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Global Solvability of Navier-Stokes Equations with Temperature in Infinite Channel

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Abstract

This paper determines the global, in time, solvability of Navier-Stokes equation coupled with temperature in a *Poincaré*-like unbounded domain. Existence, uniqueness and further regularity of solutions were obtained for the full Bousinesq system in uniformly local spaces. We have used the maximum principle for temperature to establish the cardinal properties of this illusive phenomena of turbulence.

1. Introduction

We start with the fundamental model used for problems involving the flow of fluids (Navier-Stokes Equations) given by the following:

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(1)
$$\begin{cases} \partial_t u + (u,)u - \nu\Delta u + p = f \\ divu = 0, u|_{\partial\Omega} = 0, u|_{t=0} = u_0; \end{cases}$$

where u is the fluid velocity, p is the pressure, ν is the kinematic viscosity, f is the external force and the infinite strip $\Omega = \mathbb{R} \times [-1, 1]$, ∂_t is the partial derivative with respect to time, is the space derivative in 2D, represents the laplace operator, $\partial\Omega$ is the boundary of the domain Ω.
The rigorous mathematical analysis of Navier-Stokes equations started at the beginning of the 20th century from works of the famous French mathematician J. Leray; see Leray [9]. His analysis consists of studies concerning the correct formulations of initial boundary value problems for Navier-Stokes equations, proofs of the existence and uniqueness of solutions in different functional spaces, investigation of solution's regularity, construction of asymptotics, etc. Most part of studies on mathematical theory of Navier-Stokes equations consider a bounded underlying domain (e.g with periodic boundary conditions) or the use of finite energy solutions in the whole space $\mathbb{R}^2$; only little is known about infinite energy solutions. The methods employed for the investigation of this highly important equation is the so-called *energy estimates*. In a situation where the domain is unbounded, the problem is essentially less understood. Although there exists a highly developed theory of dissipative PDEs in unbounded domains based on *weighted energy estimates*, it was however difficult to extend this result to the concrete Navier-Stokes equations in unbounded due to several principal obstacles to be explained in the sequel.
Here, we want to consider the problem of thermohydraulics in unbounded domains i.e. the coupled system of fluid (Navier-Stokes equations) and temperature in Boussinesq approximation. We consider thermal conduction between neighbouring fluid elements by including a diffusive term and introducing a material parameter known as the thermal diffusion coefficient. The influence of the temperature field on the incompressible fluid's motion is also considered by introducing a buoyancy force into the velocity evolution equation. The buoyancy force emanates from the observation that temperature variations typically lead to density variations which, in the presence of a gravitational field, lead to pressure gradients. The inclusion of density variations in the buoyancy force, while neglecting them in the continuity equation and the neglect of the local heat source due to viscous dissipation constitute the approximate formulation known as the Boussinesq equations. The equation is modeled by the following:

(2)
$$\begin{cases} \partial_t u + (u,)u - u + p = (0, 1)^t; div\ u = 0, u|_{\partial\Omega} = 0, u|_{t=0} = u_0; \\ \partial_t T + (u,)T = T,\ T|_{x_2=-1} = 1, T|_{x_2=1} = 0 \\ u|_{t=0} = u_0, T|_{t=0} = T_0, \end{cases}$$

where $T = T(t, x)$ is the temperature, $u = (u_1, u_2)$ is the velocity vector: $T_0 = 1$; and $T_1 = 0$ are respectively the temperature at the bottom plate and the top plate. We want to consider (u, T) as sufficiently regular solutions of the system. The goal would be to obtain estimates for T and u in the spaces analogous to those obtained for the solutions of the Navier Stokes system in Zelik [11, 12, 13, 14]. Existence, uniqueness, further regularity of solutions are expected to be obtained for the entire Boussinesq system. We combine the inherent problems associated with (1) and fuse them with their temperature dissipation to give (2) i.e. the problem of thermal convection (heat transfer) by an incompressible Newtonian fluid. In Anthony and Zelik [3], the local temperature of the fluid for the Navier-Stokes problem was, summarily, not taken into consideration.

Most of the complications of this equation arises from the fact that we are doing our investigation in an unbounded domain $\Omega = [-1, 1] \times \mathbb{R}$; here, we do not have finite energy. In this situation, the use of a traditional energy estimates fails because $\|u\|_{L^2} = \infty$ when we multiply (1) by the velocity vector field u.

When confronted with flows in domain like ($\Omega = \mathbb{R} \times [-1, 1]$), the space of square integrable (divergence free) vector field is not a convenient phase space to work with as we are unable to multiply our model by u because doing so the integral will not make sense. In a bounded domain, the assumption that $u \in L^2(\Omega) \Rightarrow u(x) \to 0$ as $|x| \to \infty$ is also too restrictive a decay condition. Under this choice of the phase space many hydrodynamical objects like Poiseuille flows (infinite energy), Kolmogorov flows etc.would not be captured. If the above restrictions hold on our model equation (2) we are unable to consider constant solutions, space periodic solutions etc. which will hinder us from capturing the physically relevant solutions. It was therefore important to choose a special form of exponential weight to solve this problem. Weighted energy theory need be fully developed to overcome this problem but they are also not without essential difficulties related to the emergence of cubic nonlinearity and non divergent free pressure component when multiplications are done with weights i.e. multiplying (1) by ϕu; where $\phi(x) = \phi$ is an appropriate weight function. The appearance of cubic nonlinearity is a problem because the energy equality of our modeled equation is at most quadratic with respect to u, so it was not clear how to control this cubic term in order to produce reasonable a priori estimates. This problem translates directly into (2) but it comes as a relief because equation with temperature has maximum principle which we may explore to get estimates and hence existence, uniqueness, dissipativity, further regularity of solutions etc.

The global in time estimate for the 2D Navier-Stokes equations was first obtained for bounded domains in the works of Ladyzhenskaya [7]. Later on, the unbounded domain case was treated by Abergel [1] and Babin [4], there the forcing term was required to lie in some weighted space. However the dimension estimate of the attractor in this case was independent of the weighted norm of the forcing term and it was natural to expect the existence of the global attractor for more general forces. The above mentioned difficulties stimulated the development of the alternative methods to handle the Navier-Stokes equations in unbounded domains. In particular, rather helpful is the so-called vorticity equation:

$$\partial_t w - \Delta w + (u.\nabla)w = \partial_{x_2} g_1 - \partial_{x_1} g_2; \tag{3}$$

where $w = \partial_2 u_1 - \partial_{x_1} u_2$. If the domain does not contain boundaries e.g. $\Omega = \mathbb{R}^2$ or $\Omega = \mathbb{S}^1 \times \mathbb{R}$, where $\mathbb{S}^1$ is a circle, then the maximum principles applied to (3) allows us to obtain a priori estimates for vorticity w which in turn allows us to prove global solvability for the solution $u(t)$ in uniformly local spaces. The following drawbacks exist for a priori estimates obtained for vorticity equation using maximum principles: the estimates grow linearly in time, so further estimates grow linearly in time as well; to the best of our knowledge, for the case $\Omega = \mathbb{R}^2$ has double exponential ($\sim e^{Ce^{Ct}}$) growth rate. The case of $\Omega = \mathbb{S}^1 \times \mathbb{R}$ has polynomial ($\sim t^3$) growth rate. The essential setback of this method is that it is not applicable on problems with boundaries, for instance, where $\Omega = [-1, 1] \times \mathbb{R}$ in consideration here.

Another method often used to resolve complications arising from Navier-Stokes equations in unbounded domain is the use of bifurcation analysis, see Zelik [6, 8] and reference therein. There, the basic steady state of the Navier-Stokes equations is taken slightly above the instability threshold, then the solutions remaining closed to the steady state can be described in terms of the *modulation* equations which are essentially simpler than the original Navier-Stokes problem; Ginzburg-Landau and Swift-Hohenberg are typical examples of this type of equation. See Mielkel [2] and the reference therein. Since the well-posedness and dissipativity of these modulation equations is well-understood, the standard perturbation methods sometimes allows us to obtain global in time estimates for solutions of the initial Navier-Stokes problem starting from the small neighborhood of the basic steady state. Anthony and Zelik [3] studied Navier-Stokes equations in an infinite channel with solutions obtained in uniformly local spaces. Existence, uniqueness and further regularity results were established. Further to this work, we want to build weighted theory around problems related to Navier-Stokes equation and temperature in infinite strip.

The next section provides the basic function spaces for the treatment of Navier-Stokes Equations, we also introduced the weighted energy spaces and establish their relationship with uniformly local spaces; some related results were also discussed. Section 3 began with a hightlight on the coupled system of Navier Stokes Equations and general explanation of the model equation in consideration here; the maximum principle for temperature was explored using certain properties of the truncation function. In section 4 we concluded the paper by computing estimates the entire Businessq system and obtain bounds for the solution (u, T) in the appropriate space.

2. Preliminaries

2.1. **Function Spaces for Navier-Stokes Equations.** We state here the standard mathematical spaces for Navier-Stokes equations as given by Temam [10]. The basic functional spaces is the Lebesgue space $L^2(\Omega)$ with scalar product $(u,v)=\sum_j\int_\Omega u_j(x)v_j(x)dx$, and norm $|.|=(.,.)^{\frac{1}{2}}$. We will also need the Sobolev space $\mathbb{H}^{k,p}(\Omega)$, where $k\in\mathbb{N}$ and $p\in[0,\infty)$ consisting of all $u\in L^p(\Omega)$ whose weak derivatives up to order k is in $L^p(\Omega)$. Moreover, $\mathbb{H}^{k,p}(\Omega)$ is a separable Hilbert space with norm:

$$\|u\|_{\mathbb{H}^{k,p}}\leq(\sum_{|\alpha|\leq k}\int_\Omega|D^\alpha u(x)|^p dx)^{\frac{1}{p}}.$$

Obviously, $\mathbb{H}^{k,2}(\Omega)$ is a Hilbert space where $k\in\mathbb{N}$. We will need the weak solution to problem (2); for this, we need a proper space of test functions. We take $V{=}V(\Omega)=\{\phi\in C_0^\infty(\Omega):\mathrm{div}\phi=0\ in\ \Omega\}$. The closure in $L^2(\Omega)$ and in $\mathbb{H}^{1,2}(\Omega)$ is denoted by H respectively V. The scalar product norms in those two spaces are those inherited from $L^2(\Omega)$ respectively $\mathbb{H}^{1,2}(\Omega)$.
We call Ω a *Poincaré* domain if there exist $\lambda_1>0$ such that

$$\int_\Omega\phi^2dx\leq\frac{1}{\lambda_1}\int_\Omega|\nabla\phi|^2dx,\ \phi\in C_0^\infty(\Omega). \tag{4}$$

The inequality (4) is called the *Poincaré* inequality, it can be shown that if Ω is bounded in some direction, i.e. there is a vector $b\in\mathbb{R}^2$ such that $\sup_{x\in\Omega}|(x,b)|<\infty$, then Ω is a *Poincaré* domain.
If Ω is a *Poincaré* domain, then the original norm on V is equivalent to the norm $\|.\|$ induced by the scalar product

$$((u,v))=\int_\Omega\sum_{j=1}^2\nabla u_j.\nabla v_j dx=(\nabla u,\nabla v),\quad u,v\in V. \tag{5}$$

We define next a bilinear form $a:V\times V\to\mathbb{R}$ by

$$a(u,v)=(\nabla u,\nabla v),\quad v,u\in V. \tag{6}$$

Since obviously the form a coincides with $((.,.))$ scalar product in V, it is $V-$continuous i.e. $|a(u,u)|\leq C\|u\|^2$ for some $C>0$ and $u\in V$. Hence, by the Riesz lemma, there exists a unique linear operator $A{:}V\to V'$ where V' is the dual of V such that $a(u,v)=(Au,v),\quad u,v\in V$. Moreover, the form a is obviously V-coercive i.e. it satisfies $a(u,u)\geq\alpha\|u\|^2$ for some $\alpha>0$ and $u\in V$. Therefore, by means of Lax-Milgram theorem (see Brzezniak [5], Theorem II 2.1) the operator $A{:}V\to V'$ is an isomorphism. Since V is densely and continuously embedded into H and H can be identified with its dual H', we have the following embeddings:

$$V\subset H\cong V'\subset H'. \tag{7}$$

Let us then recall that we say that the spaces V, H and V' form a Gelfand triple. Next we define an unbounded linear operator A in H as follows:

$$\begin{cases} D(A) = \{u \in V, \text{A} \in H\} \\ Au = \text{A}u, u \in D(A). \end{cases} \tag{8}$$

It is now well established that under some additional assumptions related to the regularity of the domain D, the space $D(A)$ can be characterized in terms of Sobolev spaces.

2.2. **Weighted Energy and Uniformly Local Spaces With Some Related Results.** We introduce and briefly discuss the weighted and uniformly local spaces which are the main technical tools to deal with infinite-energy solutions, see Zelik [11] for more detailed exposition. These tools will help us to obtain estimates for our equations (2) in unbounded domain $\Omega = \mathbb{R} \times (-1, 1)$. We explain the space as follows: Let us define $B^1_{x_0}$-a unit rectangle centred at $(x_0, 0)$ represented as:

$$B^1_{x_0} = (x_0 - \frac{1}{2}, x_0 + \frac{1}{2}) \times (-1, 1), x_0 \in \mathbb{R} \tag{9}$$

Let us briefly state the definition and basic properties of weight functions and weighted functional spaces as presented by Zelik [13] which will be systematically used throughout this paper (see also Zelik [6] for more details).
We start with the class of admissible weight functions.

Definition 2.1. A function $\phi \in C_{loc}(\mathbb{R})$ is a weight function of exponential growth rate $\mu > 0$ if the following inequalities hold:

$$\phi(x + y) \leq C_\phi \phi(x) e^{\mu|y|}, \quad \phi(x) > 0, \tag{10}$$

for all $x, y \in \mathbb{R}$.

The following proposition collects the following properties of those weights:

Proposition 2.2. *1. Let ϕ be a weight function of exponential growth rate μ. Then for every $\epsilon > \mu, \phi$ is a weight function of exponential growth rate ϵ (with the same constant C_ϕ).*
2. Let ϕ and ψ be weight functions of exponential growth rate μ. Then the function $\Psi_1 = \phi(x)\psi(x)$ and $\Psi_2 = \frac{\phi(x)}{\psi(x)}$ are weight functions of exponential growth rate 2μ with the constant $C_{\Psi_i} \leq C_\phi C_\psi$
3. Let ϕ be a weight function of exponential growth rate μ and let $\psi \in C_{loc}(\mathbb{R})$ satisfy

$$C_1\phi(x) \leq \psi(x) \leq C_2\phi(x), x \in \mathbb{R} \tag{11}$$

Then ψ is also a weight function of exponential growth rate μ and $C_\psi \leq C_1^{-1} C_2 C_\phi$.

4. Let $\epsilon > 0$ and $\phi(x)$ be a weight function of exponential growth rate μ. Then, the function $\phi_\epsilon(x) = \phi(\epsilon x)$ is of exponential growth rate $\epsilon\mu$ and with $C_{\phi\epsilon} = C_\phi$.

All of the assertions of the proposition are simple corollaries of estimate (1.2). The natural example of such weights is the following one:

$$\phi_{\mu,x_0} = e^{-\mu|x-x_0|}, \quad x_0 \in \mathbb{R}, \quad \mu \in \mathbb{R} \tag{12}$$

Obviously, they are of exponential growth rate $|\mu|$ and the constant $C_{\phi_{\mu,x_0}} = 1$ (independent of $x_0 \in \mathbb{R}$). However, these weights are not smooth at $x = x_0$. In order to overcome this natural drawback it is important to use the following weights:

$$\varphi_{\mu,x_0} = e^{-\mu\sqrt{1+|x-x_0|^2}}, \quad x_0 \in \mathbb{R} \tag{13}$$

Indeed, since $|x| \leq \sqrt{x^2+1} \leq |x| + 1$ then these weights satisfy:

$$e^{-|\mu|}\phi_{\mu,x_0} \leq \varphi_{\mu,x_0} \leq e^{|\mu|}\phi_{\mu,x_0}, x \in \mathbb{R} \tag{14}$$

and consequently, φ_{μ,x_0} are also weight functions of exponential growth rate μ (with $C_{\phi_{\mu,x_0}} = e^{2|\mu|}$). Moreover, in contrast to (1.4) these weights are smooth and satisfy for $\mu \leq 1$ the additional obvious inequality

$$|D_x^k \varphi_{\mu,x_0}| \leq C_k|\mu|\varphi_{\mu,x_0}, x \in \mathbb{R} \tag{15}$$

where $k \in \mathbb{N}$, D_x^k denotes a collection of all x-derivatives of order k and the constant C_k is independent of x and μ this inequality is crucial in obtaining the regularity estimates in weighted spaces (see Zelik [11]).
Another important class of weight functions is the polynomial ones:

$$\theta_{x_0}^m(x) = (1 + |x - x_0|^2)^{-\frac{m}{2}}, \quad m \in \mathbb{R} \tag{16}$$

Again these weights are of exponential growth rate μ for every $\mu > 0$ with the constant C_{θ_m,x_0} dependent on μ and m but independent of $x_0 \in \Omega$
We now introduce a class of weighted Sobolev spaces in a regular unbounded domains Ω associated with weights introduced above. We need only the case where $\Omega = \mathbb{R} \times [-1, 1]$ is a strip which obviously have regular boundary. One would like to ask why we need weighted Sobolev Spaces; recall that the uniformly local spaces encountered some deficiencies in that they are not differentiable when the supremum is involved but the weighted energy spaces resolve this problem.

Definition 2.3.

$$L^p_\phi(\Omega) = \{u \in L^p_{loc}(\Omega), \|u\|^p_{L^p_\phi} = \int \phi^p(x)|u(x)|^p dx < \infty\}$$

and

$$L^p_{b,\phi}(\Omega) = \{u \in L^p_{loc}(\Omega), \|u\|^p_{L^p_{b,\phi}} = \sup_{x_0\in\mathbb{R}} (\phi(x_0)\|u\|_{L^p(B^1_{x_0})})\}$$

The uniformly local space $L^2_b(\Omega)$ consists of all functions $u \in L^2_{loc}(\Omega)$ for which the following norm is finite

$$\|u\|_{L^2_b(\Omega)} = \sup_{x_0 \in \mathbb{R}} \|u\|_{L^2(B^1_{x_0})} < \infty$$

If $u \in L^\infty \Rightarrow u \in L^2_b$, and $\|u\|_{L^2_b} \leq C\|u\|_{L^\infty}$. This is because all functions that are bounded in L^∞ are also bounded in L^2_b but the reverse is not true.

$$\|u\|_{L^2(B^1_{x_0})} \leq |B^1_{x_0}|\|u\|_{L^\infty(B^1_{x_0})} \leq C\|u\|_{L^\infty(\Omega)}$$

Similarly, the uniformly local Sobolev spaces $H^s_b(\Omega)$ consist of all functions $u \in H^s_{loc}(\Omega)$ for which the following norm is finite:

$$\|u\|_{H^s_b(\Omega)} = \sup_{x_0 \in \mathbb{R}} \|u\|_{H^s(B^1_{x_0})} < \infty$$

where H^s is the space of all distributions whose derivative up to order s is in L^2. The following lemma establishes the relationship between the spaces L^2_ϕ and L^2_b

Lemma 2.4. *Let ϕ be a weight function of exponential growth rate, where $\phi_{x_0}(x) = \phi(x - x_0)$, satisfying $\int \phi^2 dx < \infty$ then the following inequalities hold*

$$\|u\|_{L^2_\phi} \leq C_1 \|u\|^2_{L^2_b} \cdot \int \phi^2 dx \tag{17}$$

$$\|u\|^2_{L^2_b} \leq C_2 \sup_{x_0 \in \mathbb{R}} \|u\|^2_{L^2_{\phi_{x_0}}} \tag{18}$$

where C_1 and C_2 depend only on C_ϕ and μ from definition (1.2)

Proof. If $u \in L^2_b$ then $\|u\|_{L^2(B^1_{x_0})} < C, x_0 \in \mathbb{R}$ is bounded uniformly and

$$\int_\Omega \phi^2(x)|u(x)|^2 dx = \sum_{N=-\infty}^{\infty} \int_N^{N+1} \phi^2(x)|u(x)|^2 dx.$$

$$\leq \sum \|\phi\|^2_{L^\infty[N,N+1]} \|u\|^2_{L^2[N,N+1]}$$

$$\leq \sum_N \sup_N \|u\|^2_{L^2[N,N+1]} \|\phi\|^2_{L^\infty[N,N+1]}$$

$$\leq \|u\|^2_{L^2_b} \sum_{N=-\infty}^{\infty} \|\phi\|^2_{L^\infty[N,N+1]}$$

$$\leq \|u\|^2_{L^2_b} \sum \sup_{x \in [N,N+1]} |\phi^2(x)|$$

$$= C\|u\|^2_{L^2_b} \sum \phi^2(N)$$

This follows from the integral criterion for convergence i.e $\sum_{N=-\infty}^{\infty} \phi^2(N)$ converges if and only if $\int_{-\infty}^{\infty} \phi^2(x)dx < \infty$. The opposite side of the proof takes $u \in$

$\sup_{x_0\in\mathbb{R}} \|u\|_{L^2_{\phi_{x_0}}}$ so we need to estimate the norm $\|u\|_{L^2_{B^1_{x_0}}}$ hence $\min_{x\in B^1_{x_0}} \phi_{x_0}(x) \geq c_0$ and using a specific weight function of the form

$$\phi(x+y) \leq Ce^{\alpha|x|}\phi(y) \tag{19}$$

so that

$$\phi(y) \geq C^{-1}e^{-\alpha|x|}\phi(x+y)$$

. Take $x=-y$ to get

$$\phi(y) \geq C^{-1}e^{-\alpha|x|}\phi(0) \geq C^{-1}\phi(0)$$

And hence,

$$\int_{\mathbb{R}} \phi^2_{x_0}(x)|u(x)|^2dx \geq c_0^2 \int_{B^1_{x_0}} |u(x)|^2dx = c_0^2\|u\|^2_{L^2_{B^1_{x_0}}}$$

which gives:

$$C_2 \sup_{x_0\in R} \|u\|^2_{L^2_{\phi_{x_0}}} \leq \|u\|^2_{L^2_b} \leq C_1 \sup_{c_0\in R} \|u\|^2_{L^2_{\phi_{x_0}}} \tag{20}$$

□

Before we conclude this section, we introduce a special weight function which will be essentially used in the sequel. Let

$$\phi_\epsilon(x) = (1+\epsilon^2|x|^2)^{-\frac{1}{2}} \tag{21}$$

we see that

$$\phi'_\epsilon(x) = -\frac{\epsilon^2|x|}{(1+\epsilon^2|x|^2)^{\frac{3}{2}}}$$

then we have

$$\phi'_\epsilon(x) \leq C\frac{\epsilon^2|x|}{(1+\epsilon^2|x|^2)^2}$$
$$\leq C\epsilon\frac{\epsilon|x|}{\sqrt{1+\epsilon^2x^2}}.\frac{1}{(1+\epsilon^2x^2)}$$
$$\leq C\epsilon\phi^2_\epsilon(x)$$

we have the following:

$$\phi'_\epsilon(x) \leq C\epsilon\phi^2_\epsilon(x) \tag{22}$$

This weight, in addition, to (19) (which holds for every positive μ), satisfies the following property:

$$|\phi'(x)| \leq C\phi(x)^2 \leq C\phi(x) \tag{23}$$

A bit more general are the weights $\phi(x)^N$, $N \in$, $N \neq 0$, which are also the weights of exponential growth rate μ for any $\mu > 0$ and satisfy the analog of (23) where the exponent 2 is replaced by $\frac{N+1}{N}$.

In the sequel, we will need the Sobolev embedding and interpolation inequalities for the case of weighted spaces with the embedding constants independent of $\epsilon \to 0$. Following Temam [10], such inequalities can be derived from the analogous non-weighted inequalities utilizing the isomorphism $T_{\phi_\epsilon} u = \phi_\epsilon u$ between weighted and non-weighted spaces.

Lemma 2.5. *Let ϕ_ϵ be a weight function defined by (21). Then, for all l and $1 \le p \le \infty$, the map T_{ϕ_ϵ} is an isomorphism between spaces $W^{l,p}(\Omega)$ and $W^{l,p}_{\phi}(\Omega)$ and the following inequalities hold:*

$$C_1 \|\phi u\|^2_{W^{l,p}} \le \|u\|^2_{W^{l,p}_{\phi}} \le C_2 \|\phi u\|^2_{W^{l,p}}, \tag{24}$$

where C_1 and C_2 are independent of ϵ but may depend on l and p.

3. Estimates for Temperature Coupled with Navier-Stokes

In this section, we consider the equation of coupled system of fluid and temperature in Boussinesq approximation. Using Boussinesq equation (2) in nondimensional form in the strip $\Omega = \mathbb{R} \times [0, 1]$, we have

$$\partial_t u + (u.\nabla)u + \nabla p = \Delta u + e_n T \tag{25}$$

$$u|_{x_2=1} = 0, u|_{x_2=0} = 0 \tag{26}$$

$$\nabla.u = 0 \tag{27}$$

$$\partial_t T + (u.\nabla)T - \mu \Delta T = 0 \tag{28}$$

$$T|_{x_2=1} = 0, T|_{x_2=0} = 1 \tag{29}$$

$$T|_{t=0} = T_0 \tag{30}$$

which satisfies the zero flux condition, where $u = (u_1, u_2)$ is the velocity vector, $T_0 = 1$ is the temperature at the bottom, $T_1 = 0$ is the temperature at the top, e_n is the standard coordinate basis in $\mathbb{R}^2$ and the kinematic viscosity $\mu > 0$.
We assume that (u, T) is a sufficiently regular solution of the system (25)-(30) satisfying the following properties:

$$u \in L^\infty(\mathbb{R}_+, L^2_b(\Omega)), \nabla u \in L^2_b(R_+ \times \Omega), \mathrm{div} u = 0 \tag{31}$$

and

$$T \in L^\infty(\mathbb{R}_+, L^2_b(\Omega)), \nabla T \in L^2_b(R_+ \times \Omega). \tag{32}$$

where

$$0 \le T_0 \le 1 \tag{33}$$

We want to obtain estimates for T and u in spaces (31)-(32) analogous to the ones obtained earlier for the solutions of the Navier Stokes system. The key technical tool for that is the maximum/comparison principle for temperature which we will consider in the next subsection.

3.1. **Maximum Principle for Temperature.** In this subsection, we consider equations (28)-(30) for temperature assuming that u is a given vector field satisfying (25) (u is not necesarily a solution of the Navier Stokes equation (1)). Our aim is to show that the inequality (33) at time moment $t = 0$ implies analogous inequality fore all $t \geq 0$. To justify the maximum/comparison principle, we need the following properties of the trucation functions due to Temam [10].

Lemma 3.1. *let $V \in H^1_b(\Omega)$. Then, the truncation functions $V_+(x) = \max\{V(x), 0\}$ and $V_-(x) = \max\{-V(x), 0\}$ belong to $H^1_b(\Omega)$ as well, and their distributional derivatives satisfy*

$$\nabla V_+ = (\nabla V)_+ = \begin{cases} \nabla V & \textit{if } V > 0 \\ 0 & \textit{if } V < 0 \end{cases}$$

similarly,

$$\nabla V_- = (\nabla V)_- = \begin{cases} \nabla V & \textit{if } T < 0 \\ 0 & \textit{if } T > 0 \end{cases}$$

In particular,

$$|T_-(x,t)|.|T_+(x,t)| = 0 \tag{34}$$

$$\nabla T_-(x,t).\nabla T_+(x,t) = 0 \tag{35}$$

almost everywhere.

Now, we will state the prove of the main result of this subsection.

Theorem 3.2. *Let u and T satisfy equation (25)-(30), and $0 \leq T(x,0) \leq 1$ for almost all $x \in \mathbb{R}$ then,*

$$0 \leq T(x,t) \leq 1, \ \forall (x,t) \in \mathbb{R}_+ \times \Omega \tag{36}$$

Proof. Let us first prove that $T \geq 0$ almost everywhere. To this end, we multiply (28) by $-T_-\phi^2$, where T_- is the truncation of T and $\phi = \phi(x)$ is the weight function $\phi_\epsilon(x_1) = e^{-\epsilon|x_1|}$. We integrate by parts and use $T = T_+ - T_-$ and $T_-|_{\partial\Omega}$ to obtain T_- estimate as follows:

$$\frac{1}{2}\frac{d}{dt}\|T_-\|^2_{L^2_\phi} + \mu(\nabla T_+ - \nabla T_-, \nabla T_-) + (u(\nabla T_+ - \nabla T_-), T_-) = 0 \tag{37}$$

then using (34) and (35) to obtain

$$\frac{1}{2}\frac{d}{dt}\|T_-\|^2_{L^2_\phi} + \mu\|\nabla T_-\|^2_{L^2_\phi} + (\nabla T_-, 2T_-\epsilon\phi^2) + ((u\nabla)T_-, T_-\phi^2) = 0. \tag{38}$$

By using that $\text{div}u = 0$, we simplify the nonlinearity to get the following:

$$((u\nabla)T_-, T_-\phi^2) = (u_1\partial_{x_1}T_-, T_-\phi^2) + (u_2\partial_{x_2}T_-, T_-\phi^2)$$
$$= \frac{1}{2}((u_1\partial_{x_1}T_-^2, \phi^2) + \frac{1}{2}(u_2\partial_{x_2}(T_-^2), \phi^2)$$
$$= \frac{1}{2}\{(\partial_{x_1}u_1 + \partial_{x_2}u_2), T_-^2) - (u_1T_-^2, \phi\phi')\} = -(u_1T_-^2, \phi\phi')$$

Using that ϕ satisfies (23), we see that

$$|((u, \nabla)T_-, T_-\phi^2)| \leq C\epsilon(u_1T^2, \phi^2) \tag{39}$$

We estimate the RHS of (39) as follows:

$$C\epsilon(u_1T^2, \phi^2) = C\epsilon\sum_{N=-\infty}^{\infty}\int_{\Omega_{N,N+1}}(u_1T^2, \phi^2)dx$$
$$\leq C\epsilon\sum_{N=-\infty}^{\infty}\int_{\Omega_{N,N+1}}|T_-^2u_1||\phi|^2dx$$
$$\leq C\epsilon\sum_{N=-\infty}^{\infty}\|T_-^2u_1\|_{L^1[\Omega_{N,N+1}]}\|\phi\|^2_{L^\infty[N,N+1]}$$
$$\leq C\epsilon\sum_N\sup_N\|u_1\|_{L^2[\Omega_{N,N+1}]}\|T_-^2\|_{L^2[\Omega_{N,N+1}]}\phi^2(N)$$
$$\leq C\epsilon\|u_1\|_{L_b^2}\sum_{N=-\infty}^{\infty}\|T_-\|^2_{L^4[\Omega_{N,N+1}]}\phi^2(N)$$

Using Ladyzhenskaya inequality,

$$\|T_-\|^2_{L^4(\Omega_{N,N+1})} \leq \|T_-\|_{L^2(\Omega_{N,N+1})}.\|\nabla T_-\|_{L^2(\Omega_{N,N+1})} \tag{40}$$

and

$$\|T_-\|^2_{L^2_\phi} \sim \sum_{N=-\infty}^{\infty}\phi^2(N)\|T_-\|^2_{L^2(\Omega_{N,N+1})}. \tag{41}$$

We finally obtain:

$$\sum_{N=-\infty}^{\infty}\phi^2(N)\|T_-\|^2_{L^4(\Omega_{N,N+1})}$$
$$\leq C\sum_{N=-\infty}^{\infty}\phi^2(N)\|T_-\|_{L^2(\Omega_{N,N+1})}.\|\nabla T_-\|_{L^2(\Omega_{N,N+1})} \tag{42}$$
$$\leq \delta\sum_{N=-\infty}^{\infty}\phi^2(N)\|\nabla T_-\|^2_{L^2(\Omega_{N,N+1})} + C_\delta\sum_{N=-\infty}^{\infty}\phi^2(N)\|T_-\|^2_{L^2(\Omega_{N,N+1})}$$

$$\leq \delta \|\nabla T_-\|^2_{L^2_\phi} + C_\delta \|T_-\|^2_{L^2_\phi}.$$

Thus we have the estimate for the nonlinearity:

$$|((u,\nabla)T_-, T_-\phi^2)| \leq C\|u\|^2_{L^2_b}(\delta\|\nabla T_-\|^2_{L^2_\phi} + C_\delta\|T_-\|^2_{L^2_\phi}). \tag{43}$$

Substituting this estimate into (38) we obtain:

$$\frac{d}{dt}\|T_-\|^2_{L^2_\phi} + \frac{\nu}{2}\|\nabla T_-\|^2_{L^2_\phi} \leq \delta\|u\|_{L^2_b}\|\nabla T_-\|^2_{L^2_\phi}$$

$$+C_\delta\|u\|_{L^2_b}\|T_-\|^2_{L^2_\phi} + 2\epsilon^2\nu\|T_-\|^2_{L^2_\phi} \tag{44}$$

Taking $\delta > 0$ small enough reduce to:

$$\frac{d}{dt}\|T_-\|^2_{L^2_\phi} + c_1\|\nabla T_-\|^2_{L^2_\phi} \leq c_2\|T_-\|^2_{L^2_\phi}. \tag{45}$$

Dropping the second term on the LHS of equation (45) we obtain:

$$\frac{d}{dt}\|T_-\|^2_{L^2_\phi} - c_2\|T_-\|^2_{L^2_\phi} \leq 0.$$

Applying Gronwall's inequality and taking supremum with respect to all shifts , we obtain the following estimate:

$$\|T_-\|^2_{L^2_\phi} \leq \|T_-(0)\|^2_{L^2_\phi} e^{c_2 t}. \tag{46}$$

Since we know $T_-(x,0) = 0$ we may conclude from the above that:

$$\|T_-(x,t)\|^2_{L^2_\phi} = 0 \tag{47}$$

We proceed similarly bearing in mind that $(T(x,t)-1)_+ \leq 0$ to prove that:

$$\|(T-1)_+(t)\|^2_{L^2_\phi} \leq \|(T-1)_+(0)\|^2_{L^2_\phi} e^{c_2 t}. \tag{48}$$

where $\|(T-1)_+(0)\|^2_{L^2_\phi} = 0$. By similar argument we conclude that:

$$\|(T-1)_+(t)\|^2_{L^2_\phi} = 0, \tag{49}$$

thus, (36) follows immediately. This ends the proof. □

We state a corollary that will give us a bound in time for ∇T.

Corollary 3.3. *Let T be the solution of (28) then,*

$$\|\nabla T\|_{L^2_b[0,T;\Omega]} \leq C' + C\|u\|_{L^2_b[0,T;\Omega]} \tag{50}$$

Proof. We first construct an auxiliary function $\bar{T}(x_2) = \frac{1-x_2}{2}$ which satisfies (29) so that

$$T(t, x_2) = \bar{T}(x_2) + \tilde{T}(x_2, t) \tag{51}$$

We recall (28) as repeated here:

$$\partial_t T + (u\nabla)T - \Delta T = 0.$$

Then, we substitute (51) into the equation to obtain the following:

$$\partial_t \tilde{T} + (u, \nabla)\bar{T} + (u, \nabla)\tilde{T} - \Delta\tilde{T} = 0 \tag{52}$$

which simplifies to:

$$\partial_t \tilde{T} + (u, \nabla)\tilde{T} - \Delta\tilde{T} = \frac{u_1}{2}. \tag{53}$$

We perform our usual multiplication by $\tilde{T}\phi^2$ and integrate over the domain to obtain:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\tilde{T}\|^2_{L^2_\phi} + \|\nabla\tilde{T}\|^2_{L^2_\phi} + ((u, \nabla)\tilde{T}, \tilde{T}\phi^2) \\ &\le \frac{\|u_1\|^2_{L^2_\phi}}{8} + \frac{\|\tilde{T}\|^2_{L^2_\phi}}{2} + C\epsilon^2\frac{\|\tilde{T}\|^2_{L^2_\phi}}{2} + \frac{\|\nabla\tilde{T}\|^2_{L^2_\phi}}{2} \end{aligned} \tag{54}$$

Similarly,we find the estimate for the nonlinearity in $\tilde{T}$ and u analogous to the one derived earlier in (43). Thus,

$$|((u, \nabla)\tilde{T}, \tilde{T}\phi^2)| \le C\|u\|^2_{L^2_b}\left(\delta\|\nabla\tilde{T}\|^2_{L^2_\phi} + C_\delta\|\tilde{T}\|^2_{L^2_\phi}\right). \tag{55}$$

Substituting (55) into (54) and absorbing similar terms we obtain:

$$\frac{d}{dt}\|\tilde{T}\|^2_{L^2_\phi} + \left(1 - C\|u\|^2_{L^2_b}\delta\right)\|\nabla\tilde{T}\|^2_{L^2_\phi} - \left(1 + C_\delta\|u\|^2_{L^2_b}\right)\|\tilde{T}\|^2_{L^2_\phi} \le \frac{\|u\|^2_{L^2_\phi}}{4}. \tag{56}$$

By simplifying the above we have:

$$\frac{d}{dt}\|\tilde{T}\|^2_{L^2_\phi} + C\|\nabla\tilde{T}\|^2_{L^2_\phi} \le \frac{\|u\|^2_{L^2_\phi}}{4} + C'\|\tilde{T}\|^2_{L^2_\phi}. \tag{57}$$

Integrating (57) between t and $t+1$ we obtain:

$$\begin{aligned} \|\tilde{T}(t+1)\|^2_{L^2_\phi} + \int_t^{t+1}\|\nabla\tilde{T}\|^2_{L^2_\phi}dt \le \|\tilde{T}(t)\|^2_{L^2_\phi} + \\ + \frac{1}{4}\int_t^{t+1}\|u\|^2_{L^2_\phi}dt + C'\int_t^{t+1}\|\tilde{T}\|^2_{L^2_\phi}dt. \end{aligned} \tag{58}$$

We drop the first term on the LHS of (58) and use our already obtained bounds for $T(x,t)$ as given in (36):

$$\sup_{x_0}\sup_{t}\int_t^{t+1}\|\nabla\tilde{T}\|^2_{L^2_\phi}dt \le C' + C\|u\|^2_{L^2_b[0,T;\Omega]}. \tag{59}$$

This ends the proof. □

4. Apriori Estimate for The Full Boussinesq System

In what follows, the following Lemma from Anthony and Zelik [3] is crucial:

Lemma 4.1. *Let*

$$\frac{d}{dt}(\frac{1}{2}\|u\|^2_{L^2_\phi} - (u,v_\phi)) + C\|\nabla u\|^2_{L^2_\phi}(1-\epsilon\sup_{t\in[0,T]}\|u(t)\|_{L^2_\phi})+ \\ + \gamma\|u\|^2_{L^2_\phi} \le C\|f\|^2_{L^2_\phi} + \epsilon^2\sup_{t\in[0,T]}\|u(t)\|^2_\phi \tag{60}$$

hold true for $t\in[0,T]$ *and let* ϵ *be such that*

$$1-\epsilon\sup_{t\in[0,T]}\|u(t)\|^2_{L^2_\phi} \ge 0. \tag{61}$$

Then,

$$\|u(t)\|^2_{L^2_\phi} \le C\epsilon^{-1}(\|u(0)\|_{L^2_b} + \|f\|_{L^2_b} + 1)^2 \tag{62}$$

where $t\in[0,T]$ *and* C *is independent of* $\epsilon\to 0$.

We obtain estimate for the full Boussinesq system by using the results of previous Lemma 4.1 and Corollary 3.3.

Theorem 4.2. *If velocity* u *and temperature* T *satisfy equations (25)-(30); and for*

$$0 \le T(x,0) \le 1 \tag{63}$$

the following hold true:

$$0 \le T(x,t) \le 1 \tag{64}$$

then, we can estimate (25)-(30) as follows:

$$\|\nabla T\|_{L^2_b} + \|u\|_{L^2_b} \le C(\|u_0\|_{L^2_b} + C)^2 + C \tag{65}$$

Next, we apply the usual principle of treating (25) as we treated the nonlinear Navier-Stokes equation in the previous section for the case of zero flux condition with e_nT serving as the forcing term. We sketch the steps of the proof without too many details. First, we take the scalar product of (25) with $\phi^2u - v_\phi$:

$$(\partial_t u, \phi^2u - v_\phi) + ((u.\nabla u)u, \phi^2u - v_\phi) - \nu(\Delta u, \phi^2u - v_\phi)$$

$$+(\nabla p, \phi^2 u - v_\phi) = (T, \phi^2 u - v_\phi). \tag{66}$$

And like before, we obtain:

$$\frac{d}{dt}(\frac{1}{2}\|u\|^2_{L^2_\phi} - (u, v_\phi)) + C\|\nabla u\|^2_{L^2_\phi}(1 - \epsilon \sup_{t\in[0,T]} \|u(t)\|_{L^2_\phi})$$

$$+\gamma\|u\|^2_{L^2_\phi} \leq C\|T\|^2_{L^2_\phi} + \epsilon^2 \sup_{t\in[0,T]} \|u(t)\|^2_\phi. \tag{67}$$

We argue by Lemma 4.1 that $\epsilon \sup_{t\in[0,T]} \|u(t)\|_{L^2_\phi} \leq 1$ to obtain:

$$\|u(t)\|^2_{L^2_\phi} \leq C\epsilon^{-1}(\|u(0)\|_{L^2_b} + C)^2$$

By a particular choice of $\epsilon = \frac{1}{4}\left(\|u(0)\|_{L^2_b} + C)^2\right)^{-1}$ we obtain the following estimate:

$$\|u\|_{L^2_b} \leq C(\|u(0)\|_{L^2_b} + C)^2 \tag{68}$$

Coupling estimates (50) and (68) we obtain full estimate for the Boussinesq system as in (65) repeated here:

$$\|\nabla T\|_{L^2_b} + \|u(t)\|_{L^2_b} \leq C(\|u_0\|_{L^2_b} + C)^2 + C, \ t \geq 0. \tag{69}$$

This ends the argument.

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