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Fixed Point with Weakly Compatible Function on Semigroup of Linear Operator

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Abstract

This paper consists of fixed point results on ω-order preserving partial contraction mapping as a weakly compatible function. Moreover, the results are established as semigroup of linear operator.

1. Introduction

Let X be a Banach space, $X_n \subseteq X$ be a finite set and $S \subseteq X_n$ be subspace of X, $L(x)$ be a bounded linear operator in X. Suppose $(T(t))_{t\geq 0}$ is C_0-semigroup, $\omega - OCP_n$ is ω-order preserving partial contraction mapping and $M_m(N)$ is a matrix.
Fixed point theory is concern with finding the condition to the structure that a set X must be endowed and as well as the properties of operator $T : X \to X$ in order to obtain results on existence, uniqueness and construction of a fixed point [2]. Riesz space value is introduced as our motivation for this paper where Riesz space is the norm.
A Riesz space is an ordered vector space and a lattice. Let $\mathcal{J}$ be a Riesz space with positive cone $\mathcal{J}_+ = \{x \in X : x \geq 0\}$. If (α_n) is a decreasing sequence in $\mathcal{J}$ such that $\inf \alpha_n = \alpha$, we write $\alpha_n \to \alpha$. This paper will focus on fixed point result

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on $\omega - OCP_n$ as a weakly compatible function of semigroup of linear operator. Altum *et al.* [1], obtained fixed points of weakly compatible maps satisfying a general contractive condition of integral type. Abbas and Jumyek [2], established common fixed point results for non-commuting mappings without continuity in cone metric spaces. Alata *et al.* [3], proved some results on common fixed point for generalized f-contraction Mapping. Vrabie [5] deduced C_0-Semigroup and Application, & Rauf and Akinyele [6], generated properties of ω-order preserving partial contraction mapping and its relation to C_0-semigroup. Many authors including Rauf and Yusuf [7] worked on common fixed point theorems in vector metric spaces.

2. Preliminaries

Definition 2.1 (Banach space)
A Banach space is a norm space which is complete in the sense that every Cauchy sequence converges to a particular point on the space.
Definition 2.2 (Archimedean)
The Riesz space $\mathcal{J}$ is said to be Archimedean if $\frac{1}{n}\alpha \to 0$ holds for every $\alpha \epsilon \mathcal{J}_+$
Definition 2.3 (Semigroup)
A semigroup is a pair $(S, *)$ in which S is a non empty set and * is a binary associative operation on S, that is, the equation $(x * y) * z = x * (y * z)$ holds for all $x, y, z \in S$.
Definition 2.4 ($\mathbf{C_0}$-semigroup)
A C_0-semigroup is a strongly continuous one parameter semigroup of bounded linear operator on a Banach space.
Definition 2.5 $(\omega - OCP_n)$
A transformation $\alpha \in S$ is called ω-order preserving partial contraction mapping if $\forall x, y \in Dom\ \alpha : x \leq y \Rightarrow \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise $T(0) = I$.
Definition 2.6 (Infinitesimal generator)
Let $T(t)$ be a C_0-semigroup, the infinitesimal generator of $T(t)$ (denoted by A) is given by

$$\lim_{t\to 0} A_t x = \lim_{t\to 0} \frac{T(t)x - x}{t} \forall x \in X$$

Example 1
Example of semigroup of linear operator is $\omega - OCP_n$ which is a subset of C_0-semigroup and the simplest one among them is 2×2 matrix.
Suppose A= $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and $T\,(t) = e^{\,tA}$, then $e^{\,tA} = \begin{pmatrix} e^{\,t} & e^{\,2t} \\ e^{\,t} & e^{\,2t} \end{pmatrix}$

Example 2

Example of 3×3 matrix in the same operator is A= $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ - & 2 & 3 \end{pmatrix}$ and $T(t) = e^{\,tA}$, then $e^{\,tA} = \begin{pmatrix} e^{\,t} & e^{\,2t} & e^{\,3t} \\ e^{\,t} & e^{\,2t} & e^{\,2t} \\ I & e^{\,2t} & e^{\,3t} \end{pmatrix}$

Example 3

It is well known that $\mathbb{R}^2$ is a Riesz space with coordinate ordering defined by:

$$(x_1,\ y_1) = (x_2, y_2) \Leftrightarrow x_1 = x_2 \ and \ y_1 = y_2$$

For $(x_1,\ y_1)\,,\ (x_2, y_2) \in \mathbb{R}^2$. We can equally have

$$(x_1,\ y_1, z_1) = (x_2, y_2, z_2) \Leftrightarrow x_1 = x_2,\ y_1 = y_2 \ and \ z_1 = z_2$$

for $(x_1,\ y_1, z_1)\,,\ (x_2, y_2, z_2) \in \mathbb{R}^3$ which can be generalized as $(x_1,\ y_{1,}z_1,\ \cdots P_n) = (x_2, y_2, z_2 \cdots P_{n+1}\ ) \Leftrightarrow x_1 = x_2,\ y_1 = y_2, z_1 = z_2 \ and \ P_n = P_{n+1}$ and for $(x_1,\ y_1,\ z_p,\ \cdots P_n)\,,\ (x_2, y_2,\ z_2 \cdots P_{n+1}) \in \mathbb{R}^n$

3. Main Results

In this section, fixed point results on $\omega - OCP_n$ which is considered as a weakly compatible function and also as a semigroup of linear operator are obtained.

Theorem 3.1

Let X be a Banach space with $\mathcal{J}$ Archimedean and $T : X \to X$ be an $\omega - OCP_n$ (order preserving partial contraction mapping) for which $a \in (0,1)$ whenever $\| \ Tx - Ty \ \| \leq \ a \ \| \ x - y \ \| \ \forall x, y \in X$. Then, T has a fixed point $u \in X$. Furthermore, for any given $x \in X$ there exists a limit with

$$\| \ T^n (x) - u \ \| \leq \frac{a^n}{1-a} \ \| \ x - T(x) \ \| \tag{1}$$

However, we have $T(X) \subseteq T_1(X)$. If we have another map $T_1 : X \to X$ which is also $\omega - OCP_n$ and $T_1(X)$ or $T(X)$ is E-complete subspace on X. If T_1 and T are weakly compatible, then they have unique common fixed point on X, where

$$u(x, y) \ \epsilon \{\| \ T_1x - T_1y \ \|, \ \| \ T_1x - Tx \ \|, \ \| \ T_1y - Ty \ \|, \frac{1}{2} [\| \ T_1x - T_y \ \|] + \ \| \ T_1y - Tx\|\}$$

Proof

Suppose T that maps a complete normed space onto a complete normed space is Archimedean for which $a \in (0,1)$ and for all $x \in X$ then

$\| \ Tx - Ty \ \| \leq a \ \| \ x - y \ \|$ and

$$\| \ T^n (x) - T^{n+1} \ (x) \ \| \leq a \ \| \ T^{n-1} (x) - T^n (x) \ \| \leq \cdots \leq a^n \ \| \ x - T(x) \ \|$$

Thus, for $m > n$ where $n \in \mathbb{N} \cup \{0\}$

$$\| \ T^n (x) - T^m (x) \ \| \leq \| \ T^n (x) - T^{n+1} (x) \ \| + \ \| \ T^{n+1} (x) - T^{n+2} (x) \ \| + \cdots +$$

$$\| T^{m-1}(x) - T^m(x) \| \leq a^n \| x - T(x) \| + \cdots + a^{m-1} \| x - T(x) \|$$
$$\leq a^n \| x - T(x) \| \left[1 + a + a^2 + \cdots \right] = \frac{a^n}{1-a} \| x - T(x) \|$$

hence

$$\| T^n(x) - T^m(x) \| = \frac{a^n}{1-a} \| x - T(x) \| \tag{2}$$

Suppose there exists $x, y \in X$ with $x = Tx$ and $y = Ty$, then by contraction mapping, we have

$$\| Tx - Ty \| \leq a \| x - y \|$$

and

$$\| Tx - Ty \| \leq \| x - y \|$$

Therefore $\| x - y \| = 0$ then

$$x = y$$

Since X is complete, there exist $u \in X$ such that

$$\lim_{n\to\infty} T^n(x) \quad = u$$

Moreover, the continuity of T yields

$$u = \lim_{n\to\infty} T^{n+1}(x) = \lim_{n\to\infty} T(T^n(x)) = T(u)$$

Therefore u is a fixed point of T. Hence, letting $m \to \infty$ in (2), then

$$\| T^n(x) - u \| \leq \frac{a^n}{1-a} \| x - T(x) \| \tag{3}$$

Now, we need to show that T_1 and T are weakly compatible and have a unique common fixed point on X. Since $\mathcal{J}$ is Archimedian on X and let $T_1 : X \to X$ be another self map on X such that the range of T_1 contains the range of T and the range of at least one is E- complete whenever x_n is an E- Cauchy sequence, then there exists $z \in T_1(x)$ such that $T_1 y_n \to z$. Hence there exists a sequence k_n in $\mathcal{J}$ such that $k_n \to 0$ and $\| Ty_n - z \| = k_n$. On the other hand, we can find $v \in X$ such that $T_1 v = z$. Let us show that $Tv = z$. Then

$$\| Tv - z \| \leq \| Tv - Ty_n \| + \| Ty_n - z \| \leq au \| y_n - v \| + k_{n+1}$$

For all $n \in \mathbb{N}$,

$$x(y_n, v) \in \{\| T_1 y_n - T_1 v \| , \| T_1 y_n - T y_n, \| T_1 v - Tv \|, \frac{1}{2} [\| T_1 y_n - T v\|] + \| T_1 v - T y_n \|\}$$

at least one of the following four hold:

i. $\| Tv - z \| \leq \| T_1 y_n - T_1 v \| + k_{n+1} \leq k_n + k_{n+1} = 2k_n$;

ii. $\| Tv - z \| \leq \| T_1 y_n - T y_n \| + k_{n+1} \leq \| T_1 y_n - z \| + 2k_{m+1} \leq 3k_n$;

iii. $Tv - z \| \leq a \| T_1 v - Tv \| + k_{n+1} \leq a \| Tv - z \| + k_n \Rightarrow \| Tv - z \| \leq \frac{1}{1-a} k_n$;

and

iv. $\| Tv - z \| \leq \frac{1}{2}$ $[\| T_1 y_n - Tv \|] + \| T_1 v - Ty_n] + k_{n+1}$

$$\leq \frac{1}{2} \| T_1 y_n - T_v \| + \frac{3}{2} k_{n+1}$$
$$\leq \frac{1}{2} \| T_1 y_n - z \| + \frac{1}{2} \| Tv - z \| + \frac{3}{2} k_n$$
$$\leq \frac{1}{2} \| Tv - z \| + 2k_n$$

That is $\| Tv - z \| \leq 4k_n$. Since the infimum of the sequence on the right side of last inequality is zero, then $\| Tv - z \| = 0$ i.e $Tv = z$. Therefore, z is a point of coincidence of T and T_1. If z_1 is another point of coincidence then there is $v_1 \in X$ with $z_1 = Tv_1 = T_1 v_1$. Now from our contractive condition it follows that

$$\| z - z_1 \| = \| Tv - Tv_1 \| \leq a \| v - v_1 \|$$

where

$$\| v - v_1 \| \in \{\| T_1 v - T_1 v_1 \|, \| T_1 v - Tv \|, \| T_1 v_1 - Tv_1 \|, \frac{1}{2}[\| T_1 v - Tv_1 \| + \| T_1 v_1 - Tv \|\}$$
$$= \{0, \| z - z_1 \|\}$$

Hence, $\| z - z_1 \| = 0$, implies that $z = z_1$. If T_1 and T are weakly compatible, then it is obvious that z is unique common fixed point of T and T_1. Hence the proof.

Theorem 3.2

Let $A \in \omega - OCP_n$ such that $\omega - OCP_n \in L(X)$ where $L(X)$ is a bounded linear operator from X to X then $T(t) = e^{tA}$ with $t \in \mathbb{N}$ is uniformly continuous semigroup.

Proof

Since A is bounded, we have that $||A|| < \infty$ and thus $\sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$ converges for $t \geq 0$ to the bounded linear operator $T(t)$ since $\left(\sum_{i=0}^{\infty} \frac{(t)^i}{i!}\right) \left(\sum_{j=0}^{\infty} \frac{(s)^j}{j!}\right) = \sum_{k=0}^{\infty} \frac{(t+s)^k}{k!}$. $T(0) = I$ is clear, It remains to show that $T(t)$ is a uniformly continuous semigroup, we have

$$\|T(t) - I\| = \left\| \sum_{k=1}^{\infty} \frac{t^k A^k}{k!} \right\|$$
$$\leq \sum_{k=1}^{\infty} \frac{t^k \| A \|^k}{k!}$$
$$= e^{t\|A\|} - 1$$

and $e^{t\|A\|} - 1 \to 0_+$ as $t \to 0$. Hence, it is uniformly continuous.

Note: A C_0-semigroup is said to be uniformly continuous if an only if $\lim_{t \to 0} \|T(t) - I\| = 0$.

Corollary 3.3

Let $T(t) \supseteq \omega - OCP_n($ ω-order preserving partial contraction mapping), then for each $f \in X, t \to T(t)$, f is the continuous function on $\mathbb{N} \to X$ if
(i) $\lim_{t \to 0_+} T(t)f = f$ for each $f \in X$; and
(ii) $\lim_{h \to 0} T(t+h) = T(t)$.

Proof

Let $t, h \geq 0$ and $f \in X$, then we have

$$\begin{aligned} \|T(t+h)f - T(t)f\| &= \|T(t)T(h)f - T(t)f\| \\ &\leq \|T(t)\| \, \|T(h)f - f\| \\ &\leq Me^{\omega t} \|T(h)f - f\| \end{aligned}$$

and for $t \geq h \geq 0$

$$\begin{aligned} \|T(t-h)f - T(t)f\| &= \|T(t-h)f - T(t-h+h)f\| \\ &= \|T(t-h)f - T(t-h)T(h)f\| \\ &\leq \|T(t-h)\| \, \|f - T(h)f\| \\ &\leq Me^{\omega t} \|f - T(h)h\| \end{aligned}$$

Therefore, the limit exists and the function is continuous on $\mathbb{N} \to X$.

Proposition 3.4

Let $T(t)$ be a C_0-semigroup generated by A, where $A \in \omega - OCP_n$. Then
(i) for each $f \in D(A)$, we have $T(t)f \in D(A)$; and
(ii) for each $f \in D(A)$ and $T(t)f \in D(A)$ we have

$$T(t)f = AT(t) = T(t)Af$$

Proof

(i) Let $f \in D(A)$ and fix $t \geq 0$ then $s > 0$ by definition of infinitesimal generator. The semigroup property $T(t)T(s) = T(s)T(t) = T(s+t)$ leads to

$$\begin{aligned} AST(t)f &= \frac{T(s)T(t)f - T(t)f}{s} \\ &= \frac{T(t)T(s)f - T(t)f}{s} \\ &= T(t) \, . \frac{T(s)f - f}{s} \end{aligned}$$

$$\lim_{s \to 0^+} T(t) \, . \frac{T(s)f - f}{s} \to T(t)(Af)$$

we have that $f \in D(A)$ and $T(t)$ is continuous in X

$$\lim_{s \to 0^+} AST(t) \; f = T(t)Af = AT(t)f$$

hence

$$T(t)f \in D(A)$$

(ii) From $f \in D(A)$ and $h > 0$, we have

$$\lim_{h\to 0^+} \frac{T\,(t+h) - T\,(t)}{h}$$

$$= \lim_{h\to 0^+} \frac{T\,(t+h)\,f - T(t)f}{h}$$

$$= \lim_{h\to 0^+} \frac{T\,(t)\,T(h)f - T(t)f}{h}$$

$$\lim_{h\to 0^+} \left(\frac{T\,(h) - I}{h}\right) T(t)f$$

$= AT(t)f = T(t)Af.$

Consider

$$\lim_{h\to 0^+} \frac{T\,(t-h)\,f - T(t)f}{h} = \lim_{h\to 0^+} \frac{T\,(t-h)\,f - T\,(t+h-h)\,f}{h}$$

$$= \lim_{h\to 0^+} \frac{T\,(t-h)\,f - T\,(t-h)\,T(h)f}{h} = \lim_{h\to 0^+} \left(\frac{I - T\,(h)}{h}\right) T(t-h)f$$

$= AT(t)f.$

Conclusion

In this paper, it has been shown that $\omega - OCP_n$, which is a semigroiup of linear operator has a common unique fixed point when it is considered as a weakly compatible function.

Conflict of Interest The authors declare that there are no conflicts of interest regarding the publication of the paper.

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