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# Four Steps Collocation Block Method for Solving Second Order Differential Equations

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Abstract

In this paper, a four-step block Method for numerical solution of second order differential equations using Legendre polynomials as the basic function is developed. Interpolation and collocation procedures are used by choosing interpolation points at $s = 2$ steps points using power series, while collocation points at $r = k$ step points, using a combination of power series and perturbation term gotten from the Legendre polynomials, giving rise to a polynomial of degree $r + s - 2$ and $r + s$ equations. The analysis shows that the derived scheme is stable, convergent and has region of absolute stability. Numerical examples are provided to test the performance of the method. Results obtained shows that the method is accurate and efficient when compared with existing methods in the literature.

## 1. Introduction

Numerous problems in many field of application, notably in physics, chemistry, biology, engineering and social sciences are modeled mathematically by ordinary differential equation (ODEs) e.g. series circuits, mechanical systems with several springs attached in series lead to a system of differential equation, Abualnaja[1].

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Also in diverse fields like economics, medicine, psychology, operation research and even in anthropology are modeled mathematically, Anake [2].
Interestingly, some differential equations arising from the modeling of physical phenomena, often do not have analytic solutions, hence the development of numerical method to obtain approximate solutions become necessary, Ehigie et al. [3]. To that extent, several numerical methods such as one step method, linear multistep methods, hybrid methods and block method have been developed based on the nature and type of the differential equation to be solved. Some researchers attempted the solution of

$$y^{(n)} = f\left(x, y, y', y'', \cdots, y^{(n-1)}\right), y\left(x_0\right) = y_0, y'\left(x\right) = y_1, \cdots, y^{(n-1)} = y_{n-1} \tag{1}$$

using linear multistep methods (LMMs), without reduction to system of first order ODEs, Adeniyi and Adeyefa [4]. Ehigie et al. [3] proposed a generalized 2-step continuous linear multi-step method of hybrid type for the integration of second order ordinary differential equations.
Kayode and Adebeye [5] used Chebyshev polynomials without perturbation terms as the basic function for the development of their methods. The collocation and interpolation equations are generated at both grid and off-grid points for the development of continuous hybrid linear multistep method (CHLMM) for the solution of linear and non linear ODEs.
Ademiluyi [6], Anake [2] and Bolarinwa [7] proposed single step hybrid methods for the direct numerical solution of initial value problem of second order and third orders ordinary differential equations. In these cases, their methods of implementation was in block mode with the proposed methods being efficient, adequate and suitable towards catering for the class of problem of higher order ordinary differential equations for which they were designed. Osilagun et al. [8] used four steps implicit method for the solution of general second order ODEs. Abdulganiy et al. [9] employed a maximal order block trigonometrically fitted scheme for the numerical treatment of second order ODEs with oscillating solution. Peter and Ibrahim [10] used differential transform method in solving a typhoid fever model. However, Authors like Zarina et al. [11] employed block method for generalized multistep Adamas method and backward differentiation formula in solving first-order ODEs. Yahaya and Mohammed [12] used full implicit three points backward differentiation formulae for solving first order initial value problems. Odekunle et al. [13] used a new block integrator for solving initial value problems of first order ODESs. Sunday et al. [14] employed a computational approach to verhulst-pearl model of first order ODEs, and so on.
Some of these methods have their own advantages and disadvantages over the other. For example, One step method have low order of accuracy, time consuming for large scale problems, Awoyemi [15]. Linear Multistep Methods give high

order system of accuracy and are suitable for the direct solution of (1) without necessarily reducing it to an equivalent of first order IVPs of ODEs, Adeniyi and Adeyefa [4]. Block method preserves the traditional advantages of one step methods of being self starting and permitting easy change of step length, Lambert [16]. Also, the method generates simultaneous solution at all grids points.
In the light of this, Abualnaja [1] worked on a block procedure with linear multi-step methods using Legendre polynomials for solving ODEs. They derived a block for some k-step linear multi-step methods (for k = 1,2 and 3) using power series as the interpolation equation and power series with Legendre polynomial as the perturbation term as the collocation equation. Also, Abhulimen and Aigbiremhon [17] did similar work by taking K as 4 and 5. In their work, they considered the first order initial value problem. Furthermore, Abhulimen and Aigbiremhon [22] worked on $2^{nd}$ Order Initial Value Problem of ODEs, by taking $K = 3$.
These different methods have their very desirable qualities. However, in order to create a new line of research and to also improve on some of the existing methods, this paper device a mean for the direct solution of (1) without reduction to first order ODEs. In the next section, the methodology of the work is presented and the derived methods are specified.
The plan of the paper is as follows; section 1 is introduction and section 2 is the derivation of the proposed methods is presented. In section 3, the stability and convergence analysis of the block schemes is given. In section 4, numerical examples are considered. The paper ends with conclusion.

## 2. Derivation of the methods

In this section, we derive discrete methods to solve (1) at a sequence of nodal points $x_n = x_0 + nh$ where h $> 0$ is the step-length or grid size defined by $h = x_{n+1} - x_n$ and $y(x)$ denotes the true solution to (1) while the approximate solution is denoted by the power series:

$$y(x) = c_0 x_n^0 + c_1 x_n^1 + c_2 x_n^2 + \cdots + c_k x_n^k \tag{2}$$

The proposed method depends on the perturbed collocation method with respect to the power series and with the Legendre polynomials as the perturbation term. Interpolation and collocation procedures are used by choosing interpolation point at $s = 2$ grid points and collocation points at $r = k$ step points. We have a polynomial of degree $r + s - 2$ and $(r + s)$ equations.
In the first place, we consider the approximation solution of (1) in the power series:

$$p_i(x) = x^i, i = 0, 1, \cdots, k$$

Hence (2) becomes

$$y_k(x) = c_i p_i(x) = \sum_{i=0}^{k} c_i x^i \tag{3}$$

with the second derivatives as:

$$y_k^{''}(x) = c_i p_i^{''}(x) = \sum_{i=0}^{k} i(i-1) c_i x^{i-2} \tag{4}$$

Combining equation (1) and (4), with the perturbation term, we have

$$\sum_{i=1}^{k} c_i p''(x) = f\left(x,\, y,\, y'\right) + \lambda L_k(x_{n+i}), i = 1,\ (1)\, k \tag{5}$$

where $L_k(x)$ is the Legendre polynomial of degree k, valid in $x_n \le x \le x_{n+k}$ and $\lambda$ is a perturbed parameter. In particular, we shall be dealing with case k = 4 in (3) and (5), where (3) is the interpolation equations and (5) is the collocation equations. The well known Legendre polynomials can be generated using the Rodrigues' formula $p_n(x) = \frac{1}{2^n n!} \ \frac{1}{dx^n}\left[\left(x^2-1\right)^n\right]$, where $L_0(x) = 1, L_1(x) = x, .$ The rest are computed using the recurrence formula. $L_{i+1}(x) = \frac{2i+1}{i+1}\, xL_i(x) - \frac{i}{i+1} L_{i-1}(x), i = 1, 2, \cdots$, giving by

$$\begin{array}{c} L_2(x) = \frac{1}{2}\left(3x^2-1\right) \\ L_3(x) = \frac{1}{2}\left(5x^3-3x\right) \\ L_4(x) = \frac{1}{8}\left(35x^4-30x^2+3\right) \\ L_5(x) = \frac{1}{8}\ \left(63x^5-70x^3+15x\right) \end{array} \tag{6}$$

In order to use these polynomials in the interval $[x_n, x_{n+k}]$, we define the shifted Legendre polynomials by introducing the change of variable.

$$x \ = \ \frac{2\bar{x} - (x_{n+k} + x_n)}{(x_{n+k} - x_n)} \tag{7}$$

Abualnaja [1].

Interpolating (3) at s grid points and collocating (5) at k grid points respectively lead to the following systems of equations; (8) and (9):

$$\sum_{i=0}^{k} c_i p_i(x) = y_{n+s},\ s = 0, 1 \tag{8}$$

and

$$\sum_{i=0}^{k} c_i p_i^{''}(x) = f_{n+j} + \lambda L_k(x_{n+j}), \; j = 1(1)k \tag{9}$$

**Four Step Method, (K=4)**

In this case, we take the Legendre polynomial $L_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$ from (6) and use (7) i.e. $x = \frac{2\bar{x} - (x_{n+k} + x_n)}{x_{n+k} - x_n}$, to obtain the values for

$$L_4(x_{n+1}) = {}^{-37}\!/_{128}, \; L_4(x_{n+2}) = {}^{3}\!/_{8}$$
$$L_4(x_{n+3}) = {}^{-37}\!/_{128}$$

and

$$L_4(x_{n+4}) = 1$$

In addition from equation (4) i.e. $c_0 p_0^{''}(x) = 0, \; c_1 p_1^{''}(x) = 0, \; c_2 p_2^{''}(x) = 2c_2 c_3 p_3^{''}(x) = 6c_3 x_n$and $c_4 p_4^{''}(x) = 12c_4 x_n^2$. Then equation (5) i.e. $\sum_{i=1}^{k} c_i p_1^{''}(x) = f(x, y, \; y') + \lambda L_k(x)$ will reduce to the following form:

$$0 + 0 + 2c_2 + 6c_3 x + 12c_4 x^2 = f\left(x, y, y^{'}\right) + \lambda L_k(x_{n+i}), i = 1(1)4 \tag{10}$$

now collocating equation (10) at $x_{n+i}$, $i = 1, 2, 3, 4$ and interpolating (2) at $x_{n+i}, i = 0, 1$. We obtained the following system of six equations with $c_i$,i $= 0(1)4$ and $\lambda$ which in matrix form is:

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & 0 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & 0 \\ 0 & 1 & 2 & 6x_{n+1}^2 & 12x_{n+1}^3 & {}^{37}\!/_{128} \\ 0 & 1 & 2 & 6x_{n+2}^2 & 12x_{n+2}^3 & -{}^{3}\!/_{8} \\ 0 & 1 & 2 & 6x_{n+3}^2 & 12x_{n+3}^3 & {}^{37}\!/_{128} \\ 0 & 1 & 2 & 6x_{n+4}^2 & 12x_{n+4}^3 & -1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} \tag{11}$$

Equation (11) is solved by Gaussian elimination method to obtain the value of the unknown parameters, ci,$i = 0, \; 1, \; 2, \; 3, \; 4$ and $\lambda$. Which are substituted into (2) yields a continuous implicit four step method in the form described by the formula:

$$y(x) = \alpha_0(x) y_n + \alpha_1(x) y_{n+1} + h^2 \sum_{i=1}^{k} \beta_i(x) f_{n+i} \tag{12}$$

where $k = 4$ using (12), yield the parameter, $\alpha_i, i = 0, 1$ and $\beta_i, \; i \; = 1, 2, 3$and 4 are obtained as continuous functions of $t$.

$$\alpha_0(t) = -2 - t, \qquad \alpha_1(t) = 3 + t$$

$$\beta_1(t) = \frac{h^2}{5040}\left[10518 + 5495t - 222t^2 + 80t^3 + 125t^4\right]$$

$$\beta_2(t) = \frac{h^2}{1680}\left[1662 + 1715t + 222t^2 - 220t^3 - 55t^4\right]$$

$$\beta_3(t) = \frac{h^2}{1680}\left[-402 + 595t + 618t^2 + +80t^3 - 15t^4\right]$$

$$\beta_4(t) = \frac{h^2}{5040}\left[822 + 175t + 222t^2 + 340t^3 + 85t^4\right] \quad (13)$$

where
$t = \frac{x_n - x_{n+2}}{h}$. Evaluating (13) at $t = -1, 0$ and 1 and substituting into (12) gives

(14)

$$\begin{cases} y_{n+2} - 2y_{n+1} + y_n = \frac{307}{2520}h^2 f_{n+4} - \frac{79}{280}h^2 f_{n+3} + \frac{167}{840}h^2 f_{n+2} + \frac{2323}{2520}h^2 f_{n+1} (14a) \\ y_{n+3} - 3y_{n+1} + 2y_n = \frac{137}{840}h^2 f_{n+4} - \frac{67}{280}h^2 f_{n+3} + \frac{277}{280}h^2 f_{n+2} + \frac{1753}{840}h^2 f_{n+1} (14b) \\ y_{n+4} - 4y_{n+1} + 3y_n = \frac{137}{420}h^2 f_{n+4} + \frac{73}{140}h^2 f_{n+3} + \frac{277}{140}h^2 f_{n+2} + \frac{1333}{420}h^2 f_{n+1} (14c) \end{cases}$$

Differentiating (13) at $t = -3, -2, -1, 0$ and 1, and substituting into (12) gives:

(15)

$$\begin{cases} 5040hy'_n - 5040y_{n+1} + 5040y_n = -h^2\left(1157f_{n+4} - 2001f_{n+3} - 1149f_{n+2} + 4513f_{n+1}\right) \\ 5040hy'_{n+1} - 5040y_{n+1} + 5040y_n = h^2\left[647f_{n+4} - 1311f_{n+3} - 159f_{n+2} - 3343f_{n+1}\right] \\ 1680hy'_{n+2} - 1680y_{n+1} + 1680y_n = h^2\left[137f_{n+4} - 341f_{n+3} + 831f_{n+2} + 1893f_{n+1}\right] \\ 144hy'_{n+3} - 144y_{n+1} + 144y_n = h^2\left[5f_{n+4} + 51f_{n+3} + 147f_{n+2} + 157f_{n+1}\right] \\ 5040hy'_{n+4} - 5040y_{n+1} + 5040y_n = h^2\left[1979f_{n+4} + 6033f_{n+3} + 3837f_{n+2} + 5791f_{n+1}\right] \end{cases}$$

**Formation of the block**

Now, we obtain the modified block formulae from (14) and (15) as:

$$\begin{bmatrix} -5040 & 2520 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2520 & 0 & 840 & 0 & 0 & 0 & 0 & 0 \\ -1680 & 0 & 0 & 420 & 0 & 0 & 0 & 0 \\ -5040 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5040 & 0 & 0 & 0 & 5040h & 0 & 0 & 0 \\ -1680 & 0 & 0 & 0 & 0 & 1680h & 0 & 0 \\ 144 & 0 & 0 & 0 & 0 & 0 & 144h & 0 \\ -5040 & 0 & 0 & 0 & 0 & 0 & 0 & 5040h \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y'_{n+1} \\ y'_{n+2} \\ y'_{n+3} \\ y'_{n+4} \end{bmatrix} = \begin{bmatrix} -2520 & 0 \\ -1680 & 0 \\ -1260 & 0 \\ -5040 & -5040h \\ -5040 & 0 \\ -1680 & 0 \\ -144 & 0 \\ -5040 & 0 \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix}$$

$$
(16) \qquad +h^2 \begin{bmatrix} 2423 & 501 & -711 & 307 \\ 1753 & 831 & -201 & 137 \\ 1333 & 831 & 219 & 137 \\ -4513 & 1149 & 2001 & -1157 \\ -3343 & -159 & -131 & 647 \\ 1893 & 831 & -341 & 137 \\ 157 & 147 & 51 & 5 \\ 5791 & 3837 & 6033 & 1979 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix}
$$

Talking the normalized form of (16) we have:

$$
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y'_{n+1} \\ y'_{n+2} \\ y'_{n+3} \\ y'_{n+4} \end{bmatrix} = \begin{bmatrix} 1 & h \\ 1 & 2h \\ 1 & 3h \\ 1 & 4h \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix}
$$

$$
(17) \qquad + \begin{bmatrix} \frac{4513}{5040}h^2 & -\frac{383}{1680}h^2 & -\frac{667}{1680}h^2 & \frac{1157}{5040}h^2 \\ \frac{289}{105}h^2 & -\frac{9}{35}h^2 & -\frac{113}{105}h^2 & -\frac{61}{105}h^2 \\ \frac{2673}{560}h^2 & \frac{171}{560}h^2 & -\frac{801}{560}h^2 & \frac{477}{560}h^2 \\ \frac{304}{45}h^2 & \frac{16}{15}h^2 & -\frac{16}{15}h^2 & \frac{56}{45}h^2 \\ \frac{491}{315}h & -\frac{109}{420}h & -\frac{23}{35}h & \frac{451}{1260}h \\ \frac{91}{45}h & \frac{4}{15}h & -\frac{3}{5}h & \frac{14}{45}h \\ \frac{139}{70}h & \frac{111}{140}h & -\frac{3}{70}h & \frac{37}{140}h \\ \frac{92}{45}h & \frac{8}{15}h & \frac{4}{5}h & \frac{28}{45}h \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix}
$$

which is employed to obtain values for $y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y'_{n+1}, y'_{n+2}, y'_{n+3}$ and $y'_{n+4}$, simultaneously. (17) can be written explicitly as:

(18)

$$
\begin{cases} y_{n+1} = y_n + y'_n + \frac{4513}{5040}h^2 f_{n+1} - \frac{383}{1680}h^2 f_{n+2} - \frac{667}{1680}h^2 f_{n+3} + \frac{1157}{5040}h^2 f_{n+4} \\ y_{n+2} = y_n + 2hy'_n + \frac{289}{105}h^2 f_{n+1} - \frac{9}{35}h^2 f_{n+2} - \frac{113}{105}h^2 f_{n+3} + \frac{61}{105}h^2 f_{n+4} \\ y_{n+3} = y_n + 3hy'_n + \frac{2673}{560}h^2 f_{n+1} + \frac{171}{560} f_{n+2} - \frac{801}{560}h^2 f_{n+3} + \frac{477}{560}h^2 f_{n+4} \\ y_{n+4} = y_n + 4hy'_n + \frac{304}{45}h^2 f_{n+1} + \frac{16}{15} f_{n+2} - \frac{16}{15}h^2 f_{n+3} + \frac{56}{45}h^2 f_{n+4} \\ y'_{n+1} = y'_n + \frac{491}{315}hf_{n+1} - \frac{109}{420}hf_{n+2} - \frac{23}{35}hf_{n+3} + \frac{451}{1260} f_{n+4} \\ y'_{n+2} = y'_n + \frac{91}{45}hf_{n+1} + \frac{4}{15}hf_{n+2} - \frac{3}{5}hf_{n+3} + \frac{14}{45} f_{n+4} \\ y'_{n+3} = y'_n + \frac{139}{70}hf_{n+1} + \frac{111}{140}hf_{n+2} - \frac{3}{70}hf_{n+3} + \frac{37}{140} f_{n+4} \\ y'_{n+4} = y'_n + \frac{92}{45}hf_{n+1} + \frac{8}{15}hf_{n+2} + \frac{4}{5}hf_{n+3} + \frac{28}{45} f_{n+4} \end{cases}
$$

## 3. Analysis of the Method

Basic properties of the block method and their associated main method are analyzed to establish their validity. These properties help to show the nature of convergence of the methods. These properties includes; order and error constant, consistency and zero stability. All these put together reveal the nature of convergence of the method. Also the regions of absolute stability of the methods have also been established in this section. However a brief introduction of these properties are made for a better understanding of the section.

**Order and Error Constant**

**Order of the method**

Let the linear difference operator $L$ associated with the continuous multistep method (12) be defined as:

$$L\left[y(x)_j ; h\right] = \sum_{j=0}^{k}\left\{\alpha_j y\ (x_n + jh) - h^2\beta_j y''(x_n + jh)\right\}; j = 0, 1, 2, \cdots, k \tag{19}$$

Lambert [16].

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a,\ b]$. Expanding $y(x_n + jh)$ and $y''(x_n + jh)$, $j = 0, 1, 2, 3, \cdots, k$ in Taylor series about $x_n$ and collecting like terms in $h$ and $y$ gives:

$$L[y(x); h] = C_0 y(x) + C_1 hy^{(1)}(x) + C_2 h^2 y^{(2)}(x) + \cdots + Cph^{(p)}(x) + \cdots \tag{20}$$

**Definition 1**

The difference operator $L$ and the associated implicit multi step method (12) are said to be of order $p$ if in (20) $C_0 = C_1 = C_2 = \cdots = C_p = C_{P+1} = 0$, $C_{p+2} \neq 0$.

**Definition 2**

The term $C_{p+2}$ is called the error constant and it implies that the local truncation error is given by:

$$t_{n+k} = C_{p+2} h^{P+2} y^{(P+2)}(x_n) + 0\left(h^{p+3}\right)$$

**Order of the Block**

The order of the block will be defined following the method of Chollon et al. [18], however, with some modification to accommodate general higher order ordinary differential equations and step points.

**Definition 3**

The term $\bar{C}_{P+2}$ is called the error constant and implies that the local truncation error for the implicit block formula is given by:

$$t_{n+k} = \bar{C}_{P+2} h^{p+2} y^{(p+2)}(x_n) + 0\left(h^{p+3}\right) \tag{21}$$

**Order and error constant of the new method (K = 4)**
From (14c)

$$y_{n+4} = 4y_{n+1} - 3y_n + \frac{1333}{420}h^2 f_{n+1} + \frac{277}{140}h^2 f_{n+2} + \frac{73}{140}h^2 f_{n+3} + \frac{137}{420}h^2 f_{n+4}$$

can be rewritten in the form:

$$(22)\quad y_{n+4} - 4y_{n+1} + 3y_n - h^2\left[\frac{1333}{420}f_{n+1} + \frac{277}{140}f_{n+2} + \frac{73}{140}f_{n+3} + \frac{137}{420}f_{n+4}\right] = 0$$

Expanding (22) in Taylors series form, we have

$$\sum_{j=0}^{\infty}\frac{(4)^j\, h^j}{j!}y_n^{(j)} - 4\sum_{j=0}^{\infty}\frac{(1)^j\, h^j}{j!}y_n^{(j)} + 3y_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{(j+2)}\left[\frac{1333}{420}(1)^j + \frac{277}{140}(2)^j + \frac{73}{140}(3)^j + \frac{137}{420}(4)^j\right] = 0$$

The collection of terms in powers of $h$ and $y$, lead to the following:

$$\begin{aligned}
c_0 &= 1 - 4 + 3 = 0\\
c_1 &= 4 - 4 = 0\\
c_2 &= \tfrac{16}{2} - \tfrac{4}{2} - \left(\tfrac{1333}{420} + \tfrac{277}{140} + \tfrac{73}{140} + \tfrac{137}{420}\right) = 0\\
c_3 &= \tfrac{64}{6} - \tfrac{4}{6} - \left(\tfrac{1333}{420}(1) + \tfrac{277}{140}(2) + \tfrac{73}{140}(3) + \tfrac{137}{420}(4)\right) = 0\\
c_4 &= \tfrac{256}{24} - \tfrac{4}{24} - \tfrac{1}{2!}\left(\tfrac{1333}{420}(1)^2 + \tfrac{277}{140}(2)^2 + \tfrac{73}{140}(3)^2 + \tfrac{137}{420}(4)^2\right) = 0\\
c_5 &= \tfrac{1024}{120} - \tfrac{4}{120} - \tfrac{1}{3!}\left(\tfrac{1333}{420}(1)^3 + \tfrac{277}{140}(2)^3 + \tfrac{73}{140}(3)^3 + \tfrac{137}{420}(4)^3\right) = -\tfrac{69}{140}
\end{aligned}$$

Hence, the method (14c) in equation (14) is of order $p = 3$, with error constant $c_{p+2} = -\frac{69}{140}$.

**Order and Error Constant of the block method (K =4)**
Rewrite the block form of (17) in the form:

$$(23)\qquad \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_{n+1}\\ y_{n+2}\\ y_{n+3}\\ y_{n+4}\\ y'_{n+1}\\ y'_{n+2}\\ y'_{n+3}\\ y'_{n+4} \end{bmatrix} - \begin{bmatrix} 1 & h\\ 1 & 2h\\ 1 & 3h\\ 1 & 4h\\ 0 & 1\\ 0 & 1\\ 0 & 1\\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_n\\ y'_n \end{bmatrix}$$

$$-\begin{bmatrix} \frac{4513}{5040}h^2 & -\frac{383}{1680}h^2 & -\frac{667}{1680}h^2 & \frac{1157}{5040}h^2 \\ \frac{289}{105}h^2 & -\frac{9}{35}h^2 & -\frac{113}{105}h^2 & -\frac{61}{105}h^2 \\ \frac{2673}{560}h^2 & \frac{171}{560}h^2 & -\frac{801}{560}h^2 & \frac{477}{560}h^2 \\ \frac{304}{45}h^2 & \frac{16}{15}h^2 & -\frac{16}{15}h^2 & \frac{56}{45}h^2 \\ \frac{491}{315}h & -\frac{109}{420}h & -\frac{23}{35}h & \frac{451}{1260}h \\ \frac{91}{45}h & \frac{4}{15}h & -\frac{3}{5}h & \frac{14}{45}h \\ \frac{139}{70}h & \frac{111}{140}h & -\frac{3}{70}h & \frac{37}{140}h \\ \frac{92}{45}h & \frac{8}{15}h & \frac{4}{5}h & \frac{28}{45}h \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} = 0$$

and writing (23) in explicit form, we have

(24)

$$\begin{cases} y_{n+1} - y_n - hy_n^{'} - h^2 \left[\frac{4513}{5040} f_{n+1} - \frac{383}{1680} f_{n+2} - \frac{667}{1680} f_{n+3} + \frac{1157}{5040} f_{n+4}\right] = 0 \\ y_{n+2} - y_n - 2hy_n^{'} - h^2 \left[\frac{289}{105} f_{n+1} - \frac{9}{35} f_{n+2} - \frac{113}{105} f_{n+3} + \frac{61}{105} f_{n+4}\right] = 0 \\ y_{n+3} - y_n - 3hy_n^{'} - h^2 \left[\frac{2673}{560} f_{n+1} + \frac{171}{560} f_{n+2} - \frac{801}{560} f_{n+3} + \frac{477}{560} f_{n+4}\right] = 0 \\ y_{n+4} - y_n - 4hy_n^{'} - h^2 \left[\frac{304}{45} f_{n+1} + \frac{16}{15} f_{n+2} - \frac{16}{15} f_{n+3} + \frac{56}{45} f_{n+4}\right] = 0 \\ y_{n+1}^{'} - y_n^{'} - h \left[\frac{491}{315} f_{n+1} - \frac{109}{420} f_{n+2} - \frac{23}{35} f_{n+3} + \frac{451}{1260} f_{n+4}\right] = 0 \\ y_{n+2}^{'} - y_n^{'} - h \left[\frac{91}{45} f_{n+1} + \frac{4}{15} f_{n+2} - \frac{3}{5} f_{n+3} + \frac{14}{45} f_{n+4}\right] = 0 \\ y_{n+3}^{'} - y_n^{'} - h \left[\frac{139}{70} f_{n+1} + \frac{111}{140} f_{n+2} - \frac{3}{70} f_{n+3} + \frac{37}{140} f_{n+4}\right] = 0 \\ y_{n+4}^{'} - y_n^{'} - h \left[\frac{92}{45} f_{n+1} + \frac{8}{15} f_{n+2} + \frac{4}{5} f_{n+3} + \frac{28}{45} f_{n+4}\right] = 0 \end{cases}$$

and using Taylor's series expansion on (24), we have

$$\sum_{j=0}^{\infty} \frac{(1)^j h^j}{j!} y_n^{(j)} - y_n - hy_n^{(1)} - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{4513}{5040}(1)^j - \frac{383}{1680}(2)^j - \frac{667}{1680}(3)^j + \frac{1157}{5040}(4)^j\right]$$

$$\sum_{j=0}^{\infty} \frac{(2)^j h^j}{j!} y_n^{(j)} - y_n - 2hy_n^{(1)} - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{289}{105}(1)^j - \frac{9}{35}(2)^j - \frac{113}{105}(3)^j + \frac{61}{105}(4)^j\right]$$

$$\sum_{j=0}^{\infty} \frac{(3)^j h^j}{j!} y_n^{(j)} - y_n - 3hy_n^{(1)} - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{2673}{560}(1)^j + \frac{171}{560}(2)^j - \frac{801}{560}(3)^j + \frac{477}{560}(4)^j\right]$$

$$\sum_{j=0}^{\infty} \frac{(4)^j h^j}{j!} y_n^{(j)} - y_n - 4hy_n^{(1)} - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{304}{45}(1)^j + \frac{16}{15}(2)^j - \frac{16}{15}(3)^j + \frac{56}{45}(4)^j\right]$$

$$\sum_{j=0}^{\infty} \frac{(1)^j h^j}{j!} y_n^{(j+1)} - y_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{491}{315}(1)^j - \frac{109}{420}(2)^j - \frac{23}{35}(3)^j + \frac{451}{1260}(4)^j\right]$$

$$\sum_{j=0}^{\infty} \frac{(2)^j h^j}{j!} y_n^{(j+1)} - y_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{91}{45}(1)^j + \frac{4}{15}(2)^j - \frac{3}{5}(3)^j + \frac{14}{45}(4)^j\right]$$

$$\sum_{j=0}^{\infty} \frac{(3)^j h^j}{j!} y_n^{(j+1)} - y_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{139}{70}(1)^j + \frac{111}{140}(2)^j - \frac{3}{70}(3)^j + \frac{37}{140}(4)^j\right]$$

$$\sum_{j=0}^{\infty}\frac{(4)^j h^j}{j!}y_n^{(j+1)}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{(j+2)}\left[\frac{92}{45}(1)^j+\frac{8}{15}(2)^j+\frac{4}{5}(3)^j+\frac{28}{45}(4)^j\right]$$

and collecting terms in $h$ and $y$ leads to the following:

$$\bar{c}_5=\begin{bmatrix} c_5=\frac{(1)^5}{5!}-\frac{1}{3!}\left(\frac{4513}{5040}(1)^3-\frac{383}{1680}(2)^3-\frac{667}{1680}(3)^3+\frac{1157}{5040}(4)^3\right)\\ c_5=\frac{(2)^5}{5!}-\frac{1}{3!}\left(\frac{289}{105}-\frac{9}{35}(2)^3-\frac{113}{105}(3)^3+\frac{61}{105}(4)^3\right)\\ c_5=\frac{(3)^5}{5!}-\frac{1}{3!}\left(\frac{2673}{560}(1)^3+\frac{171}{560}(2)^3-\frac{801}{560}(3)^3+\frac{477}{560}(4)^3\right)\\ c_5=\frac{(4)^5}{5!}-\frac{1}{3!}\left(\frac{304}{45}(1)^3+\frac{16}{15}(2)^3-\frac{16}{15}(3)^3+\frac{56}{45}(4)^3\right)\\ c_5=\frac{(1)^4}{4!}-\frac{1}{3!}\left(\frac{491}{315}(1)^3-\frac{109}{420}(2)^3-\frac{23}{35}(3)^3+\frac{451}{1260}(4)^3\right)\\ c_5=\frac{(2)^4}{4!}-\frac{1}{3!}\left(\frac{91}{45}(1)^3+\frac{4}{15}(2)^3-\frac{3}{5}(3)^3+\frac{14}{45}(4)^3\right)\\ c_5=\frac{(3)^4}{4!}-\frac{1}{3!}\left(\frac{139}{70}(1)^3+\frac{111}{140}(2)^3-\frac{3}{70}(3)^3+\frac{37}{140}(4)^3\right)\\ c_5=\frac{(4)^4}{4!}-\frac{1}{3!}\left(\frac{92}{45}(1)^3+\frac{8}{15}(2)^3+\frac{4}{5}(3)^3+\frac{28}{45}(4)^3\right) \end{bmatrix}=\begin{bmatrix}-\frac{503}{1008}\\ -\frac{379}{315}\\ -\frac{1023}{560}\\ -\frac{112}{45}\\ -\frac{1847}{2520}\\ -\frac{29}{45}\\ -\frac{179}{280}\\ -\frac{28}{45}\end{bmatrix}$$

Hence, the block method (17) is of order

$$p=(3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3)^T \tag{25}$$

with error constant

$$c_{p+2}=\left(-\frac{503}{1008},\ -\frac{379}{315},\ -\frac{1023}{560},\ -\frac{112}{45},\ -\frac{1847}{2520},\ -\frac{29}{45},\ -\frac{179}{280},\ -\frac{28}{45}\right)$$

**Consistency**

**Definition 4**

Given a continuous implicit multi step method (12) the first and second characteristics polynomials are defined as:

$$\rho(z)=\sum_{j=0}^{k}\alpha_j z^J \tag{26}$$

$$\sigma(z)=\sum_{j=0}^{k}\beta_j z^j \tag{27}$$

where $z$ is the principle root, $\alpha_k\neq 0$ and $\alpha_0^2+\beta_0^2\neq 0$.

**Definition 5**

The continuous implicit multi step method (12) is said to be consistent if it satisfies the following conditions:

(i) the order $p\geq 1$;

(ii) $\sum_{j=0}^{k} \alpha_j = 0$;
(iii) $\rho\,(1) \; = \rho'\,(1) = 0$; and
(iv) $\rho''\,(1) = 2!\sigma\,(1)$.

**Remark**
Condition (i) is sufficient for the associated block method to be consistent that is $p \geq 1$, Jator [19]. Recall the main method (14c) such that

$$y_{n+4} - 4y_{n+1} + 3y_n = \frac{h^2}{420}\left(137f_{n+4} + 219f_{n+3} + 831f_{n+2} + 1333f_{n+1}\right)$$

The first characteristic polynomial and second characteristic polynomial of the method above are given by:

$$p(z) = z^4 - 4z + 3$$

and

$$\sigma(z) = \frac{137z^4 + 219z^3 + 831z^2 + 1333z}{420}$$

respectively.
By definition 5, (14c) is consistent since it satisfies the following:
(i) the order of the method is $P = 4 \geq 1$;
(ii) $\alpha_0 = 3, \;\; \alpha_1 = -4, \;\; \alpha_4 = 1$;
Thus, $\sum_{j=0}^{4} \alpha_j, j = 0, 1, 4, \; \sum_{j=0}^{4} \alpha_j = 3 - 4 + 1 = 0$
$p(z) = z^4 - 4z + 3$
$p(1) = (1)^4 - 4(1) + 3 = 0$
$p^{'}(z) = 4z^3 - 4$
(iii) $p^{'}(1) = 4(1)^3 - 4 = 0$; and
(iv) $p^{''}(z) = 12z^2$
$p^{''}(1) = 12(1)^2 = 12$

$$p(z) = z^4 - 4z + 3$$
$$\sigma(z) = \frac{137z^4+219z^3+831z^2+1333z}{420}$$
$$\sigma(1) = \frac{137(1)^4+219(1)^3+831(1)^2+1333(1)}{420} = \frac{2520}{420} = 6$$
$$2!\sigma(1) = 2 \,\mathrm{X}\, 6 \; = 12$$
$$p^{''}(1) = 2!\sigma(1) \; = 12$$

The conditions i-iv are satisfied, hence the method is consistent. Similarly, the block method (17) is consistent since the order of each method in the block method is greater than one as shown in (25).

**Zero Stability**
**Definition 6**
The continuous implicit multi step method (12) is said to be zero-stable if no root of the first characteristics polynomial $\rho\,(z)$ has modulus greater than one,

and if every root of modulus one has multiplicity not greater than two, Lambert [20].

**Definition 7**
The implicit block method (17) is said to be zero stable if the roots $Z_s$, $s = 1, \cdots, n$ of the first characteristics polynomial $\bar{\rho}(z)$, defined by

$$\bar{\rho}(z) = \det\left[Z\bar{A} - \bar{E}\right] \tag{28}$$

Satisfies $|Z_s| \leq 1$ and every root with $|Z_s| = 1$ has multiplicity not exceeding two in the limit as $h \to 0$.

**Zero-stability of the block method**
From (17) and using the definition 7 as $h \to 0$

$$p(z) = \det\left[z\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right]$$

$$= \det\begin{bmatrix} z & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \end{bmatrix} = (z-1)\, z^7$$

$$(z-1)\ z^7 \tag{29}$$

Equating (29) to zero and solving for the values of z gives $z_2 = z_3 = z_4 = z_5 = 0,\ z_1 = 1$. Hence, the block method is stable.

**Zero-Stability of Main Method (14)**
The first characteristic polynomial of (14c) that is

$$y_{n+4} - 4y_{n+1} + 3y_n = h^2\left[\frac{1333}{420}f_{n+1} + \frac{277}{140}f_{n+2} + \frac{73}{140}f_{n+3} + \frac{137}{140}f_{n+4}\right]$$

is given by:

$$p(z) = z^4 - 4z + 3 \tag{30}$$

Solving (30) i.e. $p(z) = z^4 - 4z + 3 = 0$

$$z = -1 + i\sqrt{2},\ z = -1 - i\sqrt{2},\ z = 1,\ z = 1$$

The root of z of (30) for which $|z| = 1$is simple. Hence the method is zero stable as $h \to 0$ as defined by definition 6 and by the stability of the block method (17).

**Convergence**

The convergence of the continuous implicit multi step method (12) is considered in the light of the basic properties, in conjunction with the fundamental theorem of Dahlquist, Henrici [21] for linear multistep methods. In what follows, we state Dahlquist's theorem without proof.

**Theorem 3.1: Dahlquist theorem** Lambert [16].

The necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable.

**Remark**

The numerical methods derived here are considered to be convergent by theorem 3.1 as $h \to 0$.

Following theorem 3.1, the method (30) is convergence since it satisfies the necessary and sufficient conditions of consistency and zero stability.

**Region of Absolute Stability of the block method**

**Definition 8**

If the first and second characteristics polynomials of Linear Multistep Method (LMM) are $\rho$ and $\sigma$ respectively, then the polynomial equation can be written as:

$$\pi\left(r, \bar{h}\right) \Rightarrow \rho(r) - \bar{h}\sigma(r) = 0 \tag{31}$$

where $\bar{h} = (\lambda h)^2$

Then $\pi\left(r, \bar{h}\right)$ is called the stability polynomial of the method defined by $\rho$and $\sigma$, and $\bar{h} = (\lambda h)^2$ is the test equation. To get the region of absolute stability, we use the Routh-Hurwitz criterion by substituting

$$r = \frac{1+z}{1-z} \tag{32}$$

On evaluating the coefficient of the resulted polynomials, gives the region of absolute stability. To get the graph of the stability region, we make $\bar{h}$ the subject of the formula from (31) to get

$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} \tag{33}$$

which is then plotted in MATLAB environment to produce the required absolute stability region of the method that will be plotted in a graph.

Using definition 8, and expressing the first characteristic polynomial and second characteristic polynomial of (14c) as:

$$\rho(r) = r^4 - 4r + 3$$

and
$\sigma(r) = \frac{1}{420}\left[137r^4 + 219r^3 + 831r^2 + 1333r\right]$ respectively. Substituting into (31) gives:

$$r^4 - 4r + 3 - \frac{(\lambda h)^2}{420}\left[137r^4 + 219r^3 + 831r^2 + 1333r\right] = 0$$

(34)
$$\begin{cases} \left(1 - \frac{137}{420}(\lambda h)^2\right) r^4 - \frac{219}{420}(\lambda h)^2 r^3 - \frac{831}{420}(\lambda h)^2 r^2 - \\ \left(4 + \frac{1333}{420}(\lambda h)^2\right) r + 3 = 0 \end{cases}$$

Therefore (34) is the stability polynomial. To get the region of absolute stability, we use the Routh-Hurwitz criterion by substituting $r = \frac{1+z}{1-z}$ into (34) to get

$$\left(1 - \frac{137}{420}(\lambda h)^2\right)\left(\frac{1+z}{1-z}\right)^4 - \frac{219}{420}(\lambda h)^2\left(\frac{1+z}{1-z}\right)^3 - \frac{831}{420}(\lambda h)^2\left(\frac{1+z}{1-z}\right)^2$$
$$-\left(4 + \frac{1333}{420}(\lambda h)^2\right)\left(\frac{1+z}{1-z}\right) + 3 = 0$$

Simplifying and collecting like terms, we have:

$$\left(4 + \frac{4}{15}(\lambda h)^2\right) z^4 + \left(-16 - 4(\lambda h)^2\right) z^3 + \left(12 + \frac{41}{15}(\lambda h)^2\right) z^2 + 4(\lambda h)^2 z - 3(\lambda h)^2 = 0$$

Using the coefficients of $z^4,\ z^3,\ z^2,\ z^1$ and $z^0$ we have

(35)
$$\begin{cases} 4 + \frac{(\lambda h)^2}{15} > 0 \\ -16 - 4(\lambda h)^2 > 0 \\ 12 + \frac{41}{15}h^2 > 0 \\ 4(\lambda h)^2 > 0 \\ -3(\lambda h)^2 > 0 \end{cases}$$

Simplifying (35) gives an interval of $(-5.75, 0)$. To get the graph of the absolute stability region, using (33) yields

(36)
$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} = \frac{420\left(r^4 - 4r + 3\right)}{137r^4 + 219r^3 + 831r^2 + 1333r}$$

which is then plotted in MATLAB environment to produce the required absolute stability region of the method as shown below.

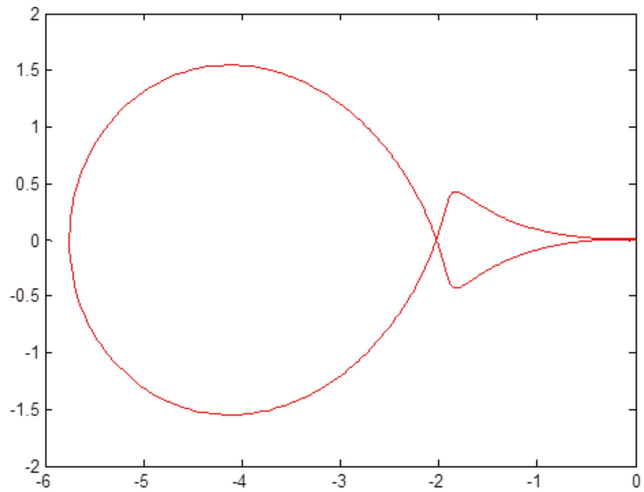


Figure 3.1: Region of absolute stability curve of the four step Method (k = 4)

The region is A-stable.

**Table: 3.1**
**Summary of the Analysis of the Methods**

| **Method** | **Order** | **Error constant** | **Zero Stability** | **Consistency** | **Interval of absolute stability** |
|---|---|---|---|---|---|
| 4SM | 3 | -4.92x$10^{-1}$ | zero stable | Consistent | -5.75, 0 |

## 4. Numerical Examples

In order to study the efficiency of the developed method, we present some numerical examples with the following three problems. The continuous implicit multi step method 3SM was applied to solve the following test problems:

**PROBLEM ONE:**
$y'' = y',\ y^{(0)} = 0,\ y'(0) = -1,\ h = 0.1$
Exact solution: $y(x) = 1 - \exp(x)$;
Source: Ehige et al. [3].

**Table: 4.1**
**SHOWING THE EXACT SOLUTION AND THE COMPUTED RESULTS FROM THE PROPOSED METHODS FOR PROBLEM ONE AND IT COMPARISM WITH** Ehigie et al. [3].

| **X values** | $y_{cx}$ | **4SM** | [3] | **Error in 4SM** | **Error in** [3] |
|---|---|---|---|---|---|
| 0.1 | -0.1051709180756 | -0.1051707971231 | -0.1048333333 | 1.209525e-07 | 3.38e-04 |
| 0.2 | -0.2214027581601 | -0.2214022036716 | -0.2206078733 | 5.544885e-07 | 7.95e-04 |
| 0.3 | -0.3498588075760 | -0.3498574425572 | -0.3484633860 | 1.365019e-06 | 1.40e-03 |
| 0.4 | -0.4918246976412 | -0.4918220705966 | -0.4896604103 | 2.627045e-06 | 2.16e-03 |
| 0.5 | -0.6487212707001 | -0.6487168441319 | -0.6455911064 | 4.426568e-06 | 3.13e-03 |
| 0.6 | -0.8221188003905 | -0.8221119377028 | -0.8177929079 | 6.862688e-06 | 4.33e-03 |
| 0.7 | -1.0137527074704 | -1.0137426580706 | -1.0079636772 | 1.004940e-06 | 5.79e-03 |
| 0.8 | -1.2255409284924 | -1.2255268108537 | -1.2179784459 | 1.411764e-05 | 7.56e-03 |
| 0.9 | -1.4596031111569 | -1.4595838935801 | -1.4499079018 | 1.921758e-05 | 9.70e-03 |
| 1.0 | -1.7182818284590 | -1.7182563072358 | -1.7060388057 | 2.552122e-05 | 1.22e-02 |

**Note:** The new method perform better than Ehigie et al. [3].

**PROBLEM TWO:**
$y'' - x\left(y'\right)^2 = 0, \quad y\left(0\right) = 1, \; y'\left(0\right) = \frac{1}{2}, \, h = \frac{0.1}{40}$
Exact solution: $y\left(x\right) = 1 + \frac{1}{2} In \, \left(\frac{2+x}{2-x}\right)$.
Source: Osilagun et al. [8].

**Table 4.2**
**SHOWING THE EXACT SOLUTION AND THE COMPUTED RESULTS FROM THE PROPOSED METHODS FOR PROBLEM TWO AND ITS COMPARISM WITH** Osilegun et al. [8].

| **X values** | $y_{cx}$ | **4SM** | [8] | **Error in 4SM** | **Error in** [8] |
|---|---|---|---|---|---|
| 0.0025 | 1.00125000065104 | 1.00125000065108 | 1.001250000186 | 3.641532e-14 | 4.650e-10 |
| 0.0050 | 1.00250000520835 | 1.00250000520844 | 1.002500003997 | 8.815171e-14 | 1.211e-09 |
| 0.0075 | 1.00375001757828 | 1.00375001757840 | 1.003750013174 | 1.338929e-13 | 4.030e-09 |
| 0.0100 | 1.00500004166729 | 1.00500004166747 | 1.005000047982 | 1.822986e-13 | 6.314e-09 |
| 0.0125 | 1.00625008138211 | 1.00625008138155 | 1.006250080358 | 5.626610e-13 | 8.462e-09 |
| 0.0150 | 1.00750014062974 | 1.00750014062845 | 1.007500025790 | 1.292300e-12 | 1.148e-09 |
| 0.0175 | 1.00875022331755 | 1.00875022331552 | 1.008750239662 | 2.028155e-12 | 1.993e-09 |
| 0.0200 | 1.01000033335334 | 1.01000033335057 | 1.010000078382 | 2.760681e-12 | 2.550e-09 |
| 0.0225 | 1.01125047464541 | 1.01125047464008 | 1.011250489037 | 5.329071e-12 | 4.256e-09 |
| 0.0250 | 1.01250065110271 | 1.01250065109482 | 1.012500610101 | 7.882361e-12 | 4.100e-08 |

**Note:** The new method perform better than Osilagun et al. [8].

**PROBLEM THREE:**
$y'' = y + xe^{3x},\ y(0) = \frac{-3}{32},\ y'(0) = \frac{-5}{32},\ h = 0.1$
Exact solution: $y(x) = \frac{4x-3}{32e^{-3x}}$.
Source: Osilagun et al. [8].

**Table: 4.3**
**SHOWING THE EXACT SOLUTION AND THE COMPUTED RESULTS FROM THE PROPOSED METHODS FOR PROBLEM THREE AND ITS COMPARISM WITH** Osilegun et al. [8].

| **X values** | $y_{cx}$ | **4SM** | [8] | **Error in 4SM** | **Error in** [8] |
|---|---|---|---|---|---|
| 0.0025 | -0.094140915761848 | -0.094140915760507 | -0.0941409131568 | 1.341510e-12 | 2.61e-09 |
| 0.0050 | -0.094532404142338 | -0.094532404139124 | -0.0945324074753 | 3.21479e-12 | 3.33e-09 |
| 0.0075 | -0.094924451608388 | -0.094924451603642 | -0.0949244224215 | 4.745607e-12 | 2.92e-08 |
| 0.0100 | -0.095317044390700 | -0.095317044384716 | -0.0953170247449 | 5.983755e-12 | 2.02e-08 |
| 0.0125 | -0.095710168480980 | -0.095710168472818 | -0.0957101793552 | 8.162346e-12 | 1.08e-08 |
| 0.0150 | -0.096103809629113 | -0.096103809618219 | -0.0961039982252 | 1.089376e-11 | 1.88e-07 |
| 0.0175 | -0.09649533403163 | -0.096497953327051 | -0.0964952920355 | 1.326500e-11 | 4.23e-08 |
| 0.0200 | -0.096892584872264 | -0.096892584856944 | -0.0968923659413 | 1.531951e-11 | 2.18e-07 |
| 0.0225 | -0.097289689232184 | -0.097287689213842 | 0.0972874625827 | 1.834202e-11 | 2.22e-06 |
| 0.0250 | -0.097683251173919 | -0.097683251151980 | -0.0976830958236 | 2.193921e-11 | 1.55e-07 |

**Note:** The new method perform better than Osilagun et al. [8].

## Discussion of the Results

The computer programs written for the implementation of the continuous implicit multi step method 4SM, was tested on numerical examples which are respectively, nonlinear, linear and stiff initial value problems of general second order ordinary differential equations in the least section.
Generally, the performance of our method as notice in table 4.1 are superior to those from methods implemented by Ehigie et al. [3], that used a 2-step continuous linear multistep method of hybrid type on moderately stiff problem one. It is observed that our method 4SM, in table 4.2, performed far better than Osilegun et al. [8], method of four steps implicit method on non-linear problem two. Also, our method 4SM, perform better than Osilegun et al. [8], method of four steps implicit method on linear problem three in table 4.3.
Finally, our scheme have been demonstrated to be more efficient in stiff problems as shown in table 4.1 of problem one.

## Conclusion

This paper illustrates the derivation, analysis and implementation of block method for solving second order initial value problem of ordinary differential equations directly.

Numerical experiments have been carried out using appropriate step size as required by each problem. Such problem which are stiff, non-linear and linear. In general, the results from numerical experiment so presented in this paper show that the new method performed effectively well when compared with other methods in the literature.

## References


[1] Abualnaja K. M. (2015). A block procedure with linear multi-step methods using Legendre polynomials for solving ODEs. Journal of Applied Mathematics. **6**, 717-732.

[2] Anake T. A. (2011). Continuous Implicit one-step methods for the solution IVPs of General Second ODEs, using power series unpublished doctoral dissertation, Convenant University, Ota.

[3] Ehigie J. O., Okunuga S. A., Sofoluwe A. B. and Akanbi M. A. (2010). On Generalized 2-step continuous linear multistep method of hybrid type for the integration of second ODEs. Journal of Scholars Research Library. **2**(6), 362-372.

[4] Adeniyi R. B. and Adeyefa E. O. (2013). Chebyshev Collocation Approach for a Continuous formation of implicit hybrid methods for IVPs in second ODEs. Journal of Mathematics, **6**(4), 9-12.

[5] Kayoed and Adebeye (2013). Two step point hybrid methods for general second order Differential Equations. African Journal of Mathematics and Computer Sciences research. **6**(10), 191-196.

[6] Ademiluyi R. A., Duromola M. K. and Bolarinwa Bolaji (2014). Modified Block Method for the direct solution of initial value problem of Fourth Order Ordinary Differential Equation. Australian Journal of Basic and Applied sciences. **8**(10), 389-394.

[7] Bolarinwa Bolaji (2012). Implicit hybrid block methods for the numerical solution of IVPs of third order ODEs. A Ph.D thesis in Mathematical Sciences Department of Federal University of Technology, Akure, Nigeria.

[8] Osilagun J. A., Adesanya A. O., Anake T. A. and Oghoyon G. J. (2009). Four steps implicit method for the solution of General second order Ordinary Differential Equations. Journal of Natural Sciences, Engineering and Technology. **8**(1), 52-61.

[9] Abdulganiy R. I., Akinfewa O. A. and Okunuga S. A. (2017). Maximal order Block Trigonometrically Fitted scheme for the Numerical Treatment of second order Initial Value Problem with oscillating solution. International Journal of Mathematical Analysis and Optimization Theory and Application. **2017**, 168-186.

[10] Peter O. J. and Ibrahim M. O. (2017). Application of Differential Transformation Method in Solving a Typhoid Fever Model. International Journal of Mathematical Analysis and Optimization Theory and Application. **2017**, 250-260.

[11] Zarina B. I., Mohammed S., Kharil I. and Zanariah M. (2005). Block method for generalized multistep Adams method and backward differentiation formula in solving first-order ODEs, Mathematika. 25-33.

[12] Yahaya Y. A. and Mohammed U. (2010). Fully implicit three point backward differentiation formulae for solution of first order initial value problems. International Journal of Numerical mathematics. **5**(3), 384-398.
[13] Odekunle M. R., Adesanya A. O. and Sunday J. (2012). A New Block integrator for the solution of initial value problems of first order ordinary differential equation. International Journal of Pure and Applied Sciences and Technology. **11**(1), 92-100.
[14] Sunday J., James A. A., Ibijola E. A., Ogunrinde R. B. and Ogunyebi S. A. (2012). A computational Approach to Verhulst-Pearl Model 10SR. Journal of Mathematics. **4**, 06-13.
[15] Awoyemi D. O. (1992). On some continuous linear multistep methods for initial value problems unpublished doctoral dissatisfaction, University of Ilorin, Ilorin, Nigeria.
[16] Lambert J. D. (1973). Computational methods in ordinary differential equation. John Wiley, New York.
[17] Abhulimen C. E. and Aigbiremhon A. A. (2017). Continuous Implicit Block Methods Using Legendre Polynomials for Numerical Solution of Ordinary Differential Equations of Order One. Journal of Mathematical Physics. **41**, 465-474.
[18] Chollon J. P., Ndam J. N. and Kunleng G. M. (2007). On some properties of the block linear multistep methods. Sciences World Journal. **2**(3), 11-17.
[19] Jator S. N. (2007). A sixeth order linear multistep method for the direct solution of ordinary differential equations. International Journal of Pure and Applied Mathematics. **40**(1), 457-472.
[20] Lambert J. D. (1991). Computational methods in ordinary differential equation. John Wiley, New York.
[21] Henrici P. (1962). Discrete variable method in Ordinary Differential Equation. John Wiley and Sons. New York.
[22] Abhulimen C. E. and Aigbiremhon A. A. (2018). Three-Steps Block Method for solving Second Order Differential Equations. International Journal of Mathematical Analysis and Optimization: Theory and Applications. **2018**, 364-381.