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A family of harmonic univalent functions defined by a linear multiplier fractional operator

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Abstract

In this paper, we use a generalized linear multiplier fractional differential operator to define a new family of harmonic univalent functions in the unit disk. Also, we obtain the coefficient conditions, convolution condition and convex combination for the class.

1. Introduction

Let A_μ denote the class of functions of the form:

(1) $$f(z) = z^{\mu+1} + \sum_{k=2}^{\infty} a_k z^{\mu+k},\ 0 \le \mu < 1$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.
If for convenience, we set $A_0 = A$, we see that A_0 is the usual class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

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which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = 0$ and $f'(0) = 1$. Moreover, we have
$f(z) \in A \Longrightarrow z^{\mu} f(z) \in A_{\mu},\ 0 \leq \mu < 1.$

Now expanding (1) by Binomial theorem, we define

$$f(z)^{\rho} = z^{\rho(\mu+1)} + \sum_{k=2}^{\infty} \rho a_k z^{(\rho-1)(\mu+1)+(\mu+k)},\ \rho > 0,\ 0 \leq \mu < 1 \tag{2}$$

2. Preliminaries

A continuous function $f = u + iv$ is a complex valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. In any simply connected domain, we can write

$$f = h + \overline{g} \tag{3}$$

The form (3) is called the local decomposition of the harmonic function f, where h and g are analytic in D. We call h and g the analytic and coanalytic part of f respectively. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'| > |g'|$ in D.
Denote by S_U, the class of functions f of the form (3) that are harmonic univalent and sense-preserving in the unit disk U. The subclasses of harmonic univalent functions have been studied by some authors for different purposes and different properties, (See Juma *et al.* (2015), Ezhilarasi *et al.* (2014), Al-shaqsi *et al.* (2010), Fadipe-Joseph and Salami (2017), Serkan *et al.* (2018), Sharma and Ravindar (2018) and Shuhai Li *et al.* (2019)). Following the linear multiplier fractional differential operator $D_{\lambda}^{m,\alpha}$ for $f(z) \in A$ defined by Al-Oboudi and Al-Omoudi (2009), Noor *et al.* (2009) extends this definition to the harmonic function $f(z) = h(z) + \overline{g(z)}$ as follows:

$$\tilde{D}_{\lambda}^{m,\alpha} f(z) = \sum_{k=1}^{\infty} \phi_{k,m}(\alpha, \lambda)(a_k z^k + \overline{b_k z^k}),\ 0 \leq \alpha < 1,\ m \in \mathbb{N}_0 = \{0, 1, 2, \cdots\} \tag{4}$$

where

$$\phi_{k,m}(\alpha, \lambda) = \left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}(1 + \lambda(k-1))\right]^m$$

and

$$\left.\begin{array}{l} h(z) = z + \sum_{k=2}^{\infty} a_k z^k \\ g(z) = \sum_{k=1}^{\infty} b_k z^k,\ |b_1| < 1 \end{array}\right\} \tag{5}$$

We note that if $g \equiv 0$ then (4) becomes the linear multiplier fractional differential operator $D_{\lambda}^{m,\alpha}$, Al-Oboudi and Al-Omoudi (2009).

In this work, we express the analytic functions h and g as:

$$\left.\begin{aligned} h(z)^\rho &= z^{\rho(\mu+1)} + \textstyle\sum_{k=2}^{\infty} \rho a_k z^{(\rho-1)(\mu+1)+(\mu+k)} \\ g(z)^\rho &= \textstyle\sum_{k=1}^{\infty} \rho b_k z^{(\rho-1)(\mu+1)+(\mu+k)},\ |b_1|<1 \end{aligned}\right\} \tag{6}$$

So that,

$$f(z)^\rho = h(z)^\rho + \overline{g(z)^\rho}$$

where

$$\left.\begin{aligned} h(z) &= z^{(\mu+1)} + \textstyle\sum_{k=2}^{\infty} a_k z^{(\mu+k)} \\ g(z) &= \textstyle\sum_{k=1}^{\infty} b_k z^{(\mu+k)},\ |b_1|<1 \text{ in the class } A_\mu \end{aligned}\right\} \tag{7}$$

and (7) reduces to (5) when $\mu = 0$.
In this present paper, we define and study a new class $\Psi_\gamma^{\rho,\mu}(m,\alpha,\zeta,\lambda)$, $(\rho > 0, 0 \leq \mu < 1, 0 \leq \alpha < 1, \zeta \geq 0, 0 \leq \gamma < 1, \lambda \geq 0)$ of harmonic univalent functions in $U = \{z \in \mathbb{C} : |z| < 1\}$ and a subclass of it.

3. THE CLASS $\Psi_\gamma^{\rho,\mu}(m,\alpha,\zeta,\lambda)$

Definition 3.1. Let $f(z)^\rho = h(z)^\rho + \overline{g(z)^\rho}$ be a harmonic function, where $h(z)^\rho$ and $g(z)^\rho$ are given by (6). Then $f(z)^\rho \in \Psi_\gamma^{\rho,\mu}(m,\alpha,\zeta,\lambda)$ if it satisfies

$$Re\left\{\zeta \tilde{D}_\lambda^{m,\alpha}\left[\frac{h(z)^\rho}{z^{\rho(\mu+1)}} + \overline{\left(\frac{g(z)^\rho}{z^{\rho(\mu+1)}}\right)}\right] + \tilde{D}_\lambda^{m,\alpha}\left[h'(z)^\rho + \overline{g'(z)^\rho}\right] - \zeta\right\} > \gamma \tag{8}$$

$\rho > 0, 0 \leq \mu < 1, 0 \leq \alpha < 1, \zeta \geq 0, 0 \leq \gamma < 1, \lambda \geq 0, z \in U$, where

$$\tilde{D}_\lambda^{m,\alpha} f(z)^\rho = \sum_{k=1}^{\infty} \phi_{k,m}(\alpha,\lambda)(a_k z^{(\rho-1)(\mu+1)+(\mu+k)} + \overline{b_k z^{(\rho-1)(\mu+1)+(\mu+k)}})$$

and

$$\phi_{k,m}(\alpha,\lambda) = \left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}(1+\lambda(k-1))\right]^m,\ m \in \mathbb{N}_0$$

Remark. We note that the class $\Psi_\gamma^{1,0}(m,\alpha,\zeta,\lambda) \equiv RH_\gamma(m,\alpha,\beta,\lambda)$ introduced and studied by Ezhilarasi *et al.*(2014).

Definition 3.2. We also let $\overline{\Psi_\gamma^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$ be the subclass of $\Psi_\gamma^{\rho,\mu}(m,\alpha,\zeta,\lambda)$ such that $h(z)^\rho$ and $g(z)^\rho$ are given by

$$\left.\begin{aligned} h(z)^\rho &= z^{\rho(\mu+1)} - \textstyle\sum_{k=2}^{\infty} \rho|a_k| z^{(\rho-1)(\mu+1)+(\mu+k)} \\ g(z)^\rho &= -\textstyle\sum_{k=1}^{\infty} \rho|b_k| z^{(\rho-1)(\mu+1)+(\mu+k)},\ |b_1|<1 \end{aligned}\right\} \tag{9}$$

Theorem 3.3. *Let $f(z)^\rho = h(z)^\rho + \overline{g(z)^\rho}$, where $h(z)^\rho$, $g(z)^\rho$ are given by (6). Furthermore, let*

$$\sum_{k=1}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)(|a_k|+|b_k|) \leq (\rho+1)(\mu+1)+\zeta-\frac{\gamma}{\rho} \tag{10}$$

$|b_1| < \frac{\rho(\mu+1)-\gamma}{\rho[\rho(\mu+1)+\zeta]} < 1$, *where* $a_1 = 1$, $\rho > 0$, $\zeta \geq 0$, $0 \leq \mu < 1$, $0 \leq \gamma < 1$ *and* $0 \leq \alpha < 1$. *Then* $f(z)^\rho$ *is harmonic univalent and sense preserving in* U *and* $f(z)^\rho \in \Psi_\gamma^{\rho,\mu}(m,\alpha,\zeta,\lambda)$.

Proof. For $z_1 \neq z_2$ and $|z_1| < |z_2| < 1$, we have by using (10),

$$|f(z_1)^\rho - f(z_2)^\rho| \geq \left| z_1^{\rho(\mu+1)} - z_2^{\rho(\mu+1)} + \sum_{k=2}^{\infty} \rho \left\{ z_1^{(\rho-1)(\mu+1)+(\mu+k)} - z_2^{(\rho-1)(\mu+1)+(\mu+k)} \right\} a_k \right|$$

$$- \left| \sum_{k=1}^{\infty} \rho \left\{ z_1^{(\rho-1)(\mu+1)+(\mu+k)} - z_2^{(\rho-1)(\mu+1)+(\mu+k)} \right\} b_k \right|$$

$$\geq |z_1^{\rho(\mu+1)} - z_2^{\rho(\mu+1)}| \left[(\mu+1) - \frac{\gamma}{\rho} - [\rho(\mu+1)+\zeta]|b_1| \right.$$

$$\left. -|z_2^\rho(\mu+1)| \sum_{k=2}^{\infty} [(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)(|a_k|+|b_k|) \right]$$

$$\geq |z_1^{\rho(\mu+1)} - z_2^{\rho(\mu+1)}| \left[(\mu+1) - \frac{\gamma}{\rho} - [\rho(\mu+1)+\zeta]|b_1| \right] (1 - |z_2|^{\rho(\mu+1)}) > 0$$

Hence, $f(z_1)^\rho \neq f(z_2)^\rho$, which shows that f^ρ is univalent in U.
We prove that f^ρ is sense preserving in U, since by using (10), $|z| < 1$,

$$|h'(z)^\rho| \geq \rho(\mu+1) - \sum_{k=2}^{\infty} \rho[(\mu+k)+(\rho-1)(\mu+1)]|a_k|$$

$$> \rho(\mu+1) - \gamma - \sum_{k=2}^{\infty} \rho[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)|a_k|$$

$$\geq \sum_{k=1}^{\infty} \rho[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)|b_k| \geq |g'(z)^\rho|$$

We now show that $f(z)^\rho \in \Psi_\gamma^{\rho,\mu}(m,\alpha,\zeta,\lambda)$. In (8), we make use of the fact that $Re\{w\} > \gamma$ if and only if $|1+w-\gamma| > |1-w+\gamma|$.
By substituting for $h(z)^\rho$ and $g(z)^\rho$ from (4) and using (10), we get

$$\left| 1 + \zeta \tilde{D}_\lambda^{m,\alpha} \left[\frac{h(z)^\rho}{z^{\rho(\mu+1)}} + \overline{\left(\frac{g(z)^\rho}{z^{\rho(\mu+1)}} \right)} \right] + \tilde{D}_\lambda^{m,\alpha} \left[h'(z)^\rho + \overline{g'(z)^\rho} \right] - \zeta - \gamma \right|$$

$$- \left| 1 - \zeta \tilde{D}_\lambda^{m,\alpha} \left[\frac{h(z)^\rho}{z^{\rho(\mu+1)}} + \overline{\left(\frac{g(z)^\rho}{z^{\rho(\mu+1)}} \right)} \right] - \tilde{D}_\lambda^{m,\alpha} \left[h'(z)^\rho + \overline{g'(z)^\rho} \right] + \zeta + \gamma \right|$$

$$= \left| 1 + \rho(\mu+1)z^{\rho(\mu+1)-1} + \rho[\zeta + \rho(\mu+1)\overline{z^{\rho(\mu+1)-1}}]\overline{b_1} - \gamma + \zeta\rho \sum_{k=2}^{\infty} \phi_{k,m}(\alpha,\lambda)(a_k z^{k-1} + \overline{b_k z^{k-1}}) \right.$$

$$+\rho\sum_{k=2}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)]\phi_{k,m}(\alpha,\lambda)(a_k z^{[(\mu+k)+(\rho-1)(\mu+1)]-1}+\overline{b_k z^{[(\mu+k)+(\rho-1)(\mu+1)]-1}})\Bigg|$$

$$-\Bigg|1-\rho(\mu+1)z^{\rho(\mu+1)-1}-\rho[\zeta+\rho(\mu+1)\overline{z^{\rho(\mu+1)-1}}]\overline{b_1}+\gamma-\zeta\rho\sum_{k=2}^{\infty}\phi_{k,m}(\alpha,\lambda)(a_k z^{k-1}+\overline{b_k z^{k-1}})$$

$$-\rho\sum_{k=2}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)]\phi_{k,m}(\alpha,\lambda)(a_k z^{[(\mu+k)+(\rho-1)(\mu+1)]-1}+\overline{b_k z^{[(\mu+k)+(\rho-1)(\mu+1)]-1}})\Bigg|$$

$$\geq 2\rho(\mu+1)-2\gamma-2\rho[\zeta+\rho(\mu+1)]|b_1|$$

$$-2\rho\sum_{k=2}^{\infty}[\zeta+(\mu+k)+(\rho-1)(\mu+1)]\phi_{k,m}(\alpha,\lambda)(|a_k|+|b_k|)|z|^{[(\mu+k)+(\rho-1)(\mu+1)]-1}$$

$$>2\left[\rho(\mu+1)+\rho[\zeta+\rho(\mu+1)]-\gamma-\rho\sum_{k=1}^{\infty}[\zeta+(\mu+k)+(\rho-1)(\mu+1)]\phi_{k,m}(\alpha,\lambda)(|a_k|+|b_k|)\right]\geq 0$$

The coefficient bound given by (10) can not be improved upon. The harmonic function given by

$$f(z)^{\rho}=z^{\rho(\mu+1)}+\sum_{k=2}^{\infty}\frac{(1-\frac{\gamma}{\rho})x_k}{[(\rho-1)(\mu+1)+(\mu+k)+\zeta]}z^{(\rho-1)(\mu+1)+(\mu+k)}$$

$$+\sum_{k=1}^{\infty}\frac{(1-\frac{\gamma}{\rho})\overline{y_k}}{[(\rho-1)(\mu+1)+(\mu+k)+\zeta]}\overline{z^{(\rho-1)(\mu+1)+(\mu+k)}}$$

where $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, $\rho>0$, $\zeta\geq 0$, $0\leq\mu<1$ and $0\leq\gamma<1$ yields the extremal.

Corollary 3.4. *Let $f(z)^1=f(z)=h(z)+\overline{g(z)}$, where $h(z)$ and $g(z)$ are given by (8). Furthermore, let*

$$\sum_{k=1}^{\infty}(k+\zeta)\phi_{k,m}(\alpha,\lambda)[|a_k|+|b_k|]\leq 2+\zeta-\gamma,\ |b_1|<\frac{1-\gamma}{1+\zeta}<1$$

where $a_1=1$, $\zeta\geq 0$, $0\leq\gamma<1$ and $0\leq\alpha<1$, $\lambda\geq 0$ and $z\in U$. Then $f(z)$ is harmonic univalent and sense preserving in U and $f(z)\in\Psi_{\gamma}^{1,0}(m,\alpha,\zeta,\lambda)$.

Proof. Put $\rho=1$ and $\mu=0$ in Theorem 3.1.

Remark. This is the result obtained by Ezhilarasi *et al.* (2014).

Corollary 3.5. *Let $f(z)^1 = f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are given by (8). Furthermore, let*

$$\sum_{k=1}^{\infty} k\phi_{k,m}(\alpha,\lambda)[|a_k| + |b_k|] \leq 2 - \gamma,\ |b_1| < 1 - \gamma < 1$$

where $a_1 = 1$, $0 \leq \gamma < 1$ and $0 \leq \alpha < 1$, $\lambda \geq 0$ and $z \in U$.
Then $f(z)$ is harmonic univalent and sense preserving in U and $f(z) \in \Psi_{\gamma}^{1,0}(m, \alpha, 0, \lambda)$.

Proof. Put $\rho = 1$ and $\zeta = \mu = 0$ in Theorem 3.1.

Remark. This is the result obtained by Noor *et al.* (2009).

We also prove that the coefficient condition given in (10) is necessary and sufficient condition for $f^{\rho} \in \overline{\Psi_{\gamma}^{\rho;\mu}(m, \alpha, \zeta, \lambda)}$.

Theorem 3.6. *Let $f(z)^{\rho} = h(z)^{\rho} + \overline{g(z)^{\rho}}$, where $h(z)^{\rho}$, $g(z)^{\rho}$ are given by (9) then $f^{\rho} \in \overline{\Psi_{\gamma}^{\rho;\mu}(m, \alpha, \zeta, \lambda)}$ if and only if*

$$\sum_{k=1}^{\infty}[(\mu + k) + (\rho - 1)(\mu + 1) + \zeta]\phi_{k,m}(\alpha,\lambda)(|a_k| + |b_k|) \leq (\rho + 1)(\mu + 1) + \zeta - \frac{\gamma}{\rho}$$

where $a_1 = 1$, $\rho > 0$, $\zeta \geq 0$, $0 \leq \mu < 1$, $0 \leq \gamma < 1$ and $0 \leq \alpha < 1$.

Proof. Suppose that $f^{\rho} \in \overline{\Psi_{\gamma}^{\rho;\mu}(m, \alpha, \zeta, \lambda)}$. Then we find from (8) with $h(z)^{\rho}$, $g(z)^{\rho}$ given by (9) that

$$Re\left\{\zeta \tilde{D}_{\lambda}^{m,\alpha}\left[\frac{h(z)^{\rho}}{z^{\rho(\mu+1)}} + \overline{\left(\frac{g(z)^{\rho}}{z^{\rho(\mu+1)}}\right)}\right] + \tilde{D}_{\lambda}^{m,\alpha}\left[h'(z)^{\rho} + \overline{g'(z)^{\rho}}\right] - \zeta\right\}$$

$$= Re\left\{\rho(\mu + 1)z^{\rho(\mu+1)-1} - \rho[\zeta + \rho(\mu + 1)\overline{z^{\rho(\mu+1)-1}}]|b_1| - \rho\sum_{k=2}^{\infty}[\zeta(|a_k|z^{k-1} + |b_k|\overline{z^{k-1}})\right.$$

$$\left. +[(\mu + k) + (\rho - 1)(\mu + 1)](|a_k|z^{[(\mu+k)+(\rho-1)(\mu+1)]-1} + |b_k|\overline{z^{[(\mu+k)+(\rho-1)(\mu+1)]-1}})]\phi_{k,m}(\alpha,\lambda)\right\} > \gamma$$

$\rho > 0$, $\zeta \geq 0$, $0 \leq \mu < 1$, $0 \leq \gamma < 1$ and $0 \leq \alpha < 1$
If we choose z to be real and let $z \to 1^{-1}$, we get

$$\rho(\mu+1) - \rho[\zeta+\rho(\mu+1)]|b_1| - \rho\sum_{k=2}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)(|a_k|+|b_k|) \geq \gamma$$

which yields the assertion (10) of Theorem 3.1.

Conversely, assume that (10) of Theorem 3.1 holds. Then we find from (8) with $h(z)^\rho$ and $g(z)^\rho$ given by (9) that on using (10), for $|z| < 1$

$$Re\left\{\zeta\tilde{D}_\lambda^{m,\alpha}\left[\frac{h(z)^\rho}{z^{\rho(\mu+1)}}+\overline{\left(\frac{g(z)^\rho}{z^{\rho(\mu+1)}}\right)}\right]+\tilde{D}_\lambda^{m,\alpha}\left[h'(z)^\rho+\overline{g'(z)^\rho}\right]-\zeta\right\}$$

$$= Re\left\{\rho(\mu+1)z^{\rho(\mu+1)-1}-\rho[\zeta+\rho(\mu+1)\overline{z^{\rho(\mu+1)-1}}]|b_1|-\rho\sum_{k=2}^{\infty}[\zeta(|a_k|z^{k-1}+|b_k|\overline{z^{k-1}})\right.$$

$$\left.+[(\mu+k)+(\rho-1)(\mu+1)](|a_k|z^{[(\mu+k)+(\rho-1)(\mu+1)]-1}+|b_k|\overline{z^{[(\mu+k)+(\rho-1)(\mu+1)]-1}})]\phi_{k,m}(\alpha,\lambda)\right\}$$

$$\geq \rho[(\rho+1)(\mu+1)+\zeta]-\rho\sum_{k=1}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)(|a_k|+|b_k|)z^{[(\mu+k)+(\rho-1)(\mu+1)]-1} > \gamma$$

Which yields that $f^\rho \in \overline{\Psi_\gamma^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$.

In the next theorem, we state and prove covering theorem for functions $f^\rho \in \overline{\Psi_\gamma^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$.

Theorem 3.7. *Let* $f^\rho \in \overline{\Psi_\gamma^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$. *Then we have that*

$$f(z)^\rho \leq (1+\rho|b_1|)r^{\rho(\mu+1)}$$

$$+\frac{\rho(2-\alpha)^m}{[(\mu+2)+(\rho-1)(\mu+1)+\zeta]2^m(1+\lambda)^m}\left[(\mu+1)-\frac{\gamma}{\rho}-[\rho(\mu+1)+\zeta]|b_1|\right]r^{(\rho-1)(\mu+1)+(\mu+2)}$$

and

$$f(z)^\rho \geq (1-\rho|b_1|)r^{\rho(\mu+1)}$$

$$-\frac{\rho(2-\alpha)^m}{[(\mu+2)+(\rho-1)(\mu+1)+\zeta]2^m(1+\lambda)^m}\left[(\mu+1)-\frac{\gamma}{\rho}-[\rho(\mu+1)+\zeta]|b_1|\right]r^{(\rho-1)(\mu+1)+(\mu+2)}$$

for $|z| = r < 1$,

Proof. Let $f(z)^\rho = h(z)^\rho + \overline{g(z)^\rho}$, where $h(z)^\rho$, $g(z)^\rho$ are given by (10). Then

$$|f(z)^\rho| = \left|z^{\rho(\mu+1)}-\sum_{k=2}^{\infty}\rho|a_k|z^{(\rho-1)(\mu+1)+(\mu+k)}-\sum_{k=1}^{\infty}\rho|b_k|z^{(\rho-1)(\mu+1)+(\mu+k)}\right|$$

$$\geq (1-\rho|b_1|)r^{\rho(\mu+1)}$$

$$-\frac{\rho(2-\alpha)^m}{[(\mu+2)+(\rho-1)(\mu+1)+\zeta]2^m(1+\lambda)^m}\sum_{k=2}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(|a_k|+|b_k|)r^{(\rho-1)(\mu+1)+(\mu+2)}$$

$$\geq (1-\rho|b_1|)r^{\rho(\mu+1)}$$

$$-\frac{\rho(2-\alpha)^m}{[(\mu+2)+(\rho-1)(\mu+1)+\zeta]2^m(1+\lambda)^m}\left[(\mu+1)-\frac{\gamma}{\rho}-[\rho(\mu+1)+\zeta]|b_1|\right]r^{(\rho-1)(\mu+1)+(\mu+2)}$$

The proof of the upper bounds of $f(z)^\rho$ is similar.

Theorem 3.8. *Let* $f(z)^\rho = h(z)^\rho + \overline{g(z)^\rho}$, *where* $h(z)^\rho$, $g(z)^\rho$ *are given by (9). Then* $f^\rho \in \overline{\Psi_\gamma^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$ *if and only if*

$$f(z)^\rho = \sum_{k=1}^{\infty} \left(\lambda_k h_k(z)^\rho + \mu_k g_k(z)^\rho\right),$$

where

$h_1(z)^\rho = z^{\rho(\mu+1)}$

$$h_k(z)^\rho = z^{\rho(\mu+1)} - \frac{(1-\frac{\gamma}{\rho})}{[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)} z^{(\rho-1)(\mu+1)+(\mu+k)} \quad k = 2,3,\cdots$$

$$g_k(z)^\rho = z^{\rho(\mu+1)} - \frac{(1-\frac{\gamma}{\rho})}{[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)} \overline{z^{(\rho-1)(\mu+1)+(\mu+k)}} \quad k = 1,2,\cdots$$

and

$$\sum_{k=1}^{\infty} (\lambda_k + \mu_k) = 1 \ \lambda_k \geq 0, \mu_k \geq 0$$

Proof. Suppose $f(z)^\rho = \sum_{k=1}^{\infty} \left(\lambda_k h_k(z)^\rho + \mu_k g_k(z)^\rho\right)$

$$= \lambda_1 h_1(z)^\rho + \mu_1 g_1(z)^\rho + \sum_{k=2}^{\infty} \left(\lambda_k h_k(z)^\rho + \mu_k g_k(z)^\rho\right)$$

$$= \sum_{k=1}^{\infty} (\lambda_k + \mu_k)\, z^{\rho(\mu+1)} - \sum_{k=2}^{\infty} \frac{(1-\frac{\gamma}{\rho})\lambda_k}{[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)} z^{(\rho-1)(\mu+1)+(\mu+k)}$$

$$- \sum_{k=1}^{\infty} \frac{(1-\frac{\gamma}{\rho})\mu_k}{[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)} \overline{z^{(\rho-1)(\mu+1)+(\mu+k)}}$$

$$= z^{\rho(\mu+1)} - \sum_{k=2}^{\infty} \rho|a_k| z^{(\rho-1)(\mu+1)+(\mu+k)} - \sum_{k=1}^{\infty} \rho|b_k| \overline{z^{(\rho-1)(\mu+1)+(\mu+k)}}$$

where

$$a_1 = 1 = \sum_{k=1}^{\infty} (\lambda_k + \mu_k)$$

$$|a_k| = \frac{(1-\frac{\gamma}{\rho})\lambda_k}{\rho[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)}, \ k \geq 2$$

$$|b_k| = \frac{(1-\frac{\gamma}{\rho})\mu_k}{\rho[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)}, \ k \geq 1$$

Since,

$$\sum_{k=2}^{\infty}\frac{[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)}{1-\frac{\gamma}{\rho}}|a_k|+\sum_{k=2}^{\infty}\frac{[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)}{1-\frac{\gamma}{\rho}}|b_k|$$

$$=\sum_{k=2}^{\infty}\frac{[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)}{1-\frac{\gamma}{\rho}}\left(\frac{1-\frac{\gamma}{\rho}}{\rho[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)}\right)\lambda_k$$

$$+\sum_{k=1}^{\infty}\frac{[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)}{1-\frac{\gamma}{\rho}}\left(\frac{1-\frac{\gamma}{\rho}}{\rho[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)}\right)\mu_k$$

$$=\sum_{k=2}^{\infty}\frac{\lambda_k}{\rho}+\sum_{k=1}^{\infty}\frac{\mu_k}{\rho}=\frac{1}{\rho}(1-\lambda_1)\leq 1$$

we have

$$\sum_{k=1}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)(|a_k|+|b_k|)=\rho(\mu+1)+\zeta+(1-\frac{\gamma}{\rho})\left[\sum_{k=2}^{\infty}\frac{\lambda_k}{\rho}+\sum_{k=1}^{\infty}\frac{\mu_k}{\rho}\right]$$

$$\leq\rho(\mu+1)+\zeta+1-\frac{\gamma}{\rho}=(\rho+1)(\mu+1)+\zeta+1-\frac{\gamma}{\rho}-\mu-1\leq(\rho+1)(\mu+1)+\zeta+1-\frac{\gamma}{\rho}$$

and so $f^{\rho}\in\overline{\Psi_{\gamma}^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$.

Conversely, suppose $f^{\rho}\in\overline{\Psi_{\gamma}^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$

Let

$$\lambda_k=\frac{\rho[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)|a_k|}{1-\frac{\gamma}{\rho}},\ k=2,3,\cdots$$

$$\mu_k=\frac{\rho[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)|b_k|}{1-\frac{\gamma}{\rho}},\ k=1,2,\cdots$$

$$\lambda_1=1-\sum_{k=2}^{\infty}\lambda_k-\sum_{k=1}^{\infty}\mu_k$$

Now,

$$f(z)^{\rho}=z^{\rho(\mu+1)}-\sum_{k=2}^{\infty}\rho|a_k|z^{(\rho-1)(\mu+1)+(\mu+k)}-\sum_{k=1}^{\infty}\rho|b_k|\overline{z^{(\rho-1)(\mu+1)+(\mu+k)}}$$

$$=z^{\rho(\mu+1)}+\sum_{k=2}^{\infty}\lambda_k\left(h_k(z)^{\rho}-z^{\rho(\mu+1)}\right)+\sum_{k=1}^{\infty}\mu_k\left(g_k(z)^{\rho}-z^{\rho(\mu+1)}\right)$$

$$=\sum_{k=2}^{\infty}\lambda_k h_k(z)^{\rho}+\sum_{k=1}^{\infty}\mu_k g_k(z)^{\rho}+z^{\rho(\mu+1)}\left[1-\sum_{k=2}^{\infty}\lambda_k-\sum_{k=1}^{\infty}\mu_k\right]$$

$$\Longrightarrow f(z)^\rho = \sum_{k=1}^{\infty} (\lambda_k h_k(z)^\rho + \mu_k g_k(z)^\rho)$$

as required.
In particular the extreme points of $\overline{\Psi_{\gamma}^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$ are $\{h_k\}$ and $\{g_k\}$.

3.1. **Convolution and Convex Combination.** We now show that the class $\overline{\Psi_{\gamma}^{\rho,\mu}}(m,\alpha,\zeta,\lambda)$ is invariant under convolution and convex combination of its elements.

For harmonic functions

$$f(z)^\rho = z^{\rho(\mu+1)} - \sum_{k=2}^{\infty} \rho|a_k| z^{(\rho-1)(\mu+1)+(\mu+k)} - \sum_{k=1}^{\infty} \rho|b_k| \overline{z^{(\rho-1)(\mu+1)+(\mu+k)}}$$

and

$$g(z)^\rho = z^{\rho(\mu+1)} - \sum_{k=2}^{\infty} \rho|c_k| z^{(\rho-1)(\mu+1)+(\mu+k)} - \sum_{k=1}^{\infty} \rho|d_k| \overline{z^{(\rho-1)(\mu+1)+(\mu+k)}}$$

The convolution (or Hadamard product) of f^ρ and g^ρ is given by
(11)

$$(f^\rho * g^\rho)(z) = f(z)^\rho * g(z)^\rho = z^{\rho(\mu+1)} - \sum_{k=2}^{\infty} \rho|a_k||c_k| z^{(\rho-1)(\mu+1)+(\mu+k)} - \sum_{k=1}^{\infty} \rho|b_k||d_k| \overline{z^{(\rho-1)(\mu+1)+(\mu+k)}}$$

Theorem 3.9. *For $0 \le \delta \le \gamma < 1$, let $f^\rho \in \overline{\Psi_{\gamma}^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$ and $g^\rho \in \overline{\Psi_{\delta}^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$. Then $f(z)^\rho * g(z)^\rho \in \overline{\Psi_{\gamma}^{\rho,\mu}(m,\alpha,\zeta,\lambda)} \subset \overline{\Psi_{\delta}^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$*

Proof. The convolution $f(z)^\rho * g(z)^\rho$ is given by (11). We want to show that the coefficients of $f(z)^\rho * g(z)^\rho$ satisfy the condition given in (10).
For $g^\rho \in \overline{\Psi_{\gamma}^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$, we note $|c_k| \le 1$, $|d_k| \le 1$.
Now for the convolution $f(z)^\rho * g(z)^\rho$, we obtain

$$\sum_{k=1}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)|a_k||c_k| + \sum_{k=1}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)|b_k||d_k|$$

$$= [\rho(\mu+1)+\zeta]|a_1||c_1| + \sum_{k=2}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)|a_k||c_k|$$

$$+ \sum_{k=1}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)|b_k||d_k|$$

$$\leq \rho(\mu+1)+\zeta+\sum_{k=2}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)|a_k|+\sum_{k=1}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)|b_k|$$

$$\leq (\rho+1)(\mu+1)+\zeta-\frac{\gamma}{\rho} \leq (\rho+1)(\mu+1)+\zeta-\frac{\delta}{\rho}$$

Thus the proof.

We now investigate the convex combination of members of $\overline{\Psi_{\gamma}^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$.

Let the functions $f_i(z)^\rho \in \overline{\Psi_{\gamma}^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$, where f_i^ρ, for $i = 1, 2, \cdots, n$, is given by

$$f_i(z)^\rho = z^{\rho(\mu+1)} - \sum_{k=2}^{\infty}\rho|a_{k,i}|z^{(\rho-1)(\mu+1)+(\mu+k)} - \sum_{k=1}^{\infty}\rho|b_{k,i}|\overline{z^{(\rho-1)(\mu+1)+(\mu+k)}} \tag{12}$$

Theorem 3.10. *Let the functions $f_i(z)^\rho$ defined by (12) be in the class $\overline{\Psi_{\gamma}^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$ for every $i = 1, 2, \cdots, n$. Then the function $t_i(z)^\rho$ defined by*

$$t_i(z)^\rho = \sum_{i=1}^{n} c_i f_i(z)^\rho \ \ (0 \leq c_i \leq 1)$$

is also in the class $\overline{\Psi_{\gamma}^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$, where $\sum_{i=1}^{n} c_i = 1$

Proof. According to the definition of $t_i(z)^\rho$, we can write

$$t_i(z)^\rho = \sum_{i=1}^{n} c_i f_i(z)^\rho$$

$$= \sum_{i=1}^{n} c_i \left[z^{\rho(\mu+1)} - \sum_{k=2}^{\infty}\rho|a_{k,i}|z^{(\rho-1)(\mu+1)+(\mu+k)} - \sum_{k=1}^{\infty}\rho|b_{k,i}|\overline{z^{(\rho-1)(\mu+1)+(\mu+k)}} \right]$$

$$= z^{\rho(\mu+1)} - \sum_{k=2}^{\infty}\rho\left(\sum_{i=1}^{n} c_i|a_{k,i}|\right) z^{(\rho-1)(\mu+1)+(\mu+k)} - \sum_{k=1}^{\infty}\rho\left(\sum_{i=1}^{n} c_i|b_{k,i}|\right)\overline{z^{(\rho-1)(\mu+1)+(\mu+k)}}$$

Using (10), we have

$$[\rho(\mu+1)+\zeta] + \sum_{k=2}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)\left(\sum_{i=1}^{n} c_i|a_{k,i}|\right)$$

$$+ \sum_{k=1}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)\left(\sum_{i=1}^{n} c_i|a_{b,i}|\right)$$

$$= \textstyle\sum_{i=1}^{n} c_i \left(\sum_{k=1}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)|a_{k,i}|\right.$$

$$\left. + \sum_{k=1}^{\infty}[(\mu+k)+(\rho-1)(\mu+1)+\zeta]\phi_{k,m}(\alpha,\lambda)|b_{k,i}|\right) \leq (\rho+1)(\mu+1)+\zeta-\frac{\gamma}{\rho}$$

since each f_i^ρ is in $\overline{\Psi_{\gamma}^{\rho,\mu}(m,\alpha,\zeta,\lambda)}$. This completes the proof.

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