*Nigerian Journal of Mathematics and Applications*
*Volume A*, **28**, (2019), 65 − 76. *Printed by Unilorin press*
©*Nig. J. Math. Appl.* *http://www.njmaman.com*

# Homotopy Perturbation with Laplace Transforms Method for Solving Singular Initial Value Problems

M. O. Olayiwola and A. Adegoke

Abstract

In this paper, the Homotopy Perturbation Method with Laplace Transform (LT-HPM) is employed in solving the Lane-Emden type of differential equations. Application is on linear and nonlinear Lane-Emden singular initial value problems. Comparison with exact solution shows considerable acceleration in convergence. The method is effective and easy to implement. The results are also presented in tabular form.

## 1. Introduction

Lane-Emden type singular initial value problems (IVPs) are applicable in various ranges of mathematical problems which are challenging in nature because of the singularity. The numerical solution is however challenging because of the singularity at the origin. Researchers have applied different types of method to solve singular initial value problems (IVPs) [1-2]. For instance, Ramos [3] and Olayiwola [4] respectively obtained a series approach to the Lane-Emden equations and comparisons with He's Homotopy perturbation method and variational

Received: 04/05/2019, Accepted: 11/11/2019, Revised: 23/12/2019. * Corresponding author.
2015 *Mathematics Subject Classification.* 34K27, & 35E15.
*Key words and phrases.* Lane-Emden, Homotopy Perturbation Method, Laplace Transform, Singular Initial Value Problem
Department of Mathematical Sciences Faculty of Basic and Applied Sciences College of Science, Engineering and Technology, Osun State University, Osogbo, Nigeria
E-mail: olayiwola.oyedunsi@uniosun.edu.ng, adebisi.adegoke@uniosun.edu.ng

Iteration method for solution of Emden-Fowler type of singular initial value problems (IVPs). Approximate solutions of the Lane-Emden equations were obtained by Homotopy Perturbation method [5-6]. The method was proposed by Huan He [7-8] and has been successfully applied to solve many types of linear and nonlinear functional equation. Applying Homotopy Perturbation method with Laplace Transform [9] is a good method for obtaining solutions of linear and nonlinear Lane-Emden type differential equation which gives more accurate solutions when compared with exact solutions. In this study, Homotopy Perturbation Method (HPM) with Laplace Transform (LT) is applied to solve the general Lane-Emden type of differential equations used in modeling and application to physical and astrophysics problems [10]. Many benefits are derived from this type of equation mostly in the area of science and engineering. Momoniat and Harley [11] obtained an approximate implicit solution by reducing the Lane-Emden equation of first order differential equation using the Lie group analysis and determining a power series solution of the reduced equation. The Homotopy Perturbation Method is spell out in the first stage, the Lane-Emden equation is given in the third stage while Homotopy Perturbation Method with Laplace Transform (LT-HPM) in the second stage. Some examples and result analysis and conclusion were also presented.

## 2 Homotopy Perturbation Method (HPM)

[13] & [14] illustrated the basic idea of this method. Also [15] considered the following general nonlinear differential equation:

$$A\left(u\right)-f\left(r\right)=0,\ r\in\Omega \tag{1}$$

with the following boundary conditions:

$$B\left(u,\partial u/\partial n\right)=0,\ r\in\Gamma \tag{2}$$

where $A$ is a general differential operator, $B$ is a boundary operator, $f\left(r\right)$ is a known analytical function and $\Gamma$ is the boundary of the domain $\Omega$.
The operator $A$ can be decomposed into a linear part and a nonlinear one designated as $L$ and $N$ respectively. Then (1) can be written in the following form:

$$L\left(u\right)+N\left(u\right)-f\left(r\right)=0 \tag{3}$$

By homotopy technique, we construct a homotopy $v\left(r,p\right):\Omega\times[0,1]\rightarrow R$ which satisfies

$$H\left(v,p\right)=\left(1-p\right)\left[L\left(v\right)-L\left(u_0\right)\right]+p\left[A\left(v\right)-f\left(r\right)\right]=0 \tag{4}$$

where $p \in [0,1]$ is an embedding parameter and $u_0$ is an initial approximation of (1) which satisfies the boundary conditions. Then, from (4) we obtained:

$$H(v,0) = L(v) - L(u_0) = 0 \tag{5}$$

$$H(v,1) = A(v) - f(r) = 0 \tag{6}$$

On changing the value of $p$ from zero to unity, $v(r,p)$ changes from $u_0(r)$ to $u(r)$; in topology this is refers to as deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic. Since $p \in [0,1]$ can be taking as a small parameter, we consider the solution of (4) as a power series of $p$ as follows:

$$v = v_0 + pv_1 + p^2 v_2 + ... + p^n v_n \tag{7}$$

Setting $p$=1 gives approximation solution for (1).

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ... + v_n \tag{8}$$

## 3 Laplace Transform Method

The Laplace Transform of a function $K(x)$ is defined by

$$K(s) = L[K(x)] = \int_0^\infty e^{-st} K(x) dx;\ x \geq 0 \tag{9}$$

The integral on the right hand side of (9) exists where $s$ is real and $L$ is the Laplace transform operator.

## 4 Homotopy Perturbation Method with Laplace Transform (LT-HPM)

In this section, we discuss the application of the LT-HPM for the solution of the Lane-Emden equation:

$$y'' + \frac{2}{x} y' + \tau(x,y) = \varphi(x),\ 0 \leq x \leq 1 \tag{10}$$

Multiplying $x$ and taking the Laplace transform on both side of (9) we obtain:

$$-s^{-2} L'(y) - y(0) + L\{x\tau(x,y) - x\varphi(x)\} = 0 \tag{11}$$

where $L$ is Laplace transform operator and $L'(y) = \frac{dL(y)}{ds}$. Then

$$L'(y) = -s^{-2} y(0) + s^{-2} L\{x\tau(x,y) - x\varphi(x)\} \tag{12}$$

and

$$\left\{\begin{array}{l} L\left(xy''\right)=-\frac{d}{ds}L\left(y''\right) \\ L\left(y''\right)=s^{2}L\left(y\right)-sy\left(0\right)-y'\left(0\right) \\ L\left(y'\right)=sL\left(y\right)-y\left(0\right) \end{array}\right. \tag{13}$$

by integrating both side of (11) with respect to $s$, we have

$$L'\left(y\right)dy=-\int s^{-2}y\left(0\right)ds+\int s^{-2}L\left\{x\tau\left(x,y\right)-x\varphi\left(x\right)\right\}ds \tag{14}$$

Taking the inverse Laplace transform on both sides of (12) and by using the initial condition $y(0)=A$, we have:

$$y\left(x\right)=A+L^{-1}\left\{\int s^{-2}L\left[x\tau\left(x,y\right)-x\varphi\left(x\right)\right]ds\right\} \tag{15}$$

We decompose $\tau\left(y,x\right)$ into two parts

$$\tau\left(y,x\right)=K\left[y\left(x\right)\right]+N\left[y\left(x\right)\right] \tag{16}$$

where $K\left[y\left(x\right)\right]$ and $N\left[y\left(x\right)\right]$ represents the linear term and the nonlinear term respectively. The Homotopy perturbation method and the He's polynomials can be used to solve (10) and to address the nonlinear term. Homotopy perturbation method with Laplace transform (LT-HPM) denotes a solution by an infinite series of components given by:

$$y\left(x\right)=\sum_{n=0}^{\infty}p^{n}y_{n}\left(x\right) \tag{17}$$

where the terms $y_{n}\left(x\right)$ are repeatedly calculated and the nonlinear $\varphi\left(y\right)$ can be given as:

$$N\left(y\right)=\sum_{n=0}^{\infty}p^{n}H_{n}\left(y\right) \tag{18}$$

where $N\left(y\right)$ is a non-linear term and $H_{n}\left(y\right)$ is the He's polynomial. For some He's polynomial $H_n$, Mishra and Nagar [16] obtained the following:

$$y\left(x\right)=-L^{-1}\left\{\left(\int s^{-2}y\left(0\right)\right)ds\right\}+L^{-1}\left\{\int s^{-2}L\left[x\tau\left(x,y\right)-x\varphi\left(x\right)\right]ds\right\} \tag{19}$$

$$H_n\left(y_0, y_1, y_2, ...y_n\right) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum p^i y_i\right)\right]_{p=0} \quad n = 0, 1, 2, ... \tag{20}$$

Putting the value of (17) and (18) in (16) we obtain:

$$\begin{array}{l}\sum p^n y_n = A+ \\ p\left\{L^{-1}\int s^{-2}\left\{\left(L\left(\left(\sum p^n H_n\left(x\right)\right) - x\left(\sum p^n y_n\left(y\right)\right)\right)\right)\right\} ds\right\}\end{array} \tag{21}$$

This is the coupling of the Laplace transformation and Homotopy Perturbation Method (LT-HPM) using He's polynomials, see [16] & [17]. Comparing the coefficient of like power of $p$, then we obtained the following approximations:

$$\begin{cases} p^0 : y_0\left(x\right) = A \\ p^1 : y_1(x) = -L^{-1}\left[\int s^{-2}\left(\left(L\left(x\left(H_0\right) - x\left(y_0\right)\right)\right)\right) ds\right] \\ p^2 : y_2(x) = -L^{-1}\left[\int s^{-2}\left(\left(L\left(x\left(H_1\right) - x\left(y_1\right)\right)\right)\right) ds\right] \\ p^3 : y_3(x) = -L^{-1}\left[\int s^{-2}\left(\left(L\left(x\left(H_2\right) - x\left(y_2\right)\right)\right)\right) ds\right] \end{cases} \tag{22}$$

If the integral on right hand side exists, then $y \geq 0$, $s$ is real and $L$ is the Laplace transform operator. In this section, we used the Laplace Transform (12) to solve Lane-Emden equations given in (10).

## 5 The Lane-Emden Equation

Lane-Emden is a differential equation, linear or nonlinear with singular initial value problems (IVPs) of the second order which could be homogenous or nonhomogeneous. The equations of this form are applicable in many field of study. In this paper, we consider Lane-Emden equation of the form:

$$y'' + \frac{2}{x}y' + a\left(y\right) = 0 \;\; 0 \leq x \leq 1 \tag{23}$$

subject to conditions,

$$y\left(0\right) = A, \;\; y'\left(0\right) = B \tag{24}$$

Also, a nonlinear type of singular initial value problem of the Lane-Emden type is of the form:

$$y'' + \frac{2}{x}y' + a\left(x, y\right) = b\left(x\right) \;\; 0 \leq x \leq 1 \tag{25}$$

## 6 Numerical Examples

**Example 1:** We consider:

$$y'' + \frac{2}{x}y' + \left(y - x^4 + x^3\right) = 21x^2 - 12x + 6.$$

$$y(0) = 0,\ y'(0) = 0$$

The exact solution is $y(x) = x^2 - x^3 + x^4$. Taking the Laplace of both sides, we obtained

$$L\left(xy''\right) + 2L\left(y'\right) + L(xy) = L\left(x^5 - x^4 + 21x^3 - 12x^2 + 6x\right) \tag{26}$$

Substitute (13) into (25)

$$\begin{aligned} &-\tfrac{d}{ds}\left(s^2L(y) - sy(0) - y'(0)\right) + 2\left(sL(y) - y(0)\right) + L(xy) = \\ &L\left(x^5 - x^4 + 21x^3 - 12x^2 + 6x\right)\end{aligned} \tag{27}$$

Integrating (27), we get:

$$\int L'(y)\,dy = -\int s^{-2}L\left(x^5 - x^4 + 21x^3 - 12x^2 + 6x\right)ds + \int s^{-2}L(xy)\,ds \tag{28}$$

Applying Laplace Inverse on both sides;

$$L(y) = \frac{x^6}{42} - \frac{x^5}{30} + \frac{21}{20}x^4 - x^3 + x^2 + L^{-1}\left(\int s^{-2}L(xy)\,ds\right) \tag{29}$$

Applying Homotopy Perturbation Method (HPM) on (29)

$$\sum_{n=0}^{\infty} p^n y_n = \frac{x^6}{42} - \frac{x^5}{30} + \frac{21}{20}x^4 - x^3 + x^2 + pL^{-1}\left(\int s^{-2}L\left(\sum_{n=0}^{\infty} p^n y_n(x)\,ds\right)\right) \tag{30}$$

equating the co-efficient to the power of p, then,

$$p_0 : y_0 = x^2 - x^3 + \frac{21}{20}x^4 - \frac{x^5}{30} + \frac{x^6}{42} \tag{31}$$

$$\left\{\begin{array}{l} p_1 : y_1 = L^{-1}\left\{\int s^{-2}L(x, y_0)(x)\,ds\right\} \\ p_2 : y_2 = L^{-1}\left\{\int s^{-2}L(x, y_1)(x)\,ds\right\} \\ p_3 : y_3 = L^{-1}\left\{\int s^{-2}L(x, y_2)(x)\,ds\right\} \\ p_4 : y_4 = L^{-1}\left\{\int s^{-2}L(x, y_3)(x)\,ds\right\} \end{array}\right\} \tag{32}$$

$$y_1 = L^{-1}\left\{\int s^{-2}L(x, y_0)(x)\,ds\right\} \tag{33}$$

$$y_1 = -\frac{x^4}{20} + \frac{x^5}{30} - \frac{x^6}{40} + \frac{x^7}{1680} - \frac{x^8}{3024} \tag{34}$$

(35) $$y_2 = L^{-1}\left\{\int s^{-2} L\left(x, y_1\right)(x)\, ds\right\}$$

(36) $$y_2 = \frac{x^6}{840} - \frac{x^7}{1680} + \frac{x^8}{2880} - \frac{x^9}{15120} + \frac{x^{10}}{332640}$$

Therefore

(37) $$y\left(x\right) = y_0 + y_1 + y_2 + y_3 + y_4 + \ldots + y_n$$

(38) $$y\left(x\right) = x^2 - x^3 + \tfrac{21}{20}x^4 - \tfrac{x^5}{30} + \tfrac{x^6}{42} - \tfrac{x^4}{20} + \tfrac{x^5}{30} - \tfrac{x^6}{40} + \tfrac{x^7}{1680} - \tfrac{x^8}{3024} + \tfrac{x^6}{840} - \tfrac{x^7}{1680} + \tfrac{x^8}{2880} - \tfrac{x^9}{15120} + \tfrac{x^{10}}{332640}$$

(39) $$y\left(x\right) = x^2 - x^3 + x^{4.}$$

Equation (39) is the exact solution to example 1.

**Example 2:** We consider:

$$y'' + \frac{2}{x}y' + y = 2\ell^x\left(x^2 + 3x + 3\right)$$
$$y\left(0\right) = 0,\ y'\left(0\right) = 0,$$

The exact solution is given as $y\left(x\right) = x^2\ell^x$. Taking the Laplace of both side yields

(40) $$L(xy'') + 2L(y') + L(xy) = L\left(6x + 12x^2 + 11x^3 + 6x^4 + \tfrac{9}{4}x^5 + \tfrac{19}{30}x^6 + \tfrac{17}{20}x^7 + \tfrac{11}{420}x^8\right)$$

Applying (13) in (40)

(41) $$-\frac{d}{ds}\left(s^2L\left(y\right) - sy\left(0\right) - y'\left(0\right)\right) + 2\left(sL\left(y\right) - y\left(0\right)\right) + L\left(xy\right) = L\left(\begin{array}{l} 6x + 12x^2 + 11x^3 + 6x^4 + \frac{9}{4}x^5 + \frac{19}{30}x^6 + \\ \frac{17}{20}x^7 + \frac{11}{420}x^8 \end{array}\right)$$

Integrating we get:

(42) $$\int L'\left(y\right)dy = -\int s^{-2}L\left(6x + 12x^2 + 11x^3 + 6x^4 + \tfrac{9}{4}x^5 + \tfrac{19}{30}x^6 + \tfrac{17}{20}x^7 + \tfrac{11}{420}x^8\right)ds + \int s^{-2}L\left(xy\right)ds$$

Applying Laplace Inverse on both sides,

$$L(y) = -L^{-1} \left\{ \int s^{-2} L\left(6x + 12x^2 + 11x^3 + 6x^4 + \tfrac{9}{4}x^5 + \tfrac{19}{30}x^6 + \tfrac{17}{120}x^7 + \tfrac{11}{420}x^8\right) ds \right\} + L^{-1}\left\{ \left( \int s^{-2} L(xy) \right) ds \right\} \tag{43}$$

Applying Homotopy Perturbation Method (HPM) on (43) we have

$$\sum_{n=0}^{\infty} p^n y_n = \tfrac{132}{453600}x^9 + \tfrac{238}{120960}x^8 + \tfrac{19}{1680}x^7 + \tfrac{3}{56}x^6 + \tfrac{1}{5}x^5 + \tfrac{11}{20}x^4 + x^3 + x^2 + pL^{-1}\left( \int s^{-2} L(xy)\, ds \right) \tag{44}$$

Equating the co-efficient to the like power of $p$, then we have

$$p_0 = y_0 = x^2 + x^3 + \frac{11}{20}x^4 + \frac{1}{5}x^5 + \frac{3}{56}x^6 + \frac{19}{1680}x^7 + \frac{238}{120960}x^8 + \frac{132}{453600}x^9 \tag{45}$$

$$\left\{ \begin{array}{l} p_1 : y_1 = L^{-1}\left\{ \int s^{-2} L(x, y_0)(x)\, ds \right\} \\ p_2 : y_2 = L^{-1}\left\{ \int s^{-2} L(x, y_1)(x)\, ds \right\} \\ p_3 : y_3 = L^{-1}\left\{ \int s^{-2} L(x, y_2)(x)\, ds \right\} \\ p_4 : y_4 = L^{-1}\left\{ \int s^{-2} L(x, y_3)(x)\, ds \right\} \end{array} \right\} \tag{46}$$

$$p_1 : y_1 = L^{-1}\left\{ \int s^{-2} L(x, y_0)(x)\, ds \right\} \tag{47}$$

This implies

$$y_1 = -\tfrac{1}{20}x^4 - \tfrac{1}{30}x^5 - \tfrac{11}{840}x^6 - \tfrac{1}{280}x^7 - \tfrac{1}{1344}x^8 - \tfrac{19}{151200}x^9 - \tfrac{238}{13305600}x^{10} - \tfrac{132}{184875200}x^{11} \tag{48}$$

$$y_2 = L^{-1}\left\{ \int s^{-2} L(x, y_1)(x)\, ds \right\} \tag{49}$$

$$y_2 = \tfrac{1}{840}x^6 + \tfrac{1}{1680}x^7 + \tfrac{11}{60480}x^8 + \tfrac{1}{25200}x^9 + \tfrac{1}{147840}x^{10} + \tfrac{19}{19958400}x^{11} + \tfrac{119}{1037836800}x^{12} + \tfrac{11}{2803940533}x^{13} \tag{50}$$

$$y_3 = L^{-1}\left\{ \int s^{-2} L(x, y_2)(x)\, ds \right\} \tag{51}$$

$$y_3 = -\tfrac{1}{60480}x^8 - \tfrac{1}{151200}x^9 - \tfrac{11}{6652800}x^{10} - \tfrac{1}{3326400}x^{11} - \tfrac{1}{23063040}x^{12} - \tfrac{19}{3632428800}x^{13} - \tfrac{119}{217945728000}x^{14} - \tfrac{11}{672945727920}x^{15} \tag{52}$$

$$y_4 = L^{-1}\left\{ \int s^{-2} L(x, y_3)(x)\, ds \right\} \tag{53}$$

$$y_4 = \frac{1}{6652800}x^{10} + \frac{1}{19958400}x^{11} + \frac{11}{1037836800}x^{12} + \frac{1}{605404800}x^{13} + \frac{1}{4843238400}x^{14} + \frac{19}{871782912000}x^{15} + \frac{119}{59281238016000}x^{16} + \frac{11}{205921392743520}x^{17} \quad (54)$$

$$y(x) = y_0 + y_1 + y_2 + y_3 + y_4 + ... + y_n \quad (55)$$

$$y(x) = x^2\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 ...\right) \quad (56)$$

Equation (56) is the numerical solution to example 2.

***Table 1: Comparison of exact solution with numerical solution of example 2*** *at n=3 and* $0 \le x \le 1$

| **x** | **Exact Solution** | **Numerical Solution** | **Absolute Error** |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0.1 | 0.01105170918 | 0.01105170918 | 0 |
| 0.2 | 0.04885611032 | 0.04885611033 | 1E-11 |
| 0.3 | 0.1214872927 | 0.1214872924 | 3E-10 |
| 0.4 | 0.2386919517 | 0.2386919477 | 4E-9 |
| 0.5 | 0.4121803178 | 0.4121802811 | 3.67E-8 |
| 0.6 | 0.6559627680 | 0.6559625404 | 2.276E-7 |
| 0.7 | 0.9867388264 | 0.9867377543 | 1.0721E-6 |
| 0.8 | 1.4243461940 | 1.4243420860 | 4.108E-6 |
| 0.9 | 1.9922785200 | 1.9922650630 | 1.3457E-5 |
| 10 | 2.7182818280 | 2.7182491500 | 3.8913E-5 |

**Example 3**: We consider:

$$y'' + \frac{2}{x}\left(y + y'\right) - y = 0$$

$$y(0) = 1,\ y'(0) = 0 - 1$$

with the exact solution $y(x) = \ell^{-x}$. Taking the Laplace of both sides yields

$$L\left(xy''\right) + 2L\left(y'\right) + L\left((2 - x)y\right) = 0 \quad (57)$$

Recall that:

$$\begin{cases} L(xy'') = -\frac{d}{ds}L(y'') \\ L(y'') = s^2L(y) - sy(0) - y'(0) \\ L(y') = sL(y) - y(0) \end{cases} \quad (58)$$

Substitute (58) into (57) we obtained:

$$-\frac{d}{ds}\left(s^{-2}L\left(y\right)-sy\left(0\right)-y'\left(0\right)\right)+2\left(sL\left(y\right)-y\left(0\right)\right)+L\left(\left(2-x\right)y\right)=0 \tag{59}$$

Integrating (59) leads to:

$$\int L\left(y\right)dy=-\int s^{-2}ds+\int s^{-2}L\left(\left(2-x\right)y\right)ds \tag{60}$$

Applying Laplace Inverse on both sides:

$$y=1+L^{-1}\left\{\int s^{-2}L\left(\left(2-x\right)y\right)ds\right\} \tag{61}$$

Applying Homotopy Perturbation Method (HPM) on (61)

$$\sum_{n=0}^{\infty}p^{n}y_{n}=1+pL^{-1}\left\{\int s^{-2}L\left(\left(2-x\right)\sum_{n=0}^{\infty}p_{n}y^{n}\right)ds\right\} \tag{62}$$

Equating the co-efficient to the power of $p$, then

$$p_0 : y_0 = 1 \tag{63}$$

$$\left\{\begin{array}{l} p_1 : y_1 = \left(\int s^{-2}L\left(\left(2-x\right)y_0\right)ds\right) \\ p_2 : y_2 = \left(\int s^{-2}L\left(\left(2-x\right)y_1\right)ds\right) \\ p_3 : y_3 = \left(\int s^{-2}L\left(\left(2-x\right)y_2\right)ds\right) \\ p_4 : y_{41} = \left(\int s^{-2}L\left(\left(2-x\right)y_3\right)ds\right) \end{array}\right. \tag{64}$$

$$y_1=\left(\int s^{-2}L\left(\left(2-x\right)y_0\right)ds\right) \tag{65}$$

$$y_1=-x+\frac{x^2}{3!} \tag{66}$$

$$y_2=\left(\int s^{-2}L\left(\left(2-x\right)y_1\right)ds\right) \tag{67}$$

$$y_2=\frac{x^2}{3}-\frac{x^3}{9}+\frac{x^4}{5!} \tag{68}$$

$$y_3=\left(\int s^{-2}L\left(\left(2-x\right)y_2\right)ds\right) \tag{69}$$

$$y_3=-\frac{1}{18}x^3+\frac{1}{36}x^4-\frac{23}{5400}x^5+\frac{1}{5040}x^6 \tag{70}$$

$$y_4=\left(\int s^{-2}L\left(\left(2-x\right)y_3\right)ds\right) \tag{71}$$

$$y_4 = \frac{1}{180}x^4 - \frac{1}{270}x^5 + \frac{7}{8100}x^6 + \frac{11}{132300}x^7 - \frac{1}{362880}x^8 \tag{72}$$

$$y(x) = y_0 + y_1 + y_2 + y_3 + y_4 + ... + y_n \tag{73}$$

$$y(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - ... \tag{74}$$

Equation (74) is the numerical solution to example 3.

***Table 2: Comparison of exact solution with numerical solution of example 3*** *at n=3 and* $0 \leq x \leq 1$

| **x** | **Exact Solution** | **Numerical Solution** | **Absolute Error** |
|---|---|---|---|
| 0 | 1 | 1 | 0 |
| 0.1 | 0.9048374180 | 0.9048374215 | 3.5E-9 |
| 0.2 | 0.8187307531 | 0.8187308544 | 1.013E-7 |
| 0.3 | 0.7408182207 | 0.7408189426 | 7.219E-7 |
| 0.4 | 0.6703200460 | 0.6703229462 | 2.9002E-6 |
| 0.5 | 0.6065306597 | 0.6065392329 | 8.5732E-6 |
| 0.6 | 0.5488116361 | 0.5488326583 | 2.10222E-5 |
| 0.7 | 0.4965853038 | 0.4966308682 | 4.55644E-5 |
| 0.8 | 0.4493289641 | 0.4494195609 | 9.05968E-5 |
| 0.9 | 0.4065696597 | 0.4067387460 | 1.690863E-4 |
| 10 | 0.3678794412 | 0.3681800359 | 3.005947E-4 |

## 7 Discussion of Results

In this article, we applied homotopy perturbation method with Laplace transform to three different numerical problems of singular initial value problems.
In example 1, the method converged rapidly to the exact solution after two iterations. In example 2, we present the absolute error in Table 1 where it was evident that the error is within the range: $3.8913 \times 10^{-5} \leq e_r \leq 0$. This is in good agreement with the exact solution. Similarly in Table 2 the absolute error of example 3 shows that the method is also elegant with error range of $3.005947 \times 10^{-4} \leq e_r \leq 0$.

## 8 Conclusion

This paper successfully applied the Homotopy Perturbation Method with Laplace Transform (LT-HPM) in obtaining the exact solutions for singular Initial Value Problems (IVPs), the Lane-Emden type of differential equations. The method shows rapid convergence to series solution when compared with the exact solutions of considered examples. The method is a good tool for singular Initial Value

Problems (IVPs) of Lane-Emden type, in example 1 it converges directly to exact solutions in just two iterations, while others yield excellent approximations. The method can be extended to the solutions of systems of Lane-Emden type of equations.

## References


[1] Bert C. V. and Zeng H. (2004). Analysis of axial vibration of compound bars by differential transformation method. J. Sound Vib. **275**, 641-647.

[2] Chen C. K. and Chen S. S. (2004). Application of the differential transformation method to a non-linear conservative system. Appl. Math. Comput. **154**, 431-441.

[3] Ramos J. I. (2008) Series Approach to the Lane-Emden equation and comparison with the Homotopy perturbation method. Chaos Solitons Fractals. **38** (2), 400-408.

[4] Olayiwola M. O. (2019). Solutions of Emden-Fowler type equation by variational iteration method. Cankaya University Journal of Sci. and Engineering. **16** (2), 001-009.

[5] Arioel P. D. (2007). The three-dimensional flow past a stretching sheet and the Homotopy perturbation method. Comput. Math. Appl. **54**, 920-925.

[6] El-Mistikawy T. M. A. (2009). Comment on the three-dimensional flow past a stretching sheet and the Homotopy perturbation method. Comput. Math. Appl. **57** (3), 404-406.

[7] He J. H. (1999). Homotopy perturbation technique. Comp. Meth. Appl. Mech. Eng. **178**, 257-262.

[8] He J. H. (2003). A coupling method Homotopy perturbation technique and a perturbation technique for nonlinear problems. Int. J. Nonlinear Mech. **35**, 73-79.

[9] Ranjee Tripathi and Htadyesh Kumar Mishra (2016). Homotopy perturbation method with Laplace Transform (LT-HPM) for solving Lane-Emden type differential equations (LET-DEs). SpringerPlus. **5**, 1859.

[10] Chandrasekhar S. (1967). An Introduction to the study of Stellar Structure. Dover, New York.

[11] Momoniat E. and Harley C. (2006). Approximate implicit solution of a Lane-Emden equation. New Aston. 520-526.

[12] Yin F., Song J., Lu F., Leng H. (2012). A coupled method of Laplace transform and Legendre wavelets for Lane-Emden type differential equations. J. Appl. Math. Article ID 163821. 16 pages.

[13] He J. H. (1999). Homotopy Perturbation technique. Comput. Methods Appl. Mech. Engrg. **178**, 257-262.

[14] He J. H. (2000). A coupling method of a homotopy technique and a perturbation technique for non-linear problems. Internat. J. Non-Linear Mech. **35**, 37-43.

[15] Biazar J., Badpeima F. and Azimi F. (2009). Application of the homotopy perturbation method to Zakharov-Kuznetove equations. Computer and Mathematics with Applications. **58** (11-12), 2391-2394.

[16] Mishra H. K. and Nagar A. K. (2012). He-Laplace method for linear and nonlinear partial differential equations. J. Appl. Math. 1-16.

[17] Mishra H. K. (2014). He-Laplace Method for the solution of two-point boundary value problems. Am. J. Math. Anal. **2** (3), 45-49.