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Numerical Approximation of Fractional Relaxation-Oscillation Equation by Aboodh Transform Method

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Abstract

In this paper, we present the approximate solutions of fractional relaxation-oscillation equations by using Aboodh Transform Method (ATM). Some examples are considered to illustrate the capability and reliability of the method. The solutions obtained are presented in the form of rapidly convergent series with easily computable terms. The results obtained by ATM are compared with the exact solutions, the solutions obtained by Iterative Decomposition Method (IDM), Optimal Homotopy Asymptotic Method (OHAM) and Generalized Taylor Matrix Method (GTM). The result revealed that the solutions obtained by ATM which is found to be exactly the same as the solution obtained by IDM are in good agreement with the known exact solutions and solution obtained by OHAM, GTM. The applicability, reliability and effectiveness of the proposed method are tested on three examples and results show that the proposed method is more effective and convenient to use and its high accuracy is evident.

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1. Introduction

Fractional differential equations (FDEs) are generalization of differential equations. Over the years FDE has become the focus of curiosity for many researchers in exclusive disciplines of applied science and engineering due to the fact that a realistic modelling of a physical phenomena can be efficiently executed by the way of utilizing fractional differential equations. Fractional differential equations have been applied in diverse fields such as psychology [3], bioengineering [15], viscoelasticity, electrochemistry, Economics [19] among others. Most fractional differential equation do not have exact analytical solutions and thus numerical methods are developed and applied in solving them. Some of those numerical methods are Adomian Decomposition Method [23], Variational iteration method [7, 20], Homotopy perturbation method [21, 12, 13, 14], Differential Transform Method, [4], Optimal Homotopy Analysis Method (OHAM) [11], Iterative Decomposition Method [18] and Generalized Taylor Matrix (GTM) [10] have been developed.
A relaxation oscillator is a class of oscillator which is based on the behavior of a physical system's return to equilibrium after being distributed [17, 8, 9]. The relaxation-oscillation equation is the primary equation of relaxation and oscillation process [25]. The standard relaxation equation in [5] is defined as

$$\frac{du}{dx} + Bu\left(x\right) = f\left(x\right) \tag{1}$$

where B represents $\frac{E}{c}$, E is the elastic modulus, c is the viscosity coefficient and $f\left(x\right)$ represents the product of E and the strain rate. When equation (1) is homogeneous i.e $f\left(x\right) = 0$, we obtain the analytic solution

$$u\left(x\right) = ce^{-Bx} \tag{2}$$

where c is the constant determined by the initial condition. The standard oscillation equation is given by [11] as:

$$\frac{d^2u}{dx^2} + Bu\left(x\right) = f(x) \tag{3}$$

where $B = \frac{k}{m} = w$, k represents the stiffness coefficient, m is the mass, w is the angular frequency. When the equation (3) is homogeneous i.e $f\left(x\right) = 0$, the resulting solution will be

$$u\left(x\right) = Ccos\sqrt{B x} + Dsin\sqrt{B x} \tag{4}$$

where C and D are constants determined by the initial conditions. The fractional derivatives are applied on the relaxation-oscillation models to indicate slow relaxation and damped oscillation [16]. The fractional-oscillation equation model can be expressed as:

$$D^{\propto}u\left(x\right) + Bu\left(x\right) = f\left(x\right),\ x > 0 \tag{5}$$

with initial conditions:

$$u(0) = a, \ if \ 0 < \propto \leq 1 \tag{6}$$

or $u(0) = \gamma$, and

$$u^1(0) = \beta \ if \ 1 < \propto \leq 2 \tag{7}$$

where β is a positive constant. For $0 < \propto \leq 2$ this equation is called the fractional relaxation equation. When $0 < \propto \leq 1$, the model represents relaxation with power law attenuation. when$1 < \propto \leq 2$, the model represents damped oscillation with viscoelastic instrinsic damping of oscillator [24,6]. This class of fractional model has been applied in electrical model of the heart [9], signal processing [24], modelling cardiac pacemakers [6].
In this paper, we propose the Aboodh Transform Method (ATM) to approximate the solution of fractional relaxation-oscillation equations. The method was introduced by [1] and had been applied to the integer order differential equation such as in [22, 2]. Unlike some of the previously mentioned numerical methods, the ATM does not require discretization or linearization. This method is very helpful in solving linear and nonlinear differential problems.
We examine the applicability, accuracy and efficiency of ATM by comparing our results with those by other methods and the exact solutions for some examples of fractional relaxation-oscillation equations.

2. Basic idea of Aboodh Transform Method (ATM)

To illustrate the basic idea of this method, we consider a general form of Fractional Relaxation Oscillation Equation

$$D^{\propto} u(x) + Bu(x) = f(x) \tag{8}$$

$$u(0) = \gamma, \ and \ u^1(0) = \beta \ if \ 1 < \propto \leq 2 \tag{9}$$

Taking Aboodh transform on both sides of (8),

$$A\left[D^{\propto} u(x) + Bu(x)\right] = A\left[f(x)\right] \tag{10}$$

Applying the linearity property of the Aboodh Transform on (10), we have

$$A\left[D^{\propto} u(x)\right] + A\left[Bu(x)\right] = A\left[f(x)\right] \tag{11}$$

Using the differential property of Aboodh Transform

$$v^{\propto} u(v) - \frac{u(0)}{v^{2-\propto}} - \frac{u^1(0)}{v^{2-\propto}} + A\left[Bu(x)\right] = A\left[f(x)\right] \tag{12}$$

Substituting the initial conditions in equation (9) into equation (12), we obtain

$$u(v) = \frac{1}{v^{\propto}} \left\{ \frac{\gamma}{v^{2-\propto}} + \frac{\beta}{v^{2-\propto}} + A\left[f(x)\right] - A\left[Bu(x)\right] \right\} \tag{13}$$

$$u(v) = \frac{1}{v^{\propto}}\left\{\frac{\gamma}{v^{2-\propto}} + \frac{\beta}{v^{2-\propto}} + A\left[f(x)\right]\right\} - \frac{1}{v^{\propto}}\left\{A\left[Bu(x)\right]\right\} \tag{14}$$

Taking Aboodh inverse on both sides of equation (14), we obtain

$$u(x) = A^{-1}\left[\frac{1}{v^{\propto}}\left\{\frac{\gamma}{v^{2-\propto}} + \frac{\beta}{v^{2-\propto}} + A\left[f(x)\right]\right\}\right] - A^{-1}\left\{\frac{1}{v^{\propto}}\left\{A\left[Bu(x)\right]\right\}\right\} \tag{15}$$

$$G(x) = A^{-1}\left[\frac{1}{v^{\propto}}\left\{\frac{\gamma}{v^{2-\propto}} + \frac{\beta}{v^{2-\propto}} + A\left[f(x)\right]\right\}\right]$$

$$u(x) = G(x) - A^{-1}\left\{\frac{1}{v^{\propto}}\left\{A\left[Bu(x)\right]\right\}\right\} \tag{16}$$

where $G(x)$ represent terms arising from the known function $f(x)$ and the prescribed initial conditions.

$$u_0(x) = G(x,t)$$

$$u_1(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}\left\{A\left[Bu_0(x)\right]\right\}\right\}$$

$$u_2(x) = -\ A^{-1}\left\{\frac{1}{v^{\propto}}\left\{A\left[u_1(x)\right]\right\}\right\}$$

$$u_3(x) = -\ A^{-1}\left\{\frac{1}{v^{\propto}}\left\{A\left[u_2(x)\right]\right\}\right\}$$

$$\vdots$$

In general, the recursive relation is given by:

$$u_n(x) = A^{-1}\left\{-\frac{1}{v^{\propto}}\left\{A\left[u_{n-1}(x)\right]\right\}\right\}$$

Then, the solution can be expressed as

$$u_n(x) = u_0(x)(x,t) + u_1(x)(x,t) + u_2(x) + u_3(x)\cdots \tag{17}$$

3. Application

In this section, the Aboodh Transform Method is implemented for solving fractional relaxation-oscillation equation. We illustrate the applicability and the effectiveness of this method with three numerical examples. The results obtained by the proposed method are compared with other known results.

Example 1: Consider the fractional relaxation-oscillation equation in [18] as:

$$D^{\propto}u(x) = -u(x),\ 0 \le x \le 1,\ 1 \le\propto\le 2 \tag{18}$$

with initial conditions

$$u(0) = 1,\ u^1(0) = 0 \tag{19}$$

The exact solution is given as $E_{\propto}(-x^{\propto})$ where $E_{\propto}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\propto k+1)}$ is the Mittag-Leffler function of order α. Taking Aboodh Transform on both sides of equation (18), we have

$$A[D^{\propto}u(x)] = A[-u(x)] \tag{20}$$

using the differential property of the Aboodh transform, we obtain

$$v^{\propto}u(v) - \frac{u(0)}{v^{2-\propto}} - \frac{u^1(0)}{v^{2-\propto}} = A[-u(x)] \tag{21}$$

Substituting the initial conditions (19) into equation (21), we have

$$v^{\propto}u(v) - \frac{1}{v^{2-\propto}} - \frac{0}{v^{3-\propto}} = A[-u(x)]$$

$$u(v) = \frac{1}{v^{\propto}}\left\{\frac{1}{v^{2-\propto}} - A[u(x)]\right\} \tag{22}$$

Now taking the Aboodh inverse on both sides of equation (22), we obtain

$$u(x) = A^{-1}\left\{\frac{1}{v^{\propto}}\left\{\frac{1}{v^{2-\propto}} - A[u(x)]\right\}\right\}$$

$$u(x) = 1 - A^{-1}\left\{\frac{1}{v^{\propto}}A[u(x)]\right\} \tag{23}$$

$$u_0(x) = 1$$

$$u_1(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A[u_0(x)]\right\}$$

$$= -A^{-1}\left\{\frac{1}{v^{\propto}}A[1]\right\}$$

$$u_1(x) = -\frac{x^{\propto}}{\Gamma(\propto+1)}$$

$$u_2(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A[u_1(x)]\right\}$$

$$u_2(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A\left[-\frac{x^{\propto}}{\Gamma(\propto+1)}\right]\right\}$$

$$u_2(x,t) = \frac{x^{2\propto}}{\Gamma(2\propto+1)}$$

$$u_3(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A\left[u_2(x)\right]\right\}$$

$$u_3(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A\left[\frac{x^{2\propto}}{\Gamma(2\propto+1)}\right]\right\}$$

$$u_3(x) = -\frac{x^{3\propto}}{\Gamma(3\propto+1)}$$

$$u_4(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A\left[u_3(x)\right]\right\}$$

$$u_4(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A\left[-\frac{x^{3\propto}}{\Gamma(3\propto+1)}\right]\right\}$$

$$u_4(x) = \frac{x^{4\propto}}{\Gamma(4\propto+1)}$$

$$u_5(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A\left[u_4(x)\right]\right\}$$

$$u_5(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A\left[\frac{x^{4\propto}}{\Gamma(4\propto+1)}\right]\right\}$$

$$u_5(x) = -\frac{x^{5\propto}}{\Gamma(5\propto+1)}$$

$$\vdots$$

$$u_n(x) = A^{-1}\left\{-\frac{1}{v^{\propto}}A\left[u_{n-1}(x)\right]\right\}$$

The rest of the components of iteration formula can be obtained by following the same procedure. Then, the solution $u(x)$ is expressed as:

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \cdots$$

$$\left\{ \begin{array}{l} u(x) = 1 - \frac{x^{\propto}}{\Gamma(\propto+1)} + \frac{x^{2\propto}}{\Gamma(2\propto+1)} - \frac{x^{3\propto}}{\Gamma(3\propto+1)} + \\ \frac{x^{4\propto}}{\Gamma(4\propto+1)} - \frac{x^{5\propto}}{\Gamma(5\propto+1)} + \cdots \end{array} \right. \tag{24}$$

Equation (24) which is the exact solution of (18) is the same as the solution obtained in [18]. When $\propto = 1$, we obtain an approximate solution

$$u(x) = 1 - \frac{x}{\Gamma(2)} + \frac{x^2}{\Gamma(3)} - \frac{x^3}{\Gamma(4)} + \frac{x^4}{\Gamma(5)} - \frac{x^5}{\Gamma(6)} + \cdots \tag{25}$$

Thus, (25) can be written in the closed form as:

$$u(x) = E_{1,\propto}(-x^{\propto}) \tag{26}$$

when $\propto = \frac{3}{2}$, then (24) becomes

$$u(x) = 1 - \frac{x^{3/2}}{\Gamma(5/2)} + \frac{x^5}{\Gamma(4)} - \frac{x^{9/2}}{\Gamma(11/2)} + \frac{x^6}{\Gamma(7)} - \frac{x^{15/2}}{\Gamma(17/2)} + \cdots \tag{27}$$

Table 1: Numerical comparison between ATM approximate solution, exact solution, IDM solution, OHAM solution and GTM approximate solution for Example 1.

x	Exact Solution	Solution by ATM	Error ATM	Solution IDM [18]	Error IDM	Solution OHAM [11]	Error OHAM	Solution GTM [10]	Error GTM
0.0	1.000000	1.000000	0.000000	1.000000	0.000000	1.000000	0.000000	1.000000	0.000000
0.1	0.976378	0.976377	9.8E-7	0.976377	9.8E-7	0.976388	1.032E-5	0.9763783	6.031E-7
0.2	0.934036	0.934034	1.829E-6	0.934034	1.829E-6	0.934057	2.096E-5	0.9340497	1.358E-5
0.3	0.880808	0.8808047	3.258E-6	0.8808047	3.258E-6	0.880831	2.292E-5	0.8808922	8.395E-5
0.4	0.820056	0.8200506	5.40E-6	0.8200506	5.40E-6	0.820071	1.48E-5	0.82036	3.037E-4
0.5	0.754049	0.7540407	8.31E-6	0.7540407	8.31E-6	0.754049	5.838E-7	0.7548718	8.23E-4
0.6	0.68453	0.6845192	1.08E-6	0.6845192	1.08E-6	0.684517	1.289E-5	0.6863845	1.854E-3
0.7	0.612922	0.6129079	1.41E-5	0.6129079	1.41E-5	0.612903	1.483E-5	0.6166007	3.679E-3
0.8	0.540417	0.5404134	3.60E-6	0.5404134	3.60E-6	0.540404	1.249E-5	0.547065	6.648E-3
0.9	0.468031	0.468040	9.0E-6	0.468040	9.0E-6	0.468031	4.365E-7	0.4792153	1.118E-2
1.0	0.396629	0.3966032	2.58E-5	0.3966032	2.58E-5	0.396632	2.577E-6	0.414413	1.77E-2

Example 2: Consider the fractional relaxation-oscillation equation in [18] as:

$$D^{\propto} u(x) = -u(x), \; 0 \leq x \leq 1, \; 0 \leq \propto \leq 1 \tag{28}$$

with initial conditions

$$u(0) = 1 \tag{29}$$

Taking Aboodh Transform on both sides of equation (28), we have

$$A[D^{\propto} u(x)] = A[-u(x)] \tag{30}$$

using the differential property of the Aboodh transform, we obtain

$$v^{\propto} u(v) - \frac{u(0)}{v^{2-\propto}} = -A[u(x)] \tag{31}$$

Substituting the initial conditions (29) into (31) we have

$$v^{\propto} u(v) - \frac{1}{v^{2-\propto}} = -A[u(x)]$$

$$u(v) = \frac{1}{v^{\propto}} \left\{ \frac{1}{v^{2-\propto}} - A[u(x)] \right\} \tag{32}$$

Now taking the Aboodh inverse on both sides of (32), we obtain

$$u(x) = A^{-1} \left\{ \frac{1}{v^{\propto}} \left\{ \frac{1}{v^{2-\propto}} - A[u(x)] \right\} \right\}$$

(33) $u(x) = 1 - A^{-1}\left\{\frac{1}{v^{\propto}}A[u(x)]\right\}$

$$u_0(x) = 1$$

$$u_1(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A[u_0(x)]\right\}$$

$$= -A^{-1}\left\{\frac{1}{v^{\propto}}A[1]\right\}$$

$$u_1(x) = -\frac{x^{\propto}}{\Gamma(\propto +1)}$$

$$u_2(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A[u_1(x)]\right\}$$

$$u_2(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A\left[-\frac{x^{\propto}}{\Gamma(\propto +1)}\right]\right\}$$

$$u_2(x,t) = \frac{x^{2\propto}}{\Gamma(2\propto +1)}$$

$$u_3(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A[u_2(x)]\right\}$$

$$u_3(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A\left[\frac{x^{2\propto}}{\Gamma(2\propto +1)}\right]\right\}$$

$$u_3(x) = -\frac{x^{3\propto}}{\Gamma(3\propto +1)}$$

$$u_4(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A[u_3(x)]\right\}$$

$$u_4(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A\left[-\frac{x^{3\propto}}{\Gamma(3\propto +1)}\right]\right\}$$

$$u_4(x) = \frac{x^{4\propto}}{\Gamma(4\propto +1)}$$

$$u_5(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A[u_4(x)]\right\}$$

$$u_5(x) = -A^{-1}\left\{\frac{1}{v^{\propto}}A\left[\frac{x^{4\propto}}{\Gamma(4\propto +1)}\right]\right\}$$

$$u_5(x) = -\frac{x^{5\propto}}{\Gamma(5\propto +1)}$$

$$\vdots$$

$$u_n(x) = A^{-1}\left\{-\frac{1}{v^{\propto}}A[u_{n-1}(x)]\right\}$$

The rest of the components of iteration formula can be obtained by following the same procedure. Then, the solution $u(x)$ is expressed as:

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \cdots$$

$$\left\{ \begin{array}{l} u(x) = 1 - \frac{x^{\propto}}{\Gamma(\propto+1)} + \frac{x^{2\propto}}{\Gamma(2\propto+1)} - \frac{x^{3\propto}}{\Gamma(3\propto+1)} + \\ \frac{x^{4\propto}}{\Gamma(4\propto+1)} - \frac{x^{5\propto}}{\Gamma(5\propto+1)} + \cdots \end{array} \right. \tag{34}$$

Equation (34) which is the exact solution of (28) is the same as the solution obtained in [18]. (34) can also be written as:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n\propto}}{\Gamma(n\alpha+1)} \tag{35}$$

when $\propto = \frac{1}{2}$, then (34) becomes

$$u(x) = 1 - \frac{x^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} + x - \frac{x^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} + \frac{x^2}{\Gamma(3)} - \frac{x^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)} + \frac{x^3}{\Gamma(4)} \tag{36}$$

Table 2: Numerical comparison between ATM approximate solution, exact solution, IDM approximate solution, OHAM approximate solution and GTM approximate solutions in Example 2 for $\alpha = \frac{1}{2}$.

x	Exact	ATM	Error ATM	IDM [18]	Error IDM	OHAM [11]	Error OHAM	GTM [10]	Error GTM
0.0	1.000000	1.000000	0.00000	1.000000	0.00000	1.000000	0.00000	1.000000	0.0000
0.1	0.723578	0.723566	1.20E-5	0.723566	1.20E-5	0.723428	1.508E-4	0.7236019	2.355E-5
0.2	0.643788	0.643771	1.7E-5	0.643771	1.7E-5	0.643727	6.078E-5	0.6440406	2.523E-4
0.3	0.592018	0.592064	4.6E-5	0.592064	4.6E-5	0.592093	7.485E-5	0.5930206	1.002E-3
0.4	0.553606	0.553684	7.8E-5	0.553684	7.8E-5	0.553738	1.313E-4	0.5562613	2.655E-3
0.5	0.523157	0.523182	2.5E-5	0.523182	2.5E-5	0.523257	1.00E-4	0.5287949	5.638E-3
0.6	0.498025	0.497935	9.0E-5	0.497935	9.0E-5	0.498038	1.357E-5	0.508438	1.041E-2
0.7	0.476703	0.476533	1.7E-4	0.476533	1.7E-4	0.476623	7.979E-5	0.4941725	1.746E-2
0.8	0.458246	0.454982	3.26E-3	0.454982	3.26E-3	0.45812	1.256E-4	0.4855662	2.732E-2
0.9	0.442021	0.441421	6.0E-4	0.441421	6.0E-4	0.441953	6.811E-4	0.4825183	4.049E-2
1.0	0.427584	0.420460	7.124E-3	0.420460	7.124E-3	0.427731	1.478E-4	0.4851336	5.755E-2

Example 3: Consider the fractional relaxation-oscillation equation in [11] as:

$$D^{\propto} u(x) = u(x) + 1, \quad x > 0, \quad 0 \leq \propto \leq 1 \tag{37}$$

with initial conditions

$$u(0) = 0 \tag{38}$$

Taking Aboodh Transform on both sides of (37), we have

$$A[D^{\propto} u(x)] = A[u(x) + 1] \tag{39}$$

using the differential property of the Aboodh transform, we obtain

$$v^{\propto}u(v) - \frac{u(0)}{v^{2-\propto}} = A[u(x)] + A[1] \tag{40}$$

Substituting the initial conditions (38) into (40), we have

$$v^{\propto}u(v) - \frac{0}{v^{2-\propto}} = A[u(x)] + \frac{1}{v^2}$$

$$u(v) = \frac{1}{v^{\propto}}\left\{\frac{1}{v^2} + A[u(x)]\right\} \tag{41}$$

Now taking the Aboodh inverse on both sides of (41), we obtain

$$u(x) = A^{-1}\left\{\frac{1}{v^{\propto}}\left\{\frac{1}{v^2} + A[u(x)]\right\}\right\}$$

$$u(x) = A^{-1}\left\{\frac{1}{v^{\propto+2}}\right\} + A^{-1}\left\{\frac{1}{v^{\propto}}A[u(x)]\right\} \tag{42}$$

$$u_0(x) = \frac{x^{\propto}}{\Gamma(\propto+1)}$$

$$u_1(x) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_0(x)]\right\}$$

$$= A^{-1}\left\{\frac{1}{v^{\propto}}A\left[\frac{x^{\propto}}{\Gamma(\propto+1)}\right]\right\}$$

$$u_1(x) = \frac{x^{2\propto}}{\Gamma(2\propto+1)}$$

$$u_2(x) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_1(x)]\right\}$$

$$u_2(x) = A^{-1}\left\{\frac{1}{v^{\propto}}A\left[\frac{x^{2\propto}}{\Gamma(2\propto+1)}\right]\right\}$$

$$u_2(x,t) = \frac{x^{3\propto}}{\Gamma(3\propto+1)}$$

$$u_3(x) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_2(x)]\right\}$$

$$u_3(x) = A^{-1}\left\{\frac{1}{v^{\propto}}A\left[\frac{x^{3\propto}}{\Gamma(3\propto+1)}\right]\right\}$$

$$u_3(x) = \frac{x^{4\propto}}{\Gamma(4\propto+1)}$$

$$u_4(x) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_3(x)]\right\}$$

$$u_4(x) = A^{-1}\left\{\frac{1}{v^{\propto}}A\left[\frac{x^{4\propto}}{\Gamma(4\propto+1)}\right]\right\}$$

$$u_4(x) = \frac{x^{5\propto}}{\Gamma(5\propto+1)}$$

$$u_5(x) = A^{-1}\left\{\frac{1}{v^{\propto}}A\left[u_4(x)\right]\right\}$$

$$u_5(x) = A^{-1}\left\{\frac{1}{v^{\propto}}A\left[\frac{x^{5\propto}}{\Gamma(5\propto+1)}\right]\right\}$$

$$u_5(x) = \frac{x^{6\propto}}{\Gamma(6\propto+1)}$$

$$\vdots$$

$$u_n(x) = A^{-1}\left\{\frac{1}{v^{\propto}}A\left[u_{n-1}(x)\right]\right\}$$

The rest of the components of iteration formula can be obtained by following the same procedure. Then, the solution $u(x)$ is expressed as:

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \cdots$$

$$\left\{\begin{array}{l} u(x) = \frac{x^{\propto}}{\Gamma(\propto+1)} + \frac{x^{2\propto}}{\Gamma(2\propto+1)} + \frac{x^{3\propto}}{\Gamma(3\propto+1)} + \\ \frac{x^{4\propto}}{\Gamma(4\propto+1)} + \frac{x^{5\propto}}{\Gamma(5\propto+1)} + \cdots \end{array}\right. \quad (43)$$

Equation (43) which is the exact solution of (37) is the same as the solution obtained in [11]. When $\propto = \frac{1}{2}$, then (43) becomes

$$u(x) = \frac{x^{1/2}}{\Gamma(3/2)} + x + \frac{x^{3/2}}{\Gamma(5/2)} + \frac{x^2}{\Gamma(3)} + \frac{x^{5/2}}{\Gamma(7/2)} + \cdots \quad (44)$$

when $\propto = \frac{3}{4}$, then (44) becomes

$$u(x) = \frac{x^{3/4}}{\Gamma(7/4)} + \frac{x^{3/2}}{\Gamma(5/2)} + \frac{x^{9/4}}{\Gamma(13/4)} + \frac{x^3}{\Gamma(4)} + \frac{x^{15/4}}{\Gamma(19/4)} + \cdots \quad (45)$$

Table 3: Numerical comparison between the ATM Approximation, the Exact Solution and OHAM approximate solution when $\propto = \frac{1}{2}$ for equation (37).

x []	Exact Solution	ATM Approximation	Error of ATM	OHAM Approximation[11]	Error of OHAM
0.0	0.0000000	0.00000000	0.00000	0.0000000	0.0000
0.1	0.4867634	0.48677499	1.159E-5	0.4863604	4.0296E-4
0.2	0.7990172	0.79903478	1.758E-5	0.7996824	6.6517E-4
0.3	1.1076992	1.10779114	9.194E-5	1.1080854	3.8621E-4
0.4	1.4300431	1.43006980	2.67E-5	1.4295176	5.2551E-4
0.5	1.7742859	1.774312181	2.628E-5	1.7730313	1.2546E-4
0.6	2.1462130	2.146226287	1.329E-5	2.1449059	1.3071E-3
0.7	2.5508027	2.550773545	2.9155E-5	2.5501704	6.321E-4
0.8	2.9928358	2.992703315	1.325E-4	2.9931588	3.2291E-4
0.9	3.4771848	3.476833805	3.509E-4	3.4777623	5.7752E-4
1.0	4.0089800	4.008208376	7.716E-4	4.0075620	1.4180E-3

Table 4: Numerical comparison between the ATM Approximation, the Exact Solution and OHAM approximate solution when $\propto = \frac{3}{4}$ for equation (37).

x	Exact Solution	ATM approximation	Error of ATM	OHAM Approximation [11]	Error of OHAM
0.0	0.0000000	0.0000000000	0.000000	0.0000000	0.00000
0.1	0.2196607	0.2196534915	7.2085E-6	0.2195919	6.8743E-5
0.2	0.4046766	0.404666352	1.1248E-5	0.4046341	4.2490E-5
0.3	0.5960496	0.5960354041	1.4196E-6	0.5960616	2.0633E-6
0.4	0.8004557	0.8004391156	1.6544E-5	0.8004642	8.6318E-6
0.5	1.0217199	1.021701767	1.8133E-5	1.0216937	2.6338E-5
0.6	1.2629144	1.262893332	2.1068E-5	1.26281424	7.2067E-5
0.7	1.5269227	1.526902045	2.066E-5	1.5268333	8.9360E-5
0.8	1.8166648	1.816643316	2.148E-5	1.8166053	5.9557E-5
0.9	2.1352133	2.135158828	5.447E-5	2.1351973	1.5736E-5
1.0	2.4858662	2.485843317	2.288E-5	2.4857900	7.6170E-5

Conclusion

In this paper, Aboodh Transform Method (ATM) was used to obtain the approximate solution of fractional relaxation-oscillation equations. The results obtained are very close to the exact solutions which is exactly the same as the IDM approximate solution. However, the result of the ATM approximate solution are compared with other previously applied methods such as OHAM and GTM . The result obtained validate the efficiency and accuracy of the proposed method for solving fractional differential equations. The method is easy to apply and accuracy can be improved by increasing the number of approximating series.

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