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Enumeration of Partial Contraction Transformation Semi-groups

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Abstract

Let $M = \{1, 2 \dots n\}$ on a partial transformation α, $dom(\alpha) : |\alpha x - \alpha y|$ whenever $x, y \in dom(\alpha)$ and $|x - y|$, then the semi-group $CPT(M)$ of all contraction partial transformation of M is denoted by CPT_n. In this article, we derive the formula for some of the (sequences) values of CPT_n and obtained that $|PC_n| = \sum_{k=0}^{n} \binom{n-2}{q-3}$, $|POD_n| = \binom{n-(p-1)}{n-p}$ and $|\alpha S| = 2^n - 1 = 3n + 3$.

1. Introduction

Fix a positive integer X and write $X = \{1, 2 \dots\}$. The partial transformation semi-group PT_n is the semi-group of all partial transformations on X. For simplicity of notations we will now use the P_n for the partial transformation semi-group as used by [1]. Since all functions between subsets X were considered and the word "Partial" does not imply that the domain is necessarily a proper subset of X. In this notion, P_n also includes all full transformation of X [2]. Let X and Y be two non-empty sets such that there exists $f : X_n \to Y_n$ then f which is a rule is said to be a transformation.

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In same assertion with the pattern adopted in [3], we let $X = \{X_1,\ X_2,\ \ldots X_n\}$ be a finite set. Transformation of X is an array of the form: $\alpha = \begin{pmatrix} X_1,\ X_2 \ldots\ X_n \\ \alpha(X_1),\ \alpha(X_1) \ldots \alpha(X_n) \end{pmatrix}$ where $\alpha\,(X_i) \in X$. The element $\alpha\,(X_i)$ is called the value of the transformation of . The fact that α is transformation of X is usually written as $\alpha : X \to X$. The composition of T_n and P_n transformation will result to another partial and full transformation of the same set respectively such that T_n is a special case of P_n. The transformation semi-group T_n on $X_n = \{1, 2, \ldots\}$ is the set of all maps $f : X_n \to X_n$ under the composition of mapping. T_n and is said to be full if $dom\,(\alpha) = X_n$; $\alpha : dom(\alpha) \subseteq X_n \to Im(\alpha) \subseteq X_n$. It is said to be partial P_n if $\alpha : dom(\alpha) \subseteq X_n \to Im(\alpha)$. The domain of α is denoted by $dom(\alpha)$ while the image of α is $Im()$. If α is a mapping from subsets of X_n, then the set of all $\alpha(S)$ if equipped with the binary operation of composition of mappings forms the P_n . [4] defined a contraction mapping as: for all $x, y \in dom\,(\alpha) : |\alpha x - \alpha y|$ implies $|x - y|$. A transformation is another name of mapping, refer to [5, 7] for an introduction to semi-group of mappings. A transformation $\alpha \in P_n$ is said to be a contraction order-preserving if $dom\,(\alpha) : x \leq y$ implies $\alpha x \leq \alpha y$. It is order-reversing if $x \leq y$ implies $\alpha x \geq \alpha y$. It is order-decreasing if $\alpha x \leq x$. It is order-preserving or reversing if $x \leq y$ implies $\alpha x \leq \alpha y$ union $x \leq y$ implies $\alpha x \geq \alpha y$. It is order-preserving and decreasing if $x \leq y$ implies $\alpha x \leq \alpha y$ intersection $x \leq y$ implies $\alpha x \leq x$.

The purpose of the current article is to investigate the order $|S|$ when S is either P_n itself or one its sub-semi-groups:

(i) $P_n = \{\alpha \in CP_n : dom\,(\alpha) = X\}$, the partial contraction mapping;
(ii) $PO_n = \{\alpha \in OCP_n : \alpha$ is contraction order-preserving$\}$;
(iii) $PR_n = \{\alpha \in ORCP_n : \alpha$ is contraction order-reversing$\}$;
(iv) $PD_n = \{\alpha \in ODCP_n : \alpha$ is contraction order-decreasing$\}$;
(v) $POD_n = \{\alpha \in OCP_n \cup ORCP_n : \alpha$ is order-preserving or reversing$\}$; and
(vi) $PC_n = \{\alpha \in OCP_n \cap ODCP_n : \alpha$ is order-preserving and decreasing$\}$.

From the above definitions, we recall that a partial transformation $\alpha \in CP_n$ is contraction partial mapping if $|\alpha x - \alpha y|$ whenever $x, y \in dom(\alpha)$ equals $|x - y|$. For example; Let $|CP_n|$ be the order of set of all contraction mappings of P_n , for $\alpha \in P_n : \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$, where $dom\,(\alpha) = (1\ 2\ 3)$ and $Im\,(\alpha) = (1\ 1\ 2)$ then we can show that $|\alpha x - \alpha y| \leq |x - y|$ whenever $x, y \in dom(\alpha)$:

$|1 - 1| \leq |1 - 2|$ implies $0 \leq 1$;
$|1 - 2| \leq |2 - 3|$ implies $1 \leq 1$; and
$|1 - 2| \leq |1 - 3|$ implies $1 \leq 2$.

Therefore, if for all $x, y \in dom(\alpha)$, α satisfy contraction inequalities. Hence, α is a contraction mapping. Let $\alpha \in P_n$. Then the elements of P_n can be represented as $\alpha(P_n)$. For economy of size, space and time viz:

P_0	P_1
$(-)$	(1)

stands for

P_0	P_1
$\binom{1}{\emptyset}$	$\binom{1}{1}$

(2 2) stands for $\binom{1\ 2}{2\ 2}$ and $(-\ 2)$ stands for $\binom{1\ 2}{\emptyset\ 2}$ and down the road in. Authors in [6] said transformation semi-group was the most promising class of semi-groups for future study. We used the structure above to generate $\alpha(CP_2)$ as follows:

P_0
$(-)$

is 1, that is $0\cap 0$,

P_0	P_1
$(-)$	(1)

is 1 on $1\cap 0$ and 1 on $1\cap 1$,

then

P_0	P_1	P_2
$(-\ -)$	(1 1) (2 2)	(1 2) (2 1)
	$(-\ 1)$ $(-\ 2)$	
	$(1-)$ $(2-)$	

is 1 on $2\cap 0$, is 6 on $2\cap 1$ and 2 on $2\cap 2$.

2. Definition of Terms

We define the following terms which features in the proofs of our results.

Definition 2.1 [Partial Transformation P_n]: Let $X_n = \{1, 2 \ldots\}$ be a natural ordering of numbers and $\alpha : dom(\alpha) \subseteq X_n$implies $Im(\alpha) \subseteq X_n$. Then partial transformation P_n is the set of all functions $\alpha : dom(\alpha) \subseteq X_n$ on X_n.

Definition 2.2 [Partial Contraction Mapping CP_n]: Let $X_n = \{1, 2 \ldots\}$. Then a transformation $\alpha \in P_n$ is said to be partial contraction mapping if for all $x, y \in dom\,(\alpha):\ |\alpha x - \alpha y| \leq |x - y|$.

Definition 2.3 [Breadth / Width, (α)]: This is the number of elements in the domain of α. That is $|dom(\alpha)| = b(\alpha)$.

Definition 2.4 [Height / Length, $h(\alpha)$]: This is the number of elements in the image sets of α. That is $h\,(\alpha) = |Im(\alpha)|$ and it is denoted by $h\,(\alpha) = p$.

Definition 2.5 [Collapse, $c(\alpha)$]: This is the order of the union of image sets of α which is greater than 2. That is

$c\,(\alpha) = \left|\cup\{t\,(\alpha)^{-1} : t \in Im\,(\alpha)\, and\, \left|t\,(\alpha)^{-1}\right| \geq 2\}\right|$. It is denoted by $c\,(\alpha) = q$.

Definition 2.6 [Right waist, $\omega^+(\alpha)$]: This is the maximum element in the image sets of $(Im(\alpha))$ of α. That is $\omega^+\,(\alpha) = \max(Im(\alpha))$. It is denoted by $\omega^+\,(\alpha) = k$.

Definition 2.7 [Left waist, $\omega^{-}(\alpha)$]: This is the minimum element in the image sets of $(Im(\alpha))$ of α. That is $\omega^{-}(\alpha) = \min(Im(\alpha))$. It is denoted by $\omega^{-}(\alpha) = k^{-}$.
Definition 2.8 [Fix, $f(\alpha)$]: This is the order of the only element that maps itself. That is, $f(\alpha) = |f(x)| = \{x \in dom(\alpha) : \alpha x = x\}$. It is denoted by $f(\alpha) = m$.

3. Results and Findings

The results of combinatorial study of any algebraic structure are theorems. In particular, we consider $|\alpha_{ij}(S)|$ and thereby obtain explicit formula for $|S|$ in each case. The formula obtained in this way are in closed form when S and is one of T_n , P_n or some of its sub-semi-groups and are expressed as sum involving binomial coefficient. The following theorems and proposition with their proves are the outcome of the research in the current article.

Theorem 3.1: Let $= PC_n$, then $|S| = \sum_{k=0}^{n} \binom{n-2}{q-3}$.
Proof: Let $\in PC_n$; such that $dom(\alpha) \subseteq X_n$ and $Im(\alpha) \subseteq X_n$. Since the empty map is sub-semi-group of PC_n such that $\alpha \in P_n$ is a bijection, then $|\alpha S| = n+1$ if $Im(\alpha) = 3$ when $n = 3$ we have $n-2$, and $q-3$ elements whenever $|Im(\alpha)| = 3$, then for $q = 3$, then $n = (n-2)$ such that $n \geq 3$ and $k = (k-3)$ such that $k \geq 3$ whenever $|Im(\alpha)| = k$. Since the mapping was defined from $X_n \rightarrow X_n$ and then $Im(\alpha)$ is that $i = n \geq 3$ then $|PC_n|$ is exactly $\binom{n-2}{q-3}$ ways. X_{24}

Proposition 3.2: Let $S = |POD_n|$ then $f(n,m) = \frac{1}{n}\binom{n}{m}\binom{n}{m-1}$

Proof: Let $\alpha \in S : X_n = \{0, 1, 2..\}$, then if $\alpha \in POD_n$, each P_n has an identity element $\{e\}$ denoted by $\{i\}$. Since a partial (α) transformation is a bijection element of the $dom(\alpha)$ which can be chosen in $\binom{n}{m}$ ways. Which is equivalent to $2n + 1 - (2n + 1)$ for the order of $|S|$. If $\in X_n$, it implies $Im(\alpha) \subseteq m$, now let $|Im(\alpha)| = \emptyset$ where $i = 1, 2, \ldots$ then $f(n,m) = |\alpha \in S : f(\alpha)| = |Im(\alpha)| = m(i)$. (the identity). Q_{17}

Theorem 3.3: Let $S{=}|PC_n|$, then $f(n,m) {=} P(1), P(2) \ldots\ldots. P(m)$.

Proof: Let $X_n = \{1,\ 4,\ 22,\ 109\ldots\}$ be irregular triangles, $X(n,m)$ read by rows, then $X(n,m)$ is the number of partition of $P(1),\ P(2), \ldots P(m)$ of n (as weakly ascending list of points) with m points p as position m (fixed points), whenever $n \geq 0$, and $0 \leq m \leq$ (column index of last non-zero term in row n). (A238352. Sloane). C_3.

Theorem 3.4: Let $S = PC_n$, then $|\alpha S| = 2^n - 1$.

Proof: Let $\alpha \in PC_n$ such that $n \in X_n = \{1,\ 2,\ \ldots\}$. An empty map $\{\}$ is an element of PC_n and k elements of $dom(\alpha)$ can be chosen from X_n in $\binom{n}{k}$ ways. Since $Im(\alpha) \subseteq X_n$ and $\sum_{k=0}^{n} \binom{n}{k}$ is equivalent to 2^n. (that is if $n = k = 0$). Now if $\alpha \in S$, then $|Im(\alpha)| = 0$ which implies that $|\alpha S| = 1$. Hence, the result. R_{18}.

Theorem 3.5: Let $= ODCP_n$, $|\alpha S| = 2^n$ for $n \geq 2$

Proof: Let $\alpha \in S$, if $x, y \in S$ such that $\alpha x \in X_n = \{1,\ 2,\ \dots\}$, then $Im\,(\alpha) \in i\ n(1)$ for each n. There are $2n$ elements whenever $Im\,(\alpha) = n \geq 2$, such that $|\alpha S| = 2^n$. The result is observed N_{14} .

Theorem 3.6: Let $= ODCP_n$, then $|\alpha S| = 3n + 3$.

Proof: Let $X_n = \{1,\ 2,\ \dots\}$, then if $X_n \to X_n$ and the image of $\alpha\ Im(\alpha)$ is such that $i = 1,\ 2,\ 3,\ \dots$.. Let $\alpha \in S$ then we observed that n elements of $dom(\alpha)$ can be chosen from X_n in $n(3)$ ways for $n = 3,\ 4,\ \dots$ and $p = 2,\ 3,\ \dots$ then the number of order of S for $\alpha \in ODCP_n$ is $3n + 3$ and for $\alpha \geq p \geq 1$, where an empty map is an element of $ODCP_n$ and p elements of domain of $\alpha\ dom(\alpha)$ can be chosen from X_n. We observed that whenever $\alpha \in S$ and $|Im(\alpha)| = 0$ then $|\alpha S| = 1$. Then we have the results . M_{13}.

Theorem 3.7: Let $= POD_n$, then $|\alpha S| = \binom{n-p}{p-1}$ and $\binom{n-(p-1)}{n-p}$.

Proof: Let $X_n = \{1,\ 2,\ \dots\}$ such that $dom(\alpha) \subseteq X_n$ and $\alpha \in POD_n$. If $f\,(n, p) = |\alpha \in S : h(\alpha)| = |Im(\alpha)| = p$, then consider $x \in dom(\alpha)$, where $\alpha x \leq \alpha y$ or $\alpha x \leq x$. If $\alpha x = i$ such that $x \in \{i = 1,\ 2, \dots\}$ and so x has $n - i + 1$ degree of freedom then there are order of S for $|\alpha S| = \binom{n-1}{p-1}$ for whenever $n = p = 2$ we have $\binom{n-1}{p-1} = 1$. For the second statement, since for contraction mapping $\alpha \in P_n$, empty mapping denoted by $\{\}$ is a subset of all mapping and that if $\alpha \in S : h\,(\alpha) = p$, irrespective of the value of $n \geq 2$ whenever $p = (n - 1)$ there is exactly two partial contraction elements of height of n order such that $|\alpha S| = \binom{n-(p-1)}{n-p}$. Hence the result, Q_{17}.

Theorem 3.8: Let $= POD_n$, then $|\alpha S| = \binom{n-(p-1)}{n-p}$ and $n\,(10) + 2$.

Proof: Let $X_n = \{1,\ 2,\ \dots\}$ such that $dom(\alpha) \subseteq X_n$ and $\alpha \in POD_n$. If $f\,(n, p) = |\alpha \in S : h(\alpha)| = |Im(\alpha)| = p$, then consider $x \in dom(\alpha)$, where $\alpha x \leq \alpha y$ or $\alpha x \leq x$. for contraction mapping $\alpha \in P_n$, empty mapping denoted by $\{\}$ is a subset of all mapping and that if $\alpha \in S : h\,(\alpha) = p$, irrespective of the value of $n \geq 2$ whenever $p = (n-1)$ where $n = 1,\ 2, \dots$ and $p = 1,\ 2 \dots$ for there is exactly two partial contraction elements of height of n order such that $|\alpha S| = \binom{n-(p-1)}{n-p}$ by applying the disjunctive rule we observe that $|\alpha S| = \binom{n-(p-1)}{n-p} = 2$. For the second statement, since for contraction mapping $\alpha \in P_n$, empty mapping denoted by $\{\}$ is a subset of all mapping and that if $\alpha \in S : h\,(\alpha) = p$, irrespective of the value of $n \geq 2$ whenever $p = (n - 1)$ there is exactly $n\,(10) + 2$ partial contraction elements of height of n order and such that $|\alpha S| = \binom{n-(p-1)}{n-p}$, $n\,(10)+2$. Hence, the result follows immediately.

Table 1: Sequence values of $\alpha(S)$ for Disjunctive (sum) rule.

$\frac{n}{\lvert Im(\alpha)\rvert} = h$	1	2	3	4	5	$\lvert\alpha S\rvert = \binom{n-(p-1)}{n-p}$
1	2					2
2	12	2				14
3	42	22	2			66
4	120	124	32	2		278
5	310	516	254	42	2	1124

Theorem 3.9: Let $= PC_n$, then $|\alpha S| = \binom{n-p}{p-1}$

Proof: Let $X_n = \{1,\ 2,\ \ldots\}$ such that $dom(\alpha) \subseteq X_n$ and $\alpha \in PC_n$. If $f\,(n,p) = |\alpha \in S : h(\alpha)| = |Im(\alpha)| = p$, then consider $x \in dom(\alpha)$, where $\alpha x \leq \alpha y$ or $\alpha x \leq x$. If $\alpha x = i$ such that $x \in \{i = 1,\ 2, \ldots\}$ and so x has $n-i+1$ degree of freedom then there are order of S for $|\alpha S| = \binom{n-1}{p-1}$ for whenever $n = p = 1$. We have $\binom{n-1}{p-1} = 1$ as observed from Theorem 3.2. Hence the result follows immediately.

Table 2: Sequence values of $\alpha(S)$ for Sequential (product) rule.

$\frac{n}{\lvert Im(\alpha)\rvert} = h$	1	2	3	4	5	$\lvert\alpha S\rvert = \binom{n-1}{p-1}$
1	1					1
2	24	1				25
3	231	99	1			331
4	1560	2852	192	1		4605
5	8835	47930	11430	315	1	68511

Table 3: Calculated values of $\alpha(S)$ for small value of n.

n	1	2	3	4	5	$6\cdots$
CP_n	1	4	22	109	594	$\cdots$
OCP_n	1	6	11	16	21	$26\cdots$
$ORCP_n$	1	4	5	6	7	$8\cdots$
$ODCP_n$	1	2	4	8	16	$32\cdots$
POD_n	1	2	2	2	2	$2\cdots$
POD_n	$\emptyset$	12	22	32	42	$52\cdots$
PC_n	$\emptyset$	1	1	2	3	$4\cdots$

Table 4: Formula for the $\alpha(S)$ generated by a semi-group $S \subset P$.

S	Formula
CP_n	$f(n,m) = P(1), P(2), \cdots, P(m)$
OCP_n	$5n+1$
$ORCP_n$	$\binom{n}{m}\binom{n+1}{m+1}$
$ODCP_n$	$2^n, 3n+3$
POD_n	$\frac{1}{n}\binom{n}{m}\binom{n}{m-1}, n(10)+2$
PC_n	$\sum_{k=0}^{n}\binom{n-2}{k-3}$

Concluding Remarks

In this section, we have that:

Remark 1: For $p = 0,\ 1$ the concept of order-preserving , reversing concide but distinct otherwise. However there exist a bijection between the two sets of P_n for $p \geq 2$ then for $f\,(n,\ m) = n + 2$ where $m = n - 1$.
Remark 2: The transformation semi-group was the most promising class of semi-groups for future study [8]. The forecast was justified because many researchers have worked extensively on the subject. Such as [1, 3] and their supervisor [4]. But few have worked on contraction mapping among them is: [11].
Remark 3: The combinatorial nature of integer sequences and their triangular arrangement arise naturally and thus make it essentially important to find their cardinalities, general formula, hence make it applicable to mathematics and science as a whole. (OIES).

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