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Common Fixed Point Theorems for Five Self Mappings in a Complete Metric Space

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Abstract

In this research work, we use five different operators P, Q, R, T, and B, to show that there exists a common fixed point among the five operators and prove the uniqueness. Furthermore, we establish that the pairs $(R, B), (R, Q), (R, T)$ and (R, P) are weakly compatible (that is they commute). The result extend and improve the work of Latpate and Dolhare (2017).

1. Introduction

Banach (1922) in his Ph.D thesis on fixed point theory established the existence of solution of an integral equation with the following condition:
$d(Tx, Ty) \leq \alpha d(x, y)$, where $x, y \in X$, T is a self mapping of X and $\alpha \in [0, 1)$. He also derived a well known theorem for a contractive mapping in a complete metric space which stated that contraction mapping has a unique fixed point. Banach fixed point theory play an important role in science and engineering, especially in the area of mathematical analysis. For instance it has been used to show the existence of solution of nonlinear Volterra integral equations and nonlinear integro-differential equations in Banach spaces.

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Weak contraction mappings are generalization of Banach contraction mappings which has been studied by several researcher and authors. Albert and Guerre-Delabriere (1997) introduced the concept of weak contraction in inner product space popularly known as the Pre-Hilbert Space as shown below:
$d(Tx, Ty) \leq d(x, y) - \psi d(x, y)$, for $x, y \in X$ where $\psi : [0, \infty] \to [0, \infty]$ is continuous and non-decreasing function with $\psi(t) = 0$ iff $t = 0$.
If $\psi = \alpha t$ where $t = d(x, y)$ and $0 < \alpha < 1$, then weak contraction reduce to Banach contraction mapping. Rhoades (2001) showed that the result of Albert and Guerre-Delabriere (1997) had proved and also valid in complete metric space. The control functions alter the distance between two metric spaces, which were first introduced by Khan *et al.* (1984). This paper is therefore focused on common fixed point theorems for five self mappings in a complete metric space.
Kanan (1968) investigated uniqueness of a fixed point in a complete metric space. Another positive contribution was done by Sesa (1986) on weakly compatible mappings (or weakly commuting mappings) and obtained some common fixed point theorems in a complete metric space. Rajesh *et al.* (2013) investigated that weakly commuting mappings are compatible, however, the converse may not be necessarily true. Rhoades (1977) compared two hundred and fifty (250) types of contractive definition and analyzed the relationship among them. Khan et al. (2005) proved some result on common fixed points and best approximations. Azam *et al.*, (2008) obtained some common fixed point of two maps in cone metric space. Jungck (1986) worked on compatible mappings and proved some common fixed point theorems in a complete metric space. He also proved that weak commuting mappings are compatible. Berinde (2009) proved common fixed point theorem for compatible quasi contractive self mappings in a metric spaces. Jungck *et al.*, (2009) proved some common fixed point theorems for weakly compatible pairs on some metric space. Rangamma and Prudvi (2012) proved the existence of coincidence points and a common fixed point theorem for four mappings in a cone metric space.
Albert and Guerre-Delabriere (1997) introduced the concept of weakly contractive maps. They confined their theorem in Hilbert space, but acknowledged that their result are true, at least for uniformly smooth and uniformly convex Banach spaces. Rhoades (2001) proved some theorems on weakly contractive maps. Berinde (2003) introduced the approximating fixed point of weak ψ-contraction in fixed points theory. Alata (2016) proved the generalization of common fixed point for self mapping in complete cone metric space . According to Latpate and Dolhare (2017) the proved of common fixed point theorem of three mappings in a complete metric space were established.
Usamot (2017) proved semi-group of order preserving and order decreasing full contraction mappings and their idempotents in metric spaces. Sedghi and Shobe

(2007) proved common fixed point theorems for four mappings in a complete metric space. Emmanuele (1980) proved fixed point theorem in a complete metric space. Dolhare (2016) proved Generalized contraction mapping and fixed point theorems in a complete metric space. Sangay and Garg (2009) proved the expansion mapping theorems in a metric space. Ritu (2016) proved common fixed point theorem for weakly compatible mappings in two metric spaces. Kavita and Manjusha (2013) proved unique common fixed point theorem for three pairs of weakly compatible mappings satisfying generalized contractive condition of integral type.

Lemma 1: Let $T, g : [a, b] \to \mathbb{R}$ be two continuous functions such that $Ta < ga$ and $Tb > gb$. Then T and g have, at lease, a coincidence point $c \in (a, b)$.
Remark: If T and g are commuting and x is a coincidence point of T and g, then $y = Tx$ is also coincidences point of T and g. It follows from $Ty = Tgx = gTx = gy$.

Proposition: Let $\phi : [0, \infty) \to [0, \infty)$, be a non-decreasing function and $\{a_n\} \subset [0, \infty)$ is a sequence such that $\phi(a_{n+1}) < \phi(a_n)$ for all $n \in \mathbb{N}$, then $a_{n+1} < a_n$ for all $n \in \mathbb{N}$. In particular, $\{a_n\}$ is convergent and $L < a_n$ for all $n \in \mathbb{N}$ (where L is the limit of $\{a_n\}$).
Proof: The Proof can be sourced from (Ravi *et al.*, 2015).

Lemma 2: Let $\psi, \phi : [0, \infty) \to [0, \infty)$ be two functions such that ψ is non-decreasing and $\phi^{-1}([0]) = [0]$ and let $t, s, r \in [0, \infty)$.

(1) If $\psi(t) \leq \psi(s) - \phi(r)$, when $t < s$ or $r = 0$; and

(2) If ψ also verifies $\psi^{-1}(\{0\}) = \{0\}$ and $\psi(t) \leq (\psi - \phi)(s)$, then $t < s = 0$. In any case, $t \leq s$.

Proof: (i) Assume that $t \geq s$ and we have to prove that $r = 0$ indeed, as ψ is nondecreasing, $\psi(s) \leq \psi(t)$.
Therefore,
$\psi(t) \leq \psi(s) - \phi(r) \leq \psi(s) \leq \psi(t)$
As a consequence, $\psi(t) = \psi(s)$ and $\phi(r) = 0$ for $r = 0$; and
(ii) Next, assume that $\psi(t) \leq (\psi - \phi)(s)$ and $t \geq s$. By item 1 above, $s = 0$.
Hence, $0 \leq \psi(t) \leq \psi(0) - \phi(0) = 0$, so $\psi(t) = 0$ and $t = 0$.
F_{alt}, denote the family of all altering distance function.

Lemma 3: From properties of ϕ function, we have that
$0 \leq \phi(s_n) \leq \psi(s_n) - \psi(t_n)$ for all $n \in \mathbb{N}$.
As ψ is continuous, then $lim_{n\to\infty}\psi(t_n) = lim_{n\to\infty}\psi(s_n) = \psi(L)$.
Therefore, $[\phi(s_n)] \to 0$ since ϕ is lower semi-continuous and $[\phi(s_n)]$ is convergent, $0 \leq \phi(t) \leq lim_{n\to\infty}inf.(s_n) = lim_{n\to\infty}\phi(s_n) = 0$. Hence $\phi(L) = 0$, which implies

that $L = 0$.

Corollary: By hypotheses (Lemma 1), also assume that $F(X^n) \subseteq g(X)$ with the following conditions:
$[U]$ For all ϕ-coincidence points $(x_1, x_2, ..., x_n), (y_1, y_2, ...y_n)$ of F and g, there exist $(\omega_1, \omega_2, ..., \omega_n) \in X^n$ such that $gx_1 \leq_i g\omega_i$ and $gy_1 \leq_i g\omega_i$ for all $i \in [1, 2, ...n]$.
$[U']$ g is injective on the set of all ϕ-coincidence point of F and g, Then F and g have unique ϕ-coincidence point.
Proof: Let $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ...y_n)$ be two arbitrary ϕ-coincidence points of F and g. By hypotheses, $gx_i = gyi$ for all $i \in [1, 2, ..., n]$ and g is injective on the set of all ϕ-coincidence points of F and g, we conclude that $x_i = y_i$ for all $i \in [1, 2, ..., n]$.

Definition 3: Let X be a set and let $F : X \to X$ be a function that maps X into itself. (such a function is often called an operator, a transformation or transform on X, and the notation $T(x)$ or Tx is often used).

Definition 4: A fixed point of F is an element $x \in X$ for which $F(x) = x$.

Example 1: Let $\{a, b\}$ be two elements of the set X . The function $F : X \to X$ defined by $F(a) = b$ and $F(b) = a$ has no fixed point, but the other three functions that maps X into itself each have one or two fixed points. More generally, let X be an arbitrary set; every constant function $F : X \to X$ mapping X into itself has a unique fixed point; and for the identity function $F(x) = x$, every point in x is a fixed point.

Example 2 (Dolhare (2016)): If F is defined on the real number by
$F(X) = X^2 - 7X + 12$.
We know that $X = [3, 4]$ are roots of the equation. Let us consider
$F(X) = X$ where $F(X) = \frac{X^2+12}{7}$, then $X = 3$ and $X = 4$ are two fixed point of $F(X)$.
Remark: Note that the definition of a fixed point requires no structure on either the set X of the function F.

Definition 5 (Dolhare (2016)): Let (X, d) be a metric space, a point $x \in X$ is said to be fixed point if $T(x) = x$, whenever $T : X \to X$. In other word, Let $F : X \to X$, then $x \in X$ is said to be fixed point of F if $F(x) = x$.

Definition 6 (Rangamma and Prudvi (2012)): Let A and S be mappings from a metric space (X, d) into itself. A and S are said to be weakly compatible if they commute at their coincidence points, that is $Ax = Sx$ for some $x \in X$

implies that $ASx = SAx$.

Definition 7: A metric space (X, d) is said to be complete metric space if every Cauchy sequence in X converges to a point of X.

Definition 8: Let X be a metric space and if F_1 and F_2 be any two maps. An element $a \in X$ is said to be a common fixed point of F_1 and F_2 if $F_1(a) = F_2(a)$.

Example 3: If $F_1(x) = sin(x)$ and $F_2(x) = tan(x)$. Then, 0 is the common fixed point of F_1 and F_2, since $F_1(0) = sin(0) = 0$ and $F_2(0) = tan(0) = 0$.

Definition 9: Let (X, d) be a complete metric space and a function $F : X \to X$ is said to be contraction map if $d(F(x), F(y)) \leq \beta d(x, y)$ where $0 < \beta < 1$.

Definition 10: A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+ = [0, \infty)$ is an altering distance function if the following properties are satisfied:

- i $\psi(t) = 0$ iff $t = 0$;
- ii ψ is monotonically nondecreasing; and
- iii ψ is continuous.

where ψ is the set of all altering distance functions (Khan et. al., 1984).

Definition 11 (Ravi *et al.* (2015)): A function $\psi : [0, \infty) \to [0, \infty)$ is called Altering distance function if the following properties are satisfied:

- i description $\psi(0) = 0$;
- ii ψ is continuous; and
- iii monotonically non decreasing.

Definition 12: A sequence $\{x_n\}$ in a metric space (X, d) is said to be convergent if there is an $x \in X$ such that $lim_{n\to\infty} d(x_n, x) = 0$.

Definition 13: Let $X = (x, d)$ and $Y = (y, d)$ be metric space. A mapping $T : X \to Y$ is said to be continuous at a point $x_0 \in X$ if every $\epsilon > 0$ there is a $\delta > 0$ such that $d(Tx, Tx_0) < \epsilon$ for all x satisfying $d(x, x_0) < \delta$.
T is said to be continuous if it is continuous at every point of X.

Definition 14: Let (X, d) be an arbitrary metric space. A mapping T on X into it self is called a strict contraction (or simply contraction) if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

Intuitively, a contraction map is one that "shrinks" distance by a factor of $k \in [0, 1)$. The mapping $T : (X, d) \to (X, d)$ is called non-expansive if $k = 1$ in above definition, that is T is non-expansive if for all $x, y \in X$

$$d(Tx, Ty) \leq Kd(x, y)$$

Clearly every contraction mapping is non-expansive, but the converse is not true.

Definition 15: Let X be a non-empty set and $T : X \to X$ be a self map, $x \in X$ is a fixed point of T if $Tx = x$ and the set of fixed point of T denoted by $F(T)$ defined as

$$F(T) = \{x \in X : Tx = x\}$$

Theorem 1.1. *(Shukla and Ruchira, 2014)*
Let (X, d) be a complete metric space and $T : X \to X$ be self mapping satisfying the inequality

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)) - \phi'(d(Tx, X)) \tag{1}$$

where $\psi, \phi, \phi' : [0, \infty) \to [0, \infty)$ are all continuous and monotone non decreasing functions with $\psi(t) = 0 = \phi(t) = \phi'(t)$ if and only if $t = 0$. Then, T has unique fixed point.

Theorem 1.2. *(Latpate and Dolhare, 2017)*
Let (X, d) be a complete metric space and A be a non-empty closed subset of X. If $P, Q, : A \to A$ such that

$$d(P_x, Q_y) \leq \frac{1}{2}(d(R_x, Q_y) + d(R_y, P_x) + d(S_x, R_y)) - \psi(d(R_x, Q_y), d(R_y, P_x)) \tag{2}$$

for any $(x, y) \in X \times X$ where a function $\psi : [0, \infty]^2 \to [0, \infty]$ is continuous and $\psi(x, y) = 0$ iff $x = y = 0$ and $R : A \to X$ which satisfies the following conditions:

i $P(A) \subseteq R(A)$, and $Q(A) \subseteq R(A)$;
ii the pair of mappings (P, R) and (Q, R) are weakly compatible; and
iii $R(A)$ is a closed subset of X.

Then P, R, and Q, have unique common fixed point. The present article extend and improve the work of Latpate and Dolhare (2017).

2. Main Result

In this section, five different operators P, Q, R, T and B would be used to prove the our main work. However, two, three and four self mappings results had been proved in the literature. The result is as shown in the theorem below.

Theorem 2.1. *:* Let (X, d) be a complete metric space and A be a non-empty closed subset of X. Suppose $P, Q, R, T, B : A \to A$ such that:

$$\begin{aligned} d(P_x, B_y) \leq \frac{1}{4}(d(R_x, B_y) + d(R_y, T_x) + d(R_x, Q_y) + d(R_y, P_x) + d(S_x, R_y)) \\ - \psi(d(R_x, B_y), d(R_y, T_x), d(R_x, Q_y), d(R_y, P_x)) \end{aligned} \tag{3}$$

for every $(x, y) \in X \times X$ where a function $\psi : [0, \infty)^2 \to [0, \infty)$ is continuous and $\psi(x, y) = 0$ if and only iff $x = y = 0$ and $R : A \to X$ which satisfies the following conditions:

i. $P(A) \subseteq R(A), Q(A) \subseteq R(A), T(A) \subseteq R(A)$ and $B(A) \subseteq R(A)$;
ii. the pair of mappings $(P, R), (Q, R), (T, R)$,and (B, R) are weakly compatible; and
iii. $R(A)$ is a closed subset of X.

Then, P, R, Q, T and B have a unique common fixed point.

Proof: Let x_0 be any arbitrary element of A as $P(A) \subseteq R(A), Q(A) \subseteq R(A), T(A) \subseteq R(A)$ and $B(A) \subseteq R(A)$.

$$\begin{aligned} y_0 &= Px_0 = Rx_1, \\ y_1 &= Qx_1 = Rx_2, \\ y_2 &= Tx_2 = Rx_3, \\ y_3 &= Bx_3 = Rx_4, \cdots \end{aligned}$$

Let $\{x_n\}$ and $\{y_n\}$ be two sequences such that

$$\begin{aligned} y_{2n} &= Px_{2n} = Rx_{2n+1}, \\ y_{2n+1} &= Qx_{2n+1} = Rx_{2n+2}, \\ y_{2n+2} &= Tx_{2n+2} = Rx_{2n+3}, \\ y_{2n+3} &= Bx_{2n+3} = Rx_{2n+4}, \cdots \end{aligned}$$

First, we shall prove that $d(y_n, y_{n+3}) \to 0$ as $n \to \infty$ as it was proved in three mappings, then $d(y_{2n}, y_{2n+3}) \to 0$ as $n \to \infty$ which can also be use in the present work. Let $n = 2k$ by inequality (3) above, we have

$$d(y_n, y_{n+3}) = d(y_{2k}, y_{2k+3}) \tag{4}$$

$$d(y_{2n+3}, y_{2n}) = d(Bx_{4k+3}, Px_{4k}) \tag{5}$$

(6)
$$\begin{aligned} \leq &\frac{1}{4}(d(Rx_{4k}, Bx_{4k+3}) + d(Rx_{4k+1}, Tx_{4k+2}) + \\ &d(Rx_{4k+2}, Qx_{4k+1}) + d(Rx_{4k+3}, Px_{4k}) + d(Sx_{4k}, Rx_{4k+3})) \\ &- \psi(d(Rx_{4k}, Bx_{4k+3}), d(Rx_{4k+1}, Tx_{4k+2}), d(Rx_{4k+2}, Qx_{4k+1}),\ d(Rx_{4k+3}, Px_{4k})) \end{aligned}$$

$$= \frac{1}{4}(d(y_{4k-1}, y_{4k+3}) + d(y_{4k}, y_{4k}) + d(y_{4k}, y_{4k}) + d(y_{4k}, y_{4k}) +$$

(7) $$d(y_{4k}, y_{4k})) - \psi(d(y_{4k-1}, y_{4k+3}), d(y_{4k}, y_{4k}), d(y_{4k}, y_{4k}), d(y_{4k}, y_{4k}))$$

$$\leq \frac{1}{4}(d(y_{4k-1}, y_{4k+3}))$$

(8) $$\frac{1}{4}(d(y_{4k-1}, y_{4k}) + d(y_{4k}, y_{4k+1}) + d(y_{4k+1}, y_{4k+2}) + d(y_{4k+2}, y_{4k+3})$$

by expanding (7) above for $n = 2k+1$ and similarly we can show that $d(y_{2n}, y_{2n+3}) \to 0$ as $n \to \infty$

(9) $$d(y_{2n}, y_{2n+3}) = d(y_{4k+4}, y_{4k+1}) \leq d(y_{4k+4}, y_{4k+3}) \leq d(y_{4k+3}, y_{4k+2}) \leq d(y_{4k+2}, y_{4k+1}) \leq d(y_{4k+1}, y_{4k})$$

Let

$$d(y_{4k+4}, y_{4k+1}) \leq d(y_{2n+3}, y_{2n})$$

then $d(y_{2n+3}, y_{2n}) = d(y_{4k+4}, y_{4k+1})$ is a non-increasing sequence of non-negative real numbers and hence converges. Let $L = \lim_{n\to\infty}(y_{4k+4}, y_{4k+1})$, from (7) we have $d(y_{4k+4}, y_{4k+1}) \leq \frac{1}{4}(d(y_{4k}, y_{4k+4}))$ and by contraction mapping, we get

(10)

$$d(y_{4k+4}, y_{4k+1}) \leq \frac{1}{4}(d(y_{4k-1}, y_{4k+1}) + (d(y_{4k+1}, y_{4k+2}) + (d(y_{4k+2}, y_{4k+3} + (d(y_{4k+3}, y_{4k+4}))$$

Letting $k \to \infty$, we obtain

(11) $$\lim_{k\to\infty} d(y_{4k+2}, y_{4k+3}) \leq \frac{1}{4} \lim_{k\to\infty} d(y_{4k}, y_{4k+1}) \leq \lim_{k\to\infty} d(y_{4k+1}, y_{4k+2}) \leq \lim_{k\to\infty} d(y_{4k+2}, y_{4k+3} \leq \lim_{k\to\infty} d(y_{4k+3}, y_{4k+4})$$

(12) $$L \leq \frac{1}{4} \lim_{k\to\infty} d(y_{4k}, y_{4k+4}) \leq L \leq L \leq L$$

(13) $$\lim_{k\to\infty} d(y_{4k}, y_{4k+4}) = 4L$$

consider

(14) $$d(y_{2n+3}, y_{2n}) = d(Px_{4k+1}, Bx_{4k+4})$$

(15) $$\leq \frac{1}{4}(d(y_{4k-1}, y_{4k}) + d(y_{4k}, y_{4k+1}) + d(y_{4k+1}, y_{4k+2}) + d(y_{4k+2}, y_{4k+3}) + d(y_{4k+3}, y_{4k+4})) - \psi(d(y_{4k-1}, y_{4k}), d(y_{4k}, y_{4k+1}), d(y_{4k+1}, y_{4k+2}), d(y_{4k+2}, y_{4k+3}))$$

Letting $k \to \infty$ and since ψ is given to be continuous, therefore we obtain

$$L \leq \frac{1}{4}(4L) - \psi(4L, 0) \tag{16}$$

this gives

$$\psi(4L, 0) = 0 \tag{17}$$

By definition of $\psi(x, y) = 0$ if $x = y = 0$, then $4L = 0, => L = 0$

$$L = \lim_{k\to\infty}(y_{4k+1}, y_{4k+4}) = 0 \tag{18}$$

Our claim is that $\{y_{4k}\}$ is a Cauchy sequence and from (5) we have

$$d(y_{4k+2}, y_{4k+4}) \leq d(y_{4k+1}, y_{4k+2}) \tag{19}$$

To prove $\{y_{4k+1}\}$ is Cauchy sequence, we only prove that the subsequence $\{y_{4k}\}$ is Cauchy. If otherwise, suppose that $\{y_{4k}\}$ is not Cauchy sequence, then there exists $\delta>0$ for which we can find subsequences $\{y_{4n(k)}\}$ and $\{y_{4m(k)}\}$ of $\{y_{4k}\}$ such that n_k is the least index for which $n_k > m_k > k$ and

$$d(y_{4m(k)},\ y_{4n(k)}) \geq \delta \tag{20}$$

This gives

$$d(y_{4m(k)}, y_{4n(k)-4}) < \delta \tag{21}$$

By contraction mapping, we have

$$\begin{aligned}\delta \leq\, & d(y_{4m(k)}, y_{4n(k)}) \leq d(y_{4m(k)}, y_{m(k)-4}) + d(y_{4m(k)-4}, y_{4m(k)-3}) + \\ & d(y_{4m(k)-3}, y_{4m(k)-2}) + d(y_{4m(k)-2}, y_{4m(k)-1}) + d(y_{4m(k)-1}, y_{4n(k)})\end{aligned} \tag{22}$$

Now as k$\to\infty$ and from (20) we obtain

$$\lim_{k\to\infty} d(y_{4m(k)}, y_{4n(k)}) = \delta \tag{23}$$

by triangle inequality, we have

$$| d(y_{4m(k)}, y_{4n(k)-1}) - d(y_{4m(k)}, y_{4n(k)}) | \leq d(y_{4n(k)-1}, y_{4n(k)}) \tag{24}$$

also

$$| d(y_{4n(k)}, y_{4m(k)-1}) - d(y_{4n(k)}, y_{4m(k)}) | \leq d(y_{4m(k)}, y_{4m(k)-1}) \tag{25}$$

again

$$| d(y_{4n(k)}, y_{4m(k)-1}) - d(y_{4n(k)}, y_{4m(k)-2}) | \leq d(y_{4m(k)-1}, y_{4m(k)-2}) \tag{26}$$

$$| d(y_{4m(k)-1}, y_{4m(k)-2}) - d(y_{4m(k)-1}, y_{4m(k)-3}) | \leq d(y_{4m(k)-2}, y_{4m(k)-3}) \tag{27}$$

$$| d(y_{4m(k)-2}, y_{4m(k)-3}) - d(y_{4m(k)-2}, y_{m(k)-4}) | \leq d(y_{4m(k)-3}, y_{4m(k)-4}) \tag{28}$$

From (20) and (24)-(28) we have

(29)
$$\lim_{k\to\infty} d(y_{4n(k)-1}, y_{4n(k)}) \le \lim_{k\to\infty} d(y_{4m(k)-1}, y_{4m(k)})$$
$$= \lim_{k\to\infty} d(y_{4m(k)-1}, y_{4m(k)-2}) = \lim_{k\to\infty} d(y_{4m(k)-2}, y_{4m(k)-3}) = \lim_{k\to\infty} d(y_{4m(k)-3}, y_{4m(k)-4})$$

inequality (3) gives

(30)
$$d(y_{4m(k)-3}, y_{4n(k)}) = d(Px_{4n(k)}, Bx_{4m(k)-3})$$

(31)
$$\le \frac{1}{4}(d(Rx_{4n(k)}, Bx_{4m(k)-3}) + d(Rx_{4m(k)-1}, Tx_{4n(k)-2})$$
$$+ d(Rx_{4m(k)-2}, Qx_{4n(k)-1}) + d(Rx_{4m(k)-3}, Px_{4n(k)}) + d(Sx_{4n(k)}, Rx_{4m(k)-3}))$$
$$- \psi(d(Rx_{4n(k)}, Bx_{4m(k)-3}), d(Rx_{4m(k)-1}, Tx_{4n(k)-2}),$$
$$d(Rx_{4m(k)-2}, Qx_{4n(k)-1}), d(Rx_{4m(k)-3}, Px_{4n(k)}))$$

(32)
$$\frac{1}{4}(d(y_{4m(k)-1}, y_{4m(k)}) + d(y_{4m(k)}, y_{4m(k)+3}))$$

$$\frac{1}{4}(d(y_{4n(k)-1}, y_{4m(k)-1}) + d(y_{4m(k)-1}, y_{4m(k)}) + d(y_{4m(k)}, y_{4m(k)+1})$$
$$+ d(y_{4m(k)+1}, y_{4m(k)+2}) + d(y_{4m(k)+2}, y_{4m(k)+3}))$$
$$- \psi(d(y_{4n(k)-1}, y_{4m(k)-1}), d(y_{4m(k)-1}, y_{4m(k)}), d(y_{4m(k)}, y_{4m(k)+1}), d(y_{4m(k)+1}, y_{4m(k)+2}))$$

Letting $k \to \infty$ in the above inequality (32) and if ψ is continuous, therefore we have

(33)
$$\delta \le \frac{1}{4}(\delta + \delta + \delta + \delta) - \psi(\delta, \delta, \delta, \delta)$$

this gives $\psi(\delta, \delta, \delta, \delta) = 0$. By assumption of $\psi(x, y) = 0$ if $x = y = 0$, then $\delta = 0$ but $\delta > 0$. This is a contradiction, therefore $\{y_n\}$ is a Cauchy sequence to prove P, Q, R, T and B have a common fixed point. Given that (X, d) is complete and $\{y_n\}$ is a Cauchy sequence. There exists $p \in X$ such that $\lim_{n\to\infty} y_n = p$, hence A is closed. There is u$\in A$ such that:
$p = Ru$ for every $n \in \mathbb{N}$, $d(Pu, y_{2n+3}) = d(Pu, Bx_{2n+3})$
since

(34)
$$d(y_{2n}, y_{2n+3}) = d(Pu, Bx_{2n+3}) = d(y_{4k}, y_{4k+3})$$

(35)
$$\le \frac{1}{4}(d(Ru, Bx_{4k+3}) + d(Rx_{4k+2}, Tu) + d(Rx_{4k+1}, Qu) +$$
$$d(Rx_{4k}, Pu) + d(Su, Rx_{4k})) - \psi(d(Ru, Bx_{4k+3}),$$
$$d(Rx_{4k+2}, Tu), d(Rx_{4k+1}, Qu), d(Rx_{4k}, Pu))$$

$$= \frac{1}{4}(d(P, y_{4k+3}) + d(y_{4k+2}, Tu) + d(y_{4k+1}, Qu) + d(y_{4k}, Pu) + d(Su, y_{4k})) \\ -\psi(d(Ru, Bx_{4k+3}), d(y_{4k+2}, Tu), d(y_{4k+1}, Qu), d(y_{4k}, Pu)) \tag{36}$$

The corresponding Rx_{2n} and y_{2n} in (35) and (36) are obtained from iteration function sequence from the beginning of the prove when k$\to \infty$

$$d(Pu, P) \leq \frac{1}{4}(d(P, Bu) + d(P, Tu) + d(P, Qu) + d(Pu, P) + d(Su, P)) \\ - \psi(d(P, Bu), d(P, Tu), d(P, Qu), d(Pu, P)) \tag{37}$$

and hence

$$\psi(0, d(P, Pu)) \leq -\frac{1}{4}(d(P, Bu) + d(P, Tu) + d(P, Qu) + d(Su, P)) \leq 0 \tag{38}$$

therefore $d(P, Pu) = 0$ since $Pu = P$. Similarly, we can show that $Bu = p, Tu = p, Qu = p$, and $Su = p$, hence

$$Bu = Tu = Qu = Ru = Pu = p \tag{39}$$

the pairs $(R, B), (R, T), (R, Q)$ and (R, P) are weakly compatible, therefore

$$Pp = Bp = Qp = Tp = Rp \tag{40}$$

Now consider

$$d(Pp, y_{4k+3}) = d(Pp, Bx_{4k+2}) \tag{41}$$

$$\leq \frac{1}{4}(d(Rp, Bx_{4k+3}) + d(Rx_{4k+1}, Tx_{4k+2}) + d(Qx_{4k+1}, Rx_{4k+2}) + \\ d(Rx_{4k+3}, Px_{4k}) + d(Sp, Rx_{4k+3})) \\ - \psi(d(Rp, Bx_{4k+3})d(Rx_{4k+1}, Tx_{4k+2})d(Qx_{4k+1}, Rx_{4k+2},) \\ d(Rx_{4k+3}, Px_{4k}), d(Sp, Rx_{4k+3})) \tag{42}$$

(43)

$$= \frac{1}{4}(d(Rp, y_{4k+3}) + d(y_{4k+2}, Tp) + d(Qp, y_{4k+1},) + d(y_{4k}, Pp) + d(Sp, y_{4k})) \\ - \psi(d(Rp, y_{4k+3}), d(y_{4k+2}, Tp), d(Qp, y_{4k+1},), d(y_{4k}, Pp))$$

As $k \to \infty$ and since

$$Pp = Rp = Qp = Tp = Bp \tag{44}$$

we have

$$d(Pp, P) \leq \frac{1}{4}(d(Pp, P) + d(P, Pp) + d(Pp, p) + d(P, Pp) + d(Sp, P)) \\ - \psi(d(Pp, P), d(P, Pp), d(Pp, p), d(P, Pp)) \tag{45}$$

hence

$$-\psi(d(Pp, P), d(P, Pp), d(Pp, p), d(P, Pp)) = 0 \tag{46}$$

Since $d(Pp, P) = 0$ implies $Pp = p$ and from

$$Pp = Qp = Rp = Tp = Bp \tag{47}$$

we have

$$Pp = Qp = Rp = Tp = Bp = p \tag{48}$$

Thus uniqueness of common fixed point is easily obtained from inequality (3).

3. Conclusion

In this article, we used five different operators P, Q, R, T and B to show the existence of a common fixed point and also find the uniqueness. Iterative function sequence, Metric space, Altering distance function, compatible mappings and completeness properties were used to prove the result therein.

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