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Variational Iteration Method for Solving Higher-order Integro-differential Equations

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Abstract

In this paper, a numerical method for the solution of integro-differential equations is discussed. The variational iteration method (VIM), which is a modified general Lagrange multiplier method is employed to solve different types of integro-differential equations. The results revealed that the VIM is very effective, simple and is of high accuracy for solving higher order integro-differential equations.

1. Introduction

Mathematical modeling of physical phenomena usually result in integro-differential equations. These equations arise in biological models, fluid mechanics and chemical kinetics [1]. The variational iteration method, which is a modified general Lagrange multiplier method was proposed by He [2,3] and has been proved by many authors [4-11] as effective and efficient.
Recently, the variational iteration method was used to solve linear and non-linear stiff differential equations [12] as well as solution of linear and nonlinear Fredholm integral equations [13].

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In this paper, the application of variational iteration method is extended to the solution of integro-differential equations of the form:

$$y^n(x) = f(x, y(x)) + \int_a^b F(x, t, y(t))dt, \ y(x_0) = y_0 \tag{1}$$

where $a < x < b$ and m and n are integers such that $n - m > 0$. The functions f and F are given while y is to be determined. It is assumed that the computational example of the type (1) has unique solution. Comparison of the numerical solutions obtained with the given exact solutions show that the method converges rapidly to the exact solutions. Hence, we conclude that the method is elegant, effective and efficient.

2 Variational Iteration Method

To present the basic and fundamentals of concept of the variational iteration method, we consider the following general nonlinear equation:

$$Ly(x) + Ny(x) = g(x) \tag{2}$$

where L is a linear operator, N is nonlinear operator and $g(x)$ is a given continuous function.
According to variational iteration method [2-12] the correction functional can be constructed as:

$$y_{K+1}(x) = y_K(x) + \int_0^x \lambda(\tau) \left[Ly_K(\tau) + N\bar{y_K}(\tau) - g(\tau) \right] d\tau \tag{3}$$

where $y_0(x)$ is an initial approximation, λ is called a general Lagrange multiplier which can be identified optimally through variational theory, the subscript K denotes the K^{th} iteration and $\bar{y_K}$ is the restricted variation.
From (3), we can easily derive a correction functional for (1) as indicated below:

$$y_{K+1}(x) = y_K(x) + \int_0^x \lambda(\tau) \left[y_K^{(n)}(\tau) - f(\tau, y_K(\tau)) - \int_a^b F(\tau, t, y_K^m(t))dt \right] d\tau \tag{4}$$

Making above stationary, the Lagrange multiplier, therefore can be identified as:

$$\lambda(\tau) = \frac{(-1)^n}{(n-1)!}(\tau - x)^{n-1} \tag{5}$$

and equation (4) becomes:

$$\begin{aligned} &y_{K+1}(x) = y_K(x) + \\ &\int_0^x \frac{(-1)^n}{(n-1)!}(\tau - x)^{n-1} \left[y_K^{(n)}(\tau) - f(\tau, y_K(\tau)) - \int_a^b F(\tau, t, y_K^m(t))dt \right] d\tau \end{aligned} \tag{6}$$

3 Application

In this section, we present some computational examples to justify the conclusion that the variational iteration method is efficient for solving (1). The implementation associated with the problems presented were implemented using maple 18.

Example 3.1: Consider the integro-differential equation:

$$y'(x) = y(x) + x - \int_0^1 xty(t)dt, \; y(0) = 1 \tag{7}$$

with exact solution:

$$y(x) = e^x$$

According to (6), we have the following iteration formular:

$$y_{K+1}(x) = y_K(x) - \int_0^1 \left[y'_K(\tau) - y_K(\tau) - \tau + \int_0^1 \tau ty(t)dt \right] d\tau \tag{8}$$

with $y_0(x) = 1$, we obtain:

$$y_1(x) = e^x + \frac{1}{4}x^2$$

$$y_2(x) = e^x - \frac{1}{32}x^2$$

$$y_3(x) = e^x + \frac{1}{256}x^2$$

$$y_4(x) = e^x - \frac{1}{2048}x^2$$

$$y_5(x) = e^x + \frac{1}{16384}x^2$$

$$y_{10}(x) = e^x - \frac{1}{536870912}x^2$$

$$y_{15}(x) = e^x + \frac{1}{17592186044416}x^2$$

$$y_{20}(x) = e^x - \frac{1}{576460752303423488}x^2$$

It is very clear that the iterations converge to the exact result.

Example 3.2: Consider the Volterra Integro-differential equation:

$$y'(x) = 1 - \int_0^x y(t)dt, \; y(0) = 0 \tag{9}$$

with exact solution

$$y(x) = \sin x$$

The iteration formulation is:

$$y_{K+1}(x) = y_K(x) - \int_0^x \left[y'_K(x) - 1 + \int_0^x y(t)dt \right] ds \tag{10}$$

and therefore

$$y_1(x) = x$$

$$y_2(x) = x - \frac{x^3}{3!}$$

$$y_3(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$y_4(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$y_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

This converges to $\sin x$ as $n \to \infty$.

Example 3.3: Consider the problem with $n = 2$ and $m = 1$ as follows:

$$\left.\begin{array}{c} y''(x) = 2(e^x + x) - xe + y(x) + \int_0^1 xty(t)dt \\ y(0) = 0,\ y'(0) = 1 \\ \lim\limits_{n\to\infty}(1 + \frac{1}{n})^n = e \end{array}\right\} \tag{11}$$

and the exact solution $y(x) = xe^x$. Using (6) with $y_0(x) = x$, we have

$$\begin{array}{l} y_{K+1}(x) = y_K(x) + \\ \int_0^x (\tau - x)\left[y''_K(\tau) - 2(e^\tau + \tau) + \tau e - y_K(\tau) - \int_0^1 \tau t y_K(t)dt\right] d\tau \end{array} \tag{12}$$

and the approximations become:

$$y_1(x) = xe^x - \frac{1}{18}(3e - 7)x^3$$

$$y_2(x) = xe^x - \frac{1}{540}(3e - 7)x^3$$

$$y_3(x) = xe^x - \frac{1}{16200}(3e - 7)x^3$$

$$y_4(x) = xe^x - \frac{1}{486000}(3e - 7)x^3$$

$$y_5(x) = xe^x - \frac{1}{14580000}(3e - 7)x^3$$

$$y_{10}(x) = xe^x - \frac{1}{354294000000000}(3e - 7)x^3$$

$$y_{15}(x) = xe^x - \frac{1}{8609344200000000000000}(3e - 7)x^3$$

$$y_{20}(x) = xe^x - \frac{1}{209207064060000000000000000000}(3e - 7)x^3$$

As $n \to \infty,\ y(x) \to xe^x$.

Example 3.4: Consider the following higher order integro-differential equation, when $n = 3$

$$\begin{array}{c} y'''(x) = \sin x + x - \frac{1}{2}\pi x + \int_0^{\frac{\pi}{2}} xty(t)dt \\ y(0) = 1,\ y'(0) = 0,\ y''(0) = -1 \end{array} \tag{13}$$

with the exact solution $y(x) = \cos x$. Using the initial approximation $1 - \frac{x^2}{2}$ which is in agreement with initial conditions, gives:

$$\begin{array}{l} y_{K+1}(x) = y_K(x) - \\ \int_0^x \frac{(\tau - x)^2}{2} \left[y_K'''(\tau) - \sin(\tau) - \tau + \frac{1}{2}\pi\tau - \int_0^{\frac{\pi}{2}} \tau ty(t)dt \right] d\tau \end{array} \tag{14}$$

The following results can be readily obtained:

$$y_1(x) = \cos x - 0.004087679182x^4$$

$$y_2(x) = \cos x - 0.000426416090x^4$$

$$y_3(x) = \cos x - 0.0000444826231x^4$$

$$y_4(x) = \cos x - 0.00000464032289x^4$$

$$y_5(x) = \cos x - 4.84064348 \times 10^{-7}x^4$$

$$y_{15}(x) = \cos x - 2.059803000 \times 10^{-11}x^4$$

$$y_{20}(x) = \cos x - 1.95563633310 \times 10^{-11}x^4$$

The absolute error defined by:

$$E_K = |y_K(x) - y_{K+1}(x)|,\ K = 1, 2, \cdots \text{ is} \leq 10^{-11} \text{ as } n \geq 15 \tag{15}$$

This is in good agreement with the exact solution $y(x) = \cos x$.

Example 3.5: We consider a case of higher order integro-differential equation with $n = 4$ and $m = 1$ as follows:

$$\begin{array}{c} y^{iv}(x) = \frac{2}{3}x^2 - \int_{-1}^1 x^2 ty(t)dt \\ y(0) = 0,\ y'(0) = 1,\ y''(0) = 0,\ y'''(0) = 0 \end{array} \tag{16}$$

with exact solution $y(x) = x$. Using the variational iteration scheme (6) we have:

$$y_{K+1}(x) = y_K(x) + \int_0^x \frac{(\tau - x)^3}{6} \left[y_K^{iv}(\tau) - \frac{2}{3}\tau^2 + \int_{-1}^1 \tau^2 t y_K(t)dt \right] d\tau \tag{17}$$

and then

$$y_1(x) = x$$

$$y_2(x) = x$$

This converges to the exact solution after the first iteration.

4 Conclusion

In this article, we used the variational iteration method to solve integro-differential equations of different order and conditions. We have also successfully compared the obtained results with the exact solution in order to justify the efficiency of the method. The solutions obtained for the five computational problems are in good agreement with their exact solutions. The clear conclusion can be easily confirmed from the numerical results that the variational iteration method provides highly reliable solution with negligible errors.
The method is remarkable, effective, efficient and converges rapidly with minima computational time. It is more accurate without the need for any transformation method.

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