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New Generalizations of Hardy-type Inequalities on Time Scales

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Abstract

This paper consists of new generalizations of Hardy-type integral inequalities and its reversed form on time scales by using modified Jensen′s inequality via convexity. Furthermore, multidimensional Hardy-type inequalities with weight functions on time scales are also investigated and proved.

1. Introduction

The concept of inequalities is of great importance in Mathematics and the study has a pre–history and history of more than a century with a lot of applications to differential and integral calculus. Hardy-type inequalities on time scales are of particular interest.

Almost a century ago, an English Mathematician called Godfrey Harold Hardy announced his famous inequality in 1920. The famous classical inequality states that if $1 < p < \infty$, $A_n = \sum_{k=1}^{n} a_k$ and $a_n = \{a_k\}$ is a sequence of non-negative real number, then

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} A_n \right|^p \leq C_p \sum_{n=1}^{\infty} |a_n|^p \tag{1}$$

Hardy (1925) proved the continuous counterpart that if $f(x)$ is a non-negative p-integrable function defined on $(0, \infty)$, and $p > 1$. Then f is integrable over the

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interval $(0, x)$ for each x and the following inequality:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx \tag{2}$$

holds, where $\left(\frac{p}{p-1}\right)^p$ is the best possible constant. Inequality (2) is called the continuous Hardy inequality. A well-known simple fact is that (2) can equivalently, via the substitution $f(x) = h(x^{1-\frac{1}{p}})x^{\frac{-1}{p}}$, be written in the form:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x h(t)dt\right)^p \frac{dx}{x} \leq \int_0^\infty h^p(x)\frac{dx}{x} \tag{3}$$

and the form even holds with equality when p=1. Observe that (3) can easily be proved by using Jensen inequality and Fubini's theorem. Hardy (1928) obtained a generalized form of (2), namely that if $p \geq 1$ and $k \neq 1$, then

$$\int_0^\infty x^{-k}\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{k-1}\right)^p \int_0^\infty x^{p-k} f^p(x)dx \tag{4}$$

where $(p \geq 1, k > 1)$ and the dual form of this inequality is obtained as:

$$\int_0^\infty x^{-k}\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{1-k}\right)^p \int_0^\infty x^{p-k} f^p(x)dx \tag{5}$$

where $(p \geq 1, k < 1)$. The constant $\left(\frac{p}{|k-1|}\right)^p$ is the best possible constant in (4) and (5).

Furthermore, Hardy *et al.* (1952) pointed out that if k and f satisfy the conditions of the results, then (4) and (5) hold in the reversed direction with $0 < p \leq 1$ respectively.

Inequality (2) was extended and generalized in many direction, if

$$T : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \tag{6}$$

where T is an integral operator of the form:

$$(Tf)(x) = \int_{-\infty}^x K(x, y)f(y)dy \tag{7}$$

Then, Hardy inequality is expressed in form of an operator as

$$\int_0^\infty (Tf)(x)^p dx \leq A(K, p)\int_0^\infty f(x)^p dx \tag{8}$$

where $A(K, p)$ is a constant independent of f, $p > 1$ and $K(x, y) = \frac{1}{x}$ and 0 otherwise.

In the last few decades, a lot of research have been carried out on Hardy-type inequalities. Some of the notable generalization are those of Agarwal *et al.* (2015), Čižmešija *et al.* (2001), Oguntuase *et al.* (2005), Rauf *et al.* (2012), Sarkaya and Yildirim (2006), Sulaiman (2012).

A *time scale* is an arbitrary non-empty closed subset of the real numbers. Thus,

$$\mathbb{R},\ \mathbb{Z},\ \mathbb{N},\ \mathbb{N}_0$$

i.e. the real numbers, the integers, the natural numbers, and the non–negative integers are examples of time scales.
while

$$\mathbb{Q},\ \mathbb{R}\backslash\mathbb{Q},\ \mathbb{C},\ (0,1),$$

i.e, the rational numbers, the irrational numbers, the complex numbers, and the open interval between 0 and 1, are not time scales. Throughout this research, we will denote a time scale by the symbol $\mathbb{T}$. We assume throughout that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.
The calculus of time scales was initiated by Stefan Hilger (1988) in order to create a theory that can unify discrete and continuous analysis. Indeed, we will introduce the delta derivative f for a function f defined on $\mathbb{T}$ and it turns out that:
(i)$f^{\Delta} = f'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$ and
(ii)$f^{\Delta} = \Delta f$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$.

2. Literature Review

This literature introduce the basic notions connected to time scales and differentiability of functions on them and we offer the above two cases as examples.
The new concept has inspired researchers to study Hardy inequality on time scales.
In time scales analysis, Bohner and Peterson (2001), (2003) and Hilger (1990) summarize and organize much of the time scale calculus. Some researchers who worked on the applications of time scales on oscillations of dynamic equations on time scales are those of Agarwal *et al.* (2014), Chen (2013), Saker (2010), Saker *et al.* (2016), Tongxing *et al.* (2011), Tuna and Kutukcu (2008).
Řehák (2005) pioneered the time scales version of Hardy inequality in an attempt to unify and extend the classical Hardy integral inequality and the discrete Hardy inequality by means of the theory of time scales and obtaining the following result:

$$\int_a^\infty \left(\frac{F^\sigma(x)}{\sigma(x)-a}\right)^p \Delta x \leq \left(\frac{p}{p-1}\right)^p \int_a^\infty f^p(x)\Delta x \tag{9}$$

where $p > 1$, $F(x) := \int_0^x f(t)\Delta t$ and f is a non–negative function. Ozakan and Yildirim (2008) gave a time scale Hardy inequality involving several functions as:

Theorem 2.1. *Let $a \geq 0$ and $f_1, f_2, \ldots, f_n, n \in \mathbb{Z}_+$ be non–negative integrable function. Define $F_k(x) \equiv \frac{1}{\sigma(x)-a}\int_a^x f_k(t)\Delta t$, $k = 1, 2, \ldots, n$. Then*

$$\int_a^\infty \left(\prod_{k-1}^{n} F_k^\sigma(x)\right)^{\frac{p}{n}} \Delta x \leq \left(\frac{p}{p-1}\right)^p \int_a^\infty \left(\frac{1}{n}\sum_{k=1}^{n} f_k(x)\right)\Delta x \tag{10}$$

Furthermore, Ozakan and Yildrim (2008) obtained the time scale Hardy-Knopp type inequality as:

Theorem 2.2. *If $u \in C_{rd}([a,b),\mathbb{R})$ is a non–negative function such that the delta integral $\int_t^b \frac{u(x)}{(x-a)(\sigma(x)-a)}\Delta x$ exists as a finite number and the function v is defined by*

$$V(t) = (t-a)\int_t^b \frac{u(x)}{(x-a)(\sigma(x)-a)}\Delta x, t \in [a,b) \tag{11}$$

If $\phi : (c,d) \to \mathbb{R}$ is continuous and convex, where $c, d \in \mathbb{R}$ then the inequality

$$\int_a^b u(x)\phi\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t\right)\frac{\Delta x}{x-a} \leq \int_a^b \nu(x)\phi(f(x))\frac{\Delta x}{x-a} \tag{12}$$

holds for all delta integrable functions $f \in C_{rd}([a,b),\mathbb{R})$ such that $f(x) \in (c,d)$.

Ozakan and Yildrim (2009) further obtained a generalization of Hardy–Knopp type inequality for several functions on time scales and also derived the Hardy-Knopp type inequality with general kernel.
Josipa *et al.* (2013) proved two different methods of Jensen's inequality on time scales for superquadratic functions and some refinement of classical inequalities on time scales.
Oguntuase and Persson (2014) derived some new Hardy type inequality using the concept of superquadratic functions and also extended Hardy-type inequalities of subquadratic functions with general kernels to the case of arbitrary time scales.
Saker *et al.* (2014) derived some new dynamics inequalities of Hardy-Type on times scales using some algebraic inequalities, Hölder's inequality and Keller's chain on time scales.
Saker *et al.* (2017) extended some integral and discrete dynamic inequalities on time scales and formulated them using chain rule, integration by parts, Hölder's inequality and Jensen's inequality on time scales to prove their results.
Throughout this research, we denote $f^\sigma := fo\sigma$, where the forward jump operator σ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. By $x : \mathbb{T} \to \mathbb{R}$ is rd-continuous, we mean x is continuous at all right-dense points $t \in \mathbb{T}$ and at all left-dense points $t \in \mathbb{T}$ left hand limits exist (finite). The graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ is defined by $\mu := \sigma(t) - t$ Also $\mathbb{T}^k := \mathbb{T} - \{m\}$ if $\mathbb{T}$ has a left-scattered maximum m, otherwise $\mathbb{T}^k := \mathbb{T}$. we will assume that $\sup\mathbb{T} = \infty$ and define the time scale

interval $[a, b]_{\mathbb{T}}$ by $[a, b]_{\mathbb{T}} := [a, b \cap \mathbb{T}]$.

Example 1. Let us briefly consider two examples $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

(1) If $\mathbb{T} = \mathbb{R}$, then we have for any $t \in \mathbb{R}$

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t$$

and similarly $\rho(t) = t$. Hence every point $t \in \mathbb{R}$ is dense. The graininess function μ turns out to be

$$\mu(t) \equiv 0 \text{ for all } t \in \mathbb{T}$$

(2) If $\mathbb{T} = \mathbb{Z}$, then we have for any $t \in \mathbb{Z}$

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf\{t+1, t+2, t+3, \ldots, \} = t+1$$

and similarly $\rho(t) = t - 1$. Hence every point $t \in \mathbb{Z}$ is isolated. The graininess function μ in this case is

$$\mu(t) \equiv 1 \text{ for all } t \in \mathbb{T}.$$

For the two cases discussed above, the graininess function is a constant function. We will see below that the graininess function plays a central role in the analysis on time scales. For the general case, many formulae will have some term containing the factor $\mu(t)$, this term is there in case $\mathbb{T} = \mathbb{Z}$ since $\mu(t) \equiv 1$. However, for the case $\mathbb{T} = \mathbb{R}$ this term disappears since $\mu(t) \equiv 0$ in this case.
A function $f : \mathbb{T} \to \mathbb{R}$ and define the so-called *delta (or Hilger) derivative* of f at a point $t \in \mathbb{T}^k$.

Example 2.

(1) If $f : \mathbb{T} \to \mathbb{R}$ is define by $f(t) = \alpha$ for all $t \in \mathbb{T}$, where $\alpha \in \mathbb{R}$ is constant, then $f^{\Delta}(t) \equiv 0$. This is clear because for any $\epsilon > 0$,

$$|f(\sigma(t)) - f(s) - 0.[\sigma(t) - s]| = |\alpha - \alpha| = 0 \leq \epsilon|\sigma(t) - s|$$

holds for all $s \in \mathbb{T}$.

(2) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then $f^{\Delta}(t) \equiv 1$. This follows since for any $\epsilon > 0$,

$$|f(\sigma(t)) - f(s) - 1.[\sigma(t) - s]| = |\sigma(t) - s - (\sigma(t) - s)| = 0 \leq \epsilon|\sigma(t) - s|$$

holds for all $s \in \mathbb{T}$.

Theorem 2.3. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following:*

(1) *If f is differentiable at t, then f is continuous at t.*
(2) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

If t is right-dense, then f is differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f(t).$$

In order to describe classes of functions that are integrable, we introduce the following theorem.

Theorem 2.4. *Assume $f : \mathbb{T} \to \mathbb{R}$.*

(1) *If f is continuous, then f is rd-continuous;*
(2) *If f is rd-continuous, then f is regulated;*
(3) *The jump operator σ is rd-continuous.;*
(4) *If f is regulated or rd-continuous, then so is f^{σ}.;*
(5) *Assume f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then fog has that property too.*

The aim of this work is to extend these results to some generalization of Hardy-type inequalities on time scales. In particular, extensions of Jensen's inequality, Minkowski's inequality are utilized.

3. Main Results

3.1. **Some adaptations of Jensen's inequality.** (for convex function)
Let φ, ψ be continuous and convex and let $h(s,t)$ be non–negative, $s \geq 0, t \geq 0$ and λ be non-decreasing. Let $-\infty \leq \xi(t) \leq \eta(t) < \infty$, and suppose φ has a continuous inverse $(\varphi)^{-1}$(which is necessarily concave).
Then,

$$\varphi^{-1}\left[\frac{\int_{\xi(t)}^{\eta(t)} h(s,t)d\lambda(s)}{\int_{\xi(t)}^{\eta(t)} d\lambda(s)}\right] \geq \left[\frac{\int_{\xi(t)}^{\eta(t)} (\varphi)^{-1} h(s,t)d\lambda(s)}{\int_{\xi(t)}^{\eta(t)} d\lambda(s)}\right] \tag{13}$$

with the inequality reversed if φ is concave. The inequality (13) is known as Jensen's inequality for convex function. Setting $\varphi(u) = u^l$, $\xi(t) = t$ and $\eta(t) = b$, then as a consequence of (13), we have for $l \geq 1$

$$\left[\frac{\int_t^b h(s,t)d\lambda(s)}{\int_t^b d\lambda(s)}\right]^{\frac{1}{l}} \geq \left[\frac{\int_t^b h(s,t)^{\frac{1}{l}} d\lambda(s)}{\int_t^b d\lambda(s)}\right] \tag{14}$$

which we write as:

$$\left[\int_t^b h(s,t)^{\frac{1}{l}} d\lambda(s)\right]^l \leq \left[\int_t^b d\lambda(s)\right]^{l-1} \left[\int_t^b h(s,t)d\lambda(s)\right] \tag{15}$$

and for $0 < l < 1$, the inequality

$$\int_t^b h(s,t)d\lambda(s) \leq \left[\int_t^b d\lambda(s)\right]^{1-l} \left[\int_t^b h(s,t)^{\frac{1}{l}} d\lambda(s)\right]^l \tag{16}$$

is the reverse of (15). However, (13) can also be written as:

$$\varphi^{-1}\left(\psi\left[\frac{\int_{\xi(t)}^{\eta(t)} h(s,t)d\lambda(s)}{\int_{\xi(t)}^{\eta(t)} d\lambda(s)}\right]\right) \geq \left[\frac{\int_{\xi(t)}^{\eta(t)} \varphi^{-1}(\psi(h(s,t))d\lambda(s)}{\int_{\xi(t)}^{\eta(t)} d\lambda(s)}\right] \tag{17}$$

Let C be a continuous function, then $\psi, \varphi \in C([\alpha, \beta], \mathbb{R})$. Suppose φ is convex, ψ is non-negative and x is non-decreasing. Then,

$$\left(\int_\epsilon^t d\lambda(s)\right)^{-\varsigma} \leq \left(\int_\epsilon^t (\psi(s))^{\frac{1}{\zeta}} d\lambda(s)\right)^{\zeta} \left(\int_\epsilon^t \psi(t)d\lambda(s)\right)^{-\varsigma} \left(\int_\epsilon^t d\lambda(s)\right)^{-\zeta}. \tag{18}$$

In this section, we let (X, ζ, μ) be a σ-finite time scale measure space, $K(x,y)$ be a non-negative and measurable on $X \times X$ and T a positive linear operator defined for non-negative functions on the measure space. Sedov (1972) and Sysoeva (1965) dealt with some weighted inequalities for a multiple Hardy operator T_n of the form:

$$T_n = \int_a^x \int_a^{x_1} \dots \int_a^{x_n} f(t)\Delta t \Delta x_n \dots \Delta_{x_1}$$

The next theorem discusses the results of Rauf *et al.* (2012) on time scales.

Theorem 3.1. *Let $\mathbb{T}$ be a time scale. Let $p_1, \dots, p_n > 0$ such that $\sum_{k=1}^{n} \frac{1}{p_k} = 1$ and suppose ω_k are weight functions on X. For a positive function f on $(0,\infty)$, we define the operator T by $\int_x K(x,y)f(y)\Delta y$. Let f_k be p-integrable positive function on $(0,\infty)$ for $k = 1, \dots, n+1$, $0 < \pi < \infty$ and $0 < \zeta \leq \varsigma < \infty$. Then, there exist weight functions ν_k, finite μ-almost everywhere on X such that:*

$$\left(\left(\int_\pi \vartheta_1 \vartheta_\zeta w_\zeta \Delta(s)\right)^{\varsigma} \Delta(s)\right)^{\frac{1}{\delta}} \leq C \left(\left(\int_\pi \vartheta_1 \vartheta_\zeta w_\zeta \Delta(s)\right)^{\zeta} \Delta(s)\right)^{\frac{1}{\delta}} \tag{19}$$

where ϑ is positive $rd-$continuous function on time scale such that $\int_\pi K(x,y)\Delta(s) \leq \infty$.

Proof: In (18), the following holds:

$$\begin{aligned} &\varphi(s) = s^{\zeta},\ \psi(s) = K(x,y)^{\frac{1}{\rho_1}} \vartheta_1 \phi(s)^{\frac{\rho_1 - 1}{\rho_1}} K(x,y)^{-\frac{\rho_1-1}{\rho_1}} \phi(s)^{-\frac{\rho_1-1}{\rho_1}}, \\ &d\lambda(s) = K(x,y)^{\frac{1}{\rho_\zeta}} \vartheta_\zeta \phi(s)^{\frac{\rho_\zeta-1}{\rho_\zeta}} K(x,y)^{-\frac{\rho_\zeta-1}{\rho_\zeta}} \phi(s)^{-\frac{\rho_\zeta-1}{\rho_\zeta}} w_\zeta \Delta(s) \end{aligned} \tag{20}$$

where ϕ is $rd-$continuous. Therefore, we have

(21)

$$\left(\frac{\int_\pi K(x,y)^{\frac{1}{\rho_1}} \vartheta_1 \phi^{\frac{\rho_1-1}{\rho_1}} K(x,y)^{-\frac{\rho_1-1}{\rho_1}} \phi^{-\frac{\rho_1-1}{\rho_1}} K(x,y)^{\frac{1}{\rho_\zeta}} \vartheta_\zeta \phi(s)^{\frac{\rho_\zeta-1}{\rho_\zeta}} K(x,y)^{-\frac{\rho_\zeta-1}{\rho_\zeta}} \phi(s)^{-\frac{\rho_\zeta-1}{\rho_\zeta}} w_\zeta \Delta(s)}{\int_\pi K(x,y)^{\frac{1}{\rho_\zeta}} \vartheta_\zeta \phi^{\frac{\rho_\zeta-1}{\rho_\zeta}} K(x,y)^{-\frac{\rho_\zeta-1}{\rho_\zeta}} \phi(s)^{-\frac{\rho_\zeta-1}{\rho_\zeta}} w_\zeta \Delta(s)}\right)^{\varsigma} \leq$$

$$\left(\frac{\int_\pi \varphi \left(K(x,y)^{\frac{1}{\rho_1}} \vartheta_1 \phi^{\frac{\rho_1-1}{\rho_1}} K(x,y)^{-\frac{\rho_1-1}{\rho_1}} \phi^{-\frac{\rho_1-1}{\rho_1}}\right)^{\frac{1}{\zeta}} K(x,y)^{\frac{1}{\rho_\zeta}} \vartheta_\zeta \phi^{\frac{\rho_\zeta-1}{\rho_\zeta}} K(x,y)^{-\frac{\rho_\zeta-1}{\rho_\zeta}} \phi(s)^{-\frac{\rho_\zeta-1}{\rho_\zeta}} w_\zeta \Delta(s)}{\int_\pi K(x,y)^{\frac{1}{\rho_\zeta}} \vartheta_\zeta \phi(s)^{\frac{\rho_\zeta-1}{\rho_\zeta}} K(x,y)^{-\frac{\rho_\zeta-1}{\rho_\zeta}} \phi(s)^{-\frac{\rho_\zeta-1}{\rho_\zeta}} w_\zeta \Delta(s)}\right)^{\zeta}$$

latter integral inequality implies

(22)

$$\left(\int_\pi K(x,y)^{\frac{1}{\rho_1}}\vartheta_1\phi^{\frac{\rho_1-1}{\rho_1}}K(x,y)^{-\frac{\rho_1-1}{\rho_1}}\phi^{-\frac{\rho_1-1}{\rho_1}}K(x,y)^{\frac{1}{\rho_\zeta}}\vartheta_\zeta\phi^{\frac{\rho_\zeta-1}{\rho_\zeta}}K(x,y)^{-\frac{\rho_\zeta-1}{\rho_\zeta}}\phi^{-\frac{\rho_\zeta-1}{\rho_\zeta}}w_\zeta\Delta(s)\right)^{\varsigma}w_\zeta$$
$$\leq\left(\int_\pi K(x,y)^{\frac{1}{\rho_\zeta}}\vartheta_\zeta\phi^{\frac{\rho_\zeta-1}{\rho_\zeta}}K(x,y)^{-\frac{\rho_\zeta-1}{\rho_\zeta}}\phi^{-\frac{\rho_\zeta-1}{\rho_\zeta}}w_\zeta\Delta(s)\right)^{\varsigma-\zeta}\times$$
$$\left(\int_\pi\varphi\left(K(x,y)^{\frac{1}{\rho_1}}\vartheta_1\phi^{\frac{\rho_1-1}{\rho_1}}K(x,y)^{-\frac{\rho_1-1}{\rho_1}}\phi^{-\frac{\rho_1-1}{\rho_1}}\right)^{\frac{1}{\zeta}}K(x,y)^{\frac{1}{\rho_\zeta}}\vartheta_\zeta\phi^{\frac{\rho_\zeta-1}{\rho_\zeta}}K(x,y)^{-\frac{\rho_\zeta-1}{\rho_\zeta}}\phi^{-\frac{\rho_\zeta-1}{\rho_\zeta}}w_\zeta\Delta(s)\right)^{\zeta}$$

By convexity, (22) becomes

(23)

$$\left(\int_\pi K(x,y)^{\frac{1}{\rho_1}}\vartheta_1\phi^{\frac{\rho_1-1}{\rho_1}}K(x,y)^{-\frac{\rho_1-1}{\rho_1}}\phi^{-\frac{\rho_1-1}{\rho_1}}K(x,y)^{\frac{1}{\rho_\zeta}}\vartheta_\zeta\phi^{\frac{\rho_\zeta-1}{\rho_\zeta}}K(x,y)^{-\frac{\rho_\zeta-1}{\rho_\zeta}}\phi^{-\frac{\rho_\zeta-1}{\rho_\zeta}}w_\zeta\Delta(s)\right)^{\varsigma}$$
$$\leq\left(\int_\pi K(x,y)^{\frac{1}{\rho_\zeta}}\vartheta_\zeta\phi^{\frac{\rho_\zeta-1}{\rho_\zeta}}K(x,y)^{-\frac{\rho_\zeta-1}{\rho_\zeta}}\phi^{-\frac{\rho_\zeta-1}{\rho_\zeta}}w_\zeta\Delta(s)\right)^{\varsigma-\zeta}\times$$
$$\left(\int_\pi K(x,y)^{\frac{1}{\rho_1}}\vartheta_1\phi^{\frac{\rho_1-1}{\rho_1}}K(x,y)^{-\frac{\rho_1-1}{\rho_1}}\phi^{-\frac{\rho_1-1}{\rho_1}}K(x,y)^{\frac{1}{\rho_\zeta}}\vartheta_\zeta\phi^{\frac{\rho_\zeta-1}{\rho_\zeta}}K(x,y)^{-\frac{\rho_\zeta-1}{\rho_\zeta}}\phi^{-\frac{\rho_\zeta-1}{\rho_\zeta}}w_\zeta\Delta(s)\right)^{\zeta}$$

with further simplification of (19) we obtain

$$\left(\int_\pi\vartheta_1\vartheta_\zeta w_\zeta\Delta(s)\right)^{\varsigma}\Delta(s)\leq\left(\int_\pi Aw_\zeta\Delta(s)\right)^{\varsigma-\zeta}\left(\int_\pi\vartheta_1\vartheta_\zeta w_\zeta\Delta(s)\right)^{\zeta}\tag{24}$$

(25)

$$\left(\left(\int_\pi\vartheta_1\vartheta_\zeta w_\zeta\Delta(s)\right)^{\varsigma}\Delta(s)\right)^{\frac{1}{\delta}}\leq\left(\int_\pi Aw_\zeta\Delta(s)\right)^{\frac{\varsigma-\zeta}{\delta}}\left(\left(\int_\pi\vartheta_1\vartheta_\zeta w_\zeta\Delta(s)\right)^{\zeta}\Delta(s)\right)^{\frac{1}{\delta}}$$

which implies

$$\left(\left(\int_\pi\vartheta_1\vartheta_\zeta w_\zeta\Delta(s)\right)^{\varsigma}\Delta(s)\right)^{\frac{1}{\delta}}\leq C\left(\left(\int_\pi\vartheta_1\vartheta_\zeta w_\zeta\Delta(s)\right)^{\zeta}\Delta(s)\right)^{\frac{1}{\delta}}\tag{26}$$

If in (24), the $\vartheta_1\vartheta_\zeta w_\zeta$ is replaced with $\vartheta_1+\vartheta_\zeta w_\zeta$, thus the theorem follows:

Theorem 3.2. *Let $\mathbb{T}$ be a time scale and $\alpha > 1$. Let $K, \pi, \vartheta, \phi \in C_{rd}([0,\infty]_{\mathbb{T}})$, where K, π, ϑ, ϕ are positive rd−continuous function on time scale such that $\int_a^b K(x,y)\Delta(s) \leq \infty$. Let ϑ_ζ be ρ–integrable positive function defined on time scale for $k = 1, \cdots, n$. Then, there exist weight functions ν_k, finite μ-almost everywhere on X such that:*

$$\left.\begin{array}{c}\left(\left(\int_a^b([\vartheta_1+\vartheta_\zeta w_\zeta]^\alpha\,\Delta(s)\right)^{\frac{1}{\alpha}}\right)^\zeta \leq \left(\int_a^b A w_\zeta \Delta(s)\right)^{\varsigma-\zeta} \\ \left[\left(\int_a^b \vartheta_1^\alpha \Delta(s)\right)^{\frac{1}{\alpha}} + \left(\int_a^b (\vartheta_\zeta w_\zeta)^\alpha \Delta(s)\right)^{\frac{1}{\alpha}}\right]^\zeta\end{array}\right\} \tag{27}$$

Proof: We apply Hölder's inequality, Bohner *et al.* (2001) with $\beta = \alpha/(\alpha-1)$. If $\left(\int_a^b [\vartheta_1+\vartheta_\zeta w_\zeta]^\alpha \Delta(s)\right)^{\frac{1}{\alpha}} = 0$ then the inequality holds. We therefore assume that $\left(\int_a^b [\vartheta_1+\vartheta_\zeta w_\zeta]^\alpha \Delta(s)\right)^{\frac{1}{\alpha}} \neq 0$, then

$$\left.\begin{array}{r}\left(\int_a^b [\vartheta_1+\vartheta_\zeta w_\zeta]^\alpha \Delta(s)\right)^\zeta \\ \leq \left(\int_a^b A w_\zeta \Delta(s)\right)^{\varsigma-\zeta}\left(\int_a^b [\vartheta_1+\vartheta_\zeta w_\zeta]^{\alpha-1}[\vartheta_1+\vartheta_\zeta w_\zeta]\Delta(s)\right)^\zeta \Delta(s) \\ = \left(\int_a^b A w_\zeta \Delta(s)\right)^{\varsigma-\zeta}\left(\int_a^b([\vartheta_1+\vartheta_\zeta w_\zeta]^{\alpha-1}\vartheta_1+[\vartheta_1+\vartheta_\zeta w_\zeta]^{\alpha-1}\vartheta_\zeta w_\zeta)\Delta(s)\right)^\zeta \Delta(s) \\ = \left(\int_a^b A w_\zeta \Delta(s)\right)^{\varsigma-\zeta}\left(\int_a^b([\vartheta_1+\vartheta_\zeta w_\zeta]^{(\alpha-1)\beta}\Delta(s)\right)^{\frac{\zeta}{\beta}}\left[\left(\int_a^b \vartheta_1^\alpha\Delta(s)\right)^{\frac{1}{\alpha}}+\left(\int_a^b(\vartheta_\zeta w_\zeta)^\alpha\Delta(s)\right)^{\frac{1}{\alpha}}\right]^\zeta \\ \leq \left(\int_a^b A w_\zeta \Delta(s)\right)^{\varsigma-\zeta}\left(\int_a^b([\vartheta_1+\vartheta_\zeta w_\zeta]^{\alpha}\Delta(s)\right)^{\frac{\zeta}{\beta}}\left[\left(\int_a^b \vartheta_1^\alpha\Delta(s)\right)^{\frac{1}{\alpha}}+\left(\int_a^b(\vartheta_\zeta w_\zeta)^\alpha\Delta(s)\right)^{\frac{1}{\alpha}}\right]^\zeta\end{array}\right\} \tag{28}$$

Dividing both sides by $\left(\int_a^b([\vartheta_1+\vartheta_\zeta w_\zeta]^\alpha \Delta(s)\right)^{\frac{\zeta}{\beta}}$, we have

$$\left.\begin{array}{c}\left(\left(\int_a^b([\vartheta_1+\vartheta_\zeta w_\zeta]^\alpha\,\Delta(s)\right)^{\frac{1}{\alpha}}\right)^\zeta \leq \left(\int_a^b A w_\zeta \Delta(s)\right)^{\varsigma-\zeta} \\ \left[\left(\int_a^b \vartheta_1^\alpha \Delta(s)\right)^{\frac{1}{\alpha}} + \left(\int_a^b (\vartheta_\zeta w_\zeta)^\alpha \Delta(s)\right)^{\frac{1}{\alpha}}\right]^\zeta\end{array}\right\} \tag{29}$$

which completes the proof.

Conclusion

Rauf *et al.* (2012) used various methods and approaches to actualized the results therein. However, we introduced the concept of Jensen's inequality with convex functions on time scales as an essential tool to refine the work.

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