}\varphi e^{(\phi-\varphi)pi}e^{-(\phi-\varphi)(j-p)i}\,d\phi\,d\varphi\,dt\,ds$$

(42)

$$=P_k^{(\alpha,\beta)}(1)P_m^{(\gamma,\delta)}(1)\mathscr{R}_\ell^{(\alpha,\beta;\gamma,\delta)}(\lambda_k,\lambda_m)$$

and this completes the proof of the first part of the theorem. For the second part, we use the definition ([20, p. 892, 8.312(1)])

$$\Gamma(z)=\int_0^1\left(\log\frac{1}{u}\right)^{z-1}du,\qquad \mathrm{Re}\,z>0, \tag{43}$$

and the proof of the theorem is now complete. □

Remark. The coefficients a_m^ℓ occur from

(44)

$$\frac{d^{2\ell}}{d\theta^{2\ell}}y\left(\frac{1-\cos\theta}{2}\right)\Big|_{\theta=0}=\frac{d^{2\ell}}{d\theta^{2\ell}}y\left(\sin^2(\theta/2)\right)\Big|_{\theta=0}=\sum_{m=1}^{\ell}\frac{a_m^\ell}{2^m}\frac{d^m}{dt^m}y(t)\Big|_{t=0},\qquad \ell\geq 1.$$

The first these coefficients are $a_1^1=-1; a_1^2=1, a_2^2=3; a_1^3=-1, a_2^3=a_3^3=-15; a_1^4=1, a_2^4=63, a_3^4=210, a_4^4=105$. It is worth mentioning that the sequence 1, 1, 3, 1, 15, 15, 1,63, 210, 105, $\cdots$ appear in the OEIS (On-Line Encyclopedia of Integer Sequences, http://oeis.org/A156289), which upon comparing with the first coefficients $\left(a_m^\ell : 1\leq m\leq \ell\right)$ can be formulated in closed-form as follows ([27]):

$$a_m^\ell=\frac{(-1)^\ell}{2^{m-1}m!}\sum_{j=1}^{m}(-1)^{m-j}\binom{2m}{m-j}j^{2\ell}, \tag{45}$$

satisfying the recurrence relation

$$(-1)^{\ell} a_m^{\ell} = (2m-1)a_{m-1}^{\ell-1} + m^2 a_m^{\ell-1} \tag{46}$$

as well as

$$a_{\ell}^{\ell} = (-1)^{\ell}(2\ell-1)!! = (-1)^{\ell}\cdot 1\cdot 3\cdot 5\cdots(2\ell-1), \quad a_1^{\ell} = (-1)^{\ell} \ (\ell \geq 1), \ a_m^{\ell} = 0 \ (1 \leq \ell < m). \tag{47}$$

Now, if we set $\ell = 1, 2, 3, \dots$ in (37) and make the double integrals the subject, one obtains the following corollary of our main Theorem 4.2.

Corollary 4.3. *Let $\alpha+\beta > -1, \alpha+\gamma > -1, \gamma+\delta > -1$ and k, m be nonnegative integers. Then for any integer $\ell \geq 1$ we have the following integral formula:*

$$\begin{aligned}&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\mathbf{A}_{k,m}^{(\alpha,\gamma,\beta)}(\phi,\varphi) + \mathbf{B}_{k,m}^{(\alpha,\gamma,\delta)}(\phi,\varphi)\right)^{\ell} \cos^{k+m+\ell}\phi \cos^{\alpha+\gamma-\ell}\varphi \, d\phi \, d\varphi \\ &= \frac{\pi^2(-1)^{\ell}(k+m-\ell)!\Gamma(\alpha+\gamma+\ell+1)}{2^{k+m+\alpha+\gamma-\ell}k!m!(2\ell-1)!!\Gamma(\alpha+1)\Gamma(\gamma+1)} \sum_{p=0}^{\ell}\binom{2\ell}{2p}\sum_{i=0}^{\ell-p}\sum_{j=0}^{p} f_i^{\ell-p}(\alpha,\beta) f_j^{p}(\delta,\gamma)[\lambda_k^{\alpha,\beta}]^i[\lambda_m^{\gamma,\delta}]^j,\end{aligned} \tag{48}$$

where f_m^{ℓ} are constant coefficients and $\left(\mathbf{A}_{k,m}^{(\alpha,\gamma,\beta)}(\phi,\varphi)\right)^{\ell}$ and $\left(\mathbf{B}_{k,m}^{(\alpha,\gamma,\delta)}(\phi,\varphi)\right)^{\ell}$ ($\ell \geq 1$) are given respectively by

$$\begin{aligned}\left(\mathbf{A}_{k,m}^{(\alpha,\gamma,\beta)}(\phi,\varphi)\right)^{\ell} &= (k+\alpha+\beta+1)(k+\alpha+\beta+2)\cdots(k+\alpha+\beta+\ell)e^{(k-m+\ell)\phi i}e^{(\alpha-\gamma-\ell)\varphi i} \\ \left(\mathbf{B}_{k,m}^{(\alpha,\gamma,\delta)}(\phi,\varphi)\right)^{\ell} &= (m+\gamma+\delta+1)(m+\gamma+\delta+2)\cdots(m+\gamma+\delta+\ell)e^{(k-m-\ell)\phi i}e^{(\alpha-\gamma+\ell)\varphi i}.\end{aligned} \tag{49}$$

Remark. The first coefficients $f_m^{\ell}(a,b)$ are given below.

$$\begin{aligned}f_1^1(a,b) &= c_1^1(a,b) = -\frac{1}{2(a+1)}, \quad f_1^2(a,b) = c_1^1(a,b) + c_1^2(a,b) = -\frac{3(a+b+2)}{4(a+1)(b+2)} \\ f_2^2(a,b) &= c_2^2(a,b) = \frac{3}{4(a+1)(b+2)} \\ f_1^3(a,b) &= c_1^1(a,b) + c_1^2(a,b) + c_1^3(a,b) = -\frac{5a^2+3ab+25a+15ab+15b^2+39b+30}{4(a+1)(a+2)(a+3)} \\ f_2^3(a,b) &= c_2^2(a,b) + c_2^3(a,b) = \frac{3(7a+15b+16)}{8(a+1)(a+2)(a+3)} \\ f_3^3(a,b) &= c_3^3(a,b) = -\frac{15}{8(a+1)(a+2)(a+3)}\end{aligned} \tag{50}$$

$$
\begin{aligned}
f_1^4(a,b) =& c_1^1(a,b)+c_1^2(a,b)+c_1^3(a,b)+c_1^4(a,b)\\
=& -\frac{3(a+b+2)\left(9a^2+80ab+63a+105b^2+215b+108\right)}{8(a+1)(a+2)(a+3)(a+4)}\\
f_2^4(a,b) =& c_2^2(a,b)+c_2^3(a,b)+c_2^4(a,b)\\
=& \frac{3\left(63a^2+380ab+326a+385b^2+890b+436\right)}{16(a+1)(a+2)(a+3)(a+4)}\\
f_3^4(a,b) =& c_3^3(a,b)+c_3^4(a,b) = -\frac{15(8a+21b+18)}{8(a+1)(a+2)(a+3)(a+4)}\\
f_4^4(a,b) =& c_4^4(a,b) = \frac{105}{16(a+1)(a+2)(a+3)(a+4)}.
\end{aligned} \tag{51}
$$

Now, if we replace $\alpha=\beta$ and $\gamma=\delta$ by $\nu-1/2$ and $\rho-1/2$ respectively in Proposition 4.1 we obtain the important special case of Gegenbauer polynomials.

Corollary 4.4. *Let $\nu>0, \nu+\rho>0, \rho>0$ and k,m be nonnegative integers. Then*

$$
\begin{aligned}
&C_k^{\nu}(\cos\theta)C_m^{\rho}(\cos\vartheta)\\
=&\widetilde{h}_{k,m}^{\nu;\rho}\int_0^1\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\log\frac{1}{s}\right)^{k+2\nu-1}\left(\log\frac{1}{t}\right)^{m+2\rho-1}e^{(k-m)\phi i}e^{(\nu-\rho)\varphi i}\cos^{k+m}\phi\\
&\times\cos^{\nu+\rho-1}\varphi\, L_{k+m}^{\nu+\rho-1}\left(\Phi\left(\sin^2(\theta/2)\log(1/s),\sin^2(\vartheta/2)\log(1/t);\phi,\varphi\right)\right)\,d\phi\,d\varphi\,dt\,ds,
\end{aligned} \tag{52}
$$

where the constant $\widetilde{h}_{k,m}^{\nu;\rho}$ is given by

$$
\widetilde{h}_{k,m}^{\nu;\rho}=\frac{2^{k+m+\nu+\rho-1}\Gamma\left(k+2\nu\right)\Gamma\left(m+2\rho\right)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\rho+\frac{1}{2}\right)}{\pi^2\Gamma(k+2\nu)\Gamma(m+2\rho)\Gamma(k+m+\nu+\rho)\Gamma(2\nu)\Gamma(2\rho)} \tag{53}
$$

and $\Phi(u,v;\phi,\varphi)$ is as in (34).

Also upon replacing $\alpha=\beta$ and $\gamma=\delta$ by $\nu-1/2$ and $\rho-1/2$ respectively in Theorem 4.2 we obtain the important special case of Gegenbauer polynomials.

Theorem 4.5 (Integral-spectral identity)**.** *Let $\nu>0, \nu+\rho>0, \rho>0$ and k,m be nonnegative integers. Then for any integer $\ell\geq 1$ we have the following integral formula:*

$$
\begin{aligned}
&\sum_{j=1}^{\ell}\sum_{p=0}^{j}\frac{a_j^{\ell}/2^j\binom{j}{p}}{(k+m-j)!\Gamma\left(\nu+\gamma+j+\frac{1}{2}\right)}\int_0^1\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\log\frac{1}{s}\right)^{k+2\nu+p-1}\left(\log\frac{1}{t}\right)^{m+2\rho+j-p-1}\\
&\times e^{(k-m+2p-j)\phi i}e^{(\nu-\rho-2p+j)\varphi i}\cos^{k+m+j}\phi\cos^{\nu+\rho-j-1}\varphi\,d\phi\,d\varphi\,dt\,ds\\
&=\widetilde{H}_{k,m}^{\nu;\rho}\mathscr{S}_{\ell}^{(\nu;\rho)}\left(\widetilde{\lambda}_k,\widetilde{\lambda}_m\right),
\end{aligned} \tag{54}
$$

where a_m^ℓ *are as in* (45), $\mathscr{S}_\ell^{(\nu;\rho)}\left(\widetilde{\lambda}_k,\widetilde{\lambda}_m\right) := \mathscr{R}_\ell^{(\nu-1/2,\nu-1/2;\rho-1/2,\rho-1/2)}\left(\lambda_k,\lambda_m\right)$ *is the spectral polynomial in Proposition* 3.2 *and the constant* $\widetilde{H}_{k,m}^{\nu;\rho} := H_{k,m}^{\nu-1/2,\nu-1/2;\rho-1/2,\rho-1/2}$ *is given by*

$$\widetilde{H}_{k,m}^{\nu;\rho} = \frac{\pi^2\Gamma(k+2\nu)\Gamma(m+2\rho)}{2^{k+m+\nu+\rho-1}k!m!\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\rho+\frac{1}{2}\right)}. \tag{55}$$

Moreover,

$$\begin{aligned}&\sum_{j=1}^{\ell}\sum_{p=0}^{j} a_j^\ell \frac{j!\Gamma(k+2\nu+p)\Gamma(m+2\rho+j-p)}{2^j p!(j-p)!(k+m-j)!\Gamma(\nu+\rho+j)}\\ &\times\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(k-m+2p-j)\phi i}e^{(\nu-\rho-2p+j)\varphi i}\cos^{k+m+j}\phi\cos^{\nu+\rho-j-1}\varphi\,d\phi\,d\varphi\\ &= \widetilde{H}_{k,m}^{\nu;\rho}\mathscr{S}_\ell^{(\nu;\rho)}\left(\widetilde{\lambda}_k,\widetilde{\lambda}_m\right).\end{aligned} \tag{56}$$

Similarly, in this special case of Gegenbauer polynomials, upon setting $\ell = 1,2,3,\ldots$ in (56) and making the double integrals the subject we obtain the following corollary.

Corollary 4.6. *Let* $\nu > 0, \nu+\rho > 0, \rho > 0$ *and* k,m *be nonnegative integers Then for any integer* $\ell \geq 1$ *we have the following integral formula:*

$$\begin{aligned}&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\mathbf{C}_{k,m}^{\nu,\rho}(\phi,\varphi)+\mathbf{D}_{k,m}^{\nu,\rho}(\phi,\varphi)\right)^\ell\cos^{k+m+\ell}\phi\cos^{\nu+\rho-\ell-1}\varphi\,d\phi\,d\varphi\\ &= \frac{\pi^2(-1)^\ell(k+m-\ell)!\Gamma(\nu+\rho+\ell)}{(2\ell-1)!!2^{k+m+\nu+\rho-\ell-1}k!m!\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\rho+\frac{1}{2}\right)}\sum_{p=0}^{\ell}\binom{2\ell}{2p}\sum_{i=0}^{\ell-p}\sum_{j=0}^{p}\widetilde{f}_i^{\ell-p}(\nu)\widetilde{f}_j^p(\rho)[\widetilde{\lambda}_k^\nu]^i[\widetilde{\lambda}_m^\rho]^j\end{aligned} \tag{57}$$

where f_m^ℓ *are constant coefficients and* $\left(\mathbf{C}_{k,m}^{\nu,\rho}(\phi,\varphi)\right)^\ell$ *and* $\left(\mathbf{D}_{k,m}^{\nu,\rho}(\phi,\varphi)\right)^\ell$ $(\ell \geq 1)$ *are given by*

$$\begin{aligned}\left(\mathbf{C}_{k,m}^{\nu,\rho}(\phi,\varphi)\right)^\ell &= (k+2\nu)(k+2\nu+1)\cdots(k+2\nu+\ell-1)e^{(k-m+\ell)\phi i}e^{(\nu-\rho-\ell)\varphi i}\\ \left(\mathbf{D}_{k,m}^{\nu,\rho}(\phi,\varphi)\right)^\ell &= (m+2\rho)(m+2\rho+1)\cdots(m+2\rho+\ell-1)e^{(k-m-\ell)\phi i}e^{(\nu-\rho+\ell)\varphi i}.\end{aligned} \tag{58}$$

Remark. The first coefficients $\widetilde{f}_m^\ell$ are given below.

$$
\begin{aligned}
&\widetilde{f}_1^1(a) = \widetilde{c}_1^1(a) = -\frac{1}{2a+1}, \quad \widetilde{f}_1^2(a) = \widetilde{c}_1^1(a) + \widetilde{c}_1^2(a) = -\frac{3}{2a+3} \\
&\widetilde{f}_2^2(a) = \widetilde{c}_2^2(a) = \frac{3}{(2a+1)(2a+3)} \\
&\widetilde{f}_1^3(a) = \widetilde{c}_1^1(a) + \widetilde{c}_1^2(a) + \widetilde{c}_1^3(a) = -\frac{76a^2+52a+15}{(2a+1)(2a+3)(2a+5)} \\
&\widetilde{f}_2^3(a) = \widetilde{c}_2^2(a) + \widetilde{c}_2^3(a) = \frac{3(22a+5)}{(2a+1)(2a+3)(2a+5)} \\
&\widetilde{f}_3^3(a) = \widetilde{c}_3^3(a) = -\frac{15}{(2a+1)(2a+3)(2a+5)} \qquad (59) \\
&\widetilde{f}_1^4(a) = \widetilde{c}_1^1(a) + \widetilde{c}_1^2(a) + \widetilde{c}_1^3(a) + \widetilde{c}_1^4(a) = -\frac{3\left(388a^2+168a+35\right)}{(2a+3)(2a+5)(2a+7)} \\
&\widetilde{f}_2^4(a) = \widetilde{c}_2^2(a) + \widetilde{c}_2^3(a) + \widetilde{c}_2^4(a) = \frac{3\left(828a^2+388a+35\right)}{(2a+1)(2a+3)(2a+5)(2a+7)} \\
&\widetilde{f}_3^4(a) = \widetilde{c}_3^3(a) + \widetilde{c}_3^4(a) = -\frac{15(58a+7)}{(2a+1)(2a+3)(2a+5)(2a+7)} \\
&\widetilde{f}_4^4(a) = \widetilde{c}_4^4(a) = \frac{105}{(2a+1)(2a+3)(2a+5)(2a+7)}.
\end{aligned}
$$

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