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Determination of Consistent Performance of Lax-Friedrichs Method in Solving Seepage Problem

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Abstract

We developed a consistent Lax-Friedrichs method for numerical solution of the seepage problem subject to some initial and boundary conditions. We also compared the result obtained using numerical method (Lax-Friedrichs) with the result obtained using Alternating Direction Implicit finite difference method (ADIS). The scheme developed was found to be stable for $\phi < 1$, and unstable for $\phi \geq 1$. Since the model used in this work was evaluated at a selected discrete point with the region of integration, the surface of the plots becomes naturally not smooth. We also deduced from our result that for any given value of x, the solution $u(x,t)$ increases to almost one as $t \to \infty$. Also, for a given value of t, the solution $u(x,t)$ decreases indefinitely as $x \to \infty$, and we observed that the seepage gradually goes to null as the mesh size becomes smaller, which shows that the fluid flow seepage problem may eventually stop.

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Abbreviations. Alternating Direction Implicit Finite Difference Scheme(ADIS); Crank Nicholson Method(CNM); Explicit Finite Difference Method (EFDM); Forward in time, centered in space (FTCS); Implicit Finite Difference Method (IFDM); Lax-Friedrichs Method (L_xFM)
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1. Introduction

The origins of ambitious efforts to understanding the mechanics of fluids date back to Ancient Greece when Archimedes discovered the famous principle of buoyancy bearing his name more than 2000 years ago. Ever since, outstanding scientists have devoted their research to the formulation of principles elucidating the physics of fluid flow in manifold phenomena. The introduction of infinitesimal calculus into fluid mechanics in the 18th century by Leonhard Euler amongst others constituted a landmark in the development of a sound mathematical framework and laid the foundation for the Navier-Stokes equations which are widely used for the modeling of fluid flow nowadays. However, technical innovations in the course of the industrial revolution necessitated the derivation of rules of thumb in order to make statements about the behaviour of mechanical systems since partial differential equations providing no analytical solutions were of little use when it came to engineering challenges in those days. In the recent decades, emphasis has been placed on the investigation of coupled models incorporating different flow domains, motivated by environmental and technical issues. In this spirit, we will deal with the coupling of surface and subsurface flow in this article, which is of great interest in many hydrological applications such as the interaction between surface run-off and groundwater.

In this research work, the Lax-Friedrichs method is applied to deal with steady seepage analysis in homogeneous isotropic medium. The Lax-Friedrichs method, named after Peter Lax and Kurt O. Friedrichs, is a numerical method for the solution of hyperbolic partial differential equations based on finite differences. The method can be described as the FTCS (forward in time, centered in space) scheme with an artificial viscosity term of $\frac{1}{2}$. One can view the Lax-Friedrichs method as an alternative to Godunov's scheme, where one avoids solving a Riemann problem at each cell interface, at the expense of adding artificial viscosity. This work is therefore structured towards solving the Seepage Laplace equation using the Lax-Friedrichs Approach and it is structured into several sections, which include the analytical solution to the problems and the numerical solution to seepage equation using Lax-Friedrichs approach.

2. Mathematical Formulation

In this section, we present some preliminary results and concepts in Finite Difference Methods (FDMs) for the solution of the Partial Differential Equations (PDEs). We also present conditions necessary for the stability and convergence of FDMs.

2.1. **Finite Difference Method (FDM).** The FDM is a numerical approximation used for solving certain differential equations. The FDM obtain the value of a derivative by evaluating the differential equation that is satisfied by the derivative. The PDE is substituted using the Taylor series approximation of the function near the points of interest. This usually resulted to a set of difference equations and thus can be solved iteratively. In order to define a finite difference, we need a grid defined on Ω. Let $N+1$ be given and define $h = \frac{1}{N+1}$. Then we define the grid as the collection of points

$$S = \{(nh, mh) : \ n, m = 0, 1, \ldots, N+1\}$$

We denote the interior points as S_I

$$S_I = \{(nh, mh) : \ n, m = 1, 2, \ldots, N\}$$

and the boundary points as $S_b = S \setminus S_I$.

Furthermore, in order to write a computer code for a FDM, we need to write the equivalent linear system that the finite difference gives rise to. To do that, we enumerate the vertices S_I. We define $\phi_1, \phi_2, \ldots, \phi_N$ as ($\mathcal{T}$ is the domain) follows;

$$\phi_{n+(m-1)N} = (nh, mh) \text{ for all } n, m = 1, \ldots, N$$

We define $V(\phi_k) = V_k$ for all $k = 1, \ldots, N^{\mathcal{T}}$

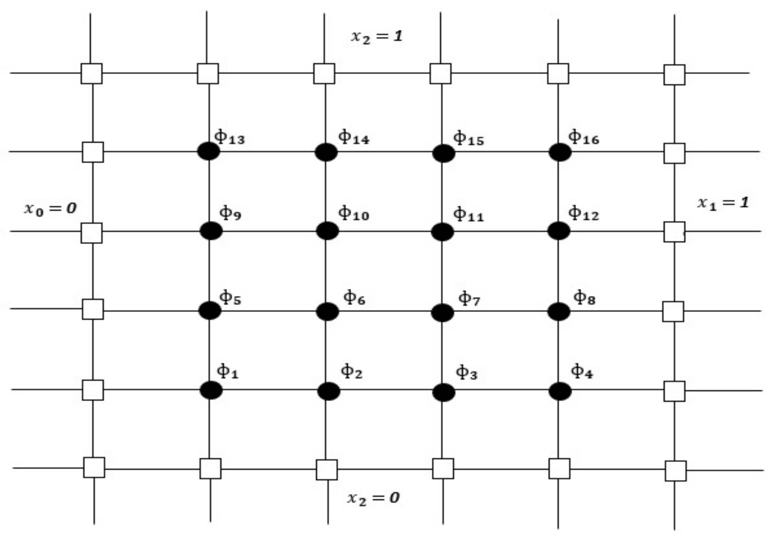

Figure 1: Grid $N = 4$ grid points. S_b are in square and the S_I are in circle

For the purpose of this research, we shall briefly discuss the implicit finite difference method, the explicit finite difference method, the Crank Nicholson method and the Lax Friedrichs method. To do this, we shall consider one-dimensional heat equation presented in subsection 3.2.

2.2. **Illustrative Model Problem.** Consider the one-dimensional heat equation,

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where k is a constant coefficient. In order to obtain an approximation for the model problem (1) by FDMs, we first need to define a set of grid points in the domain of interest, say Ω. According to (Hull, 2006), the following steps are considered:

(i) select or choose a step size $h = \frac{\Omega_2 - \Omega_1}{N}$, where N is an integer and a time step Δt; and

(ii) Draw a set of vertical and horizontal lines across the domain Ω, and identify all the points of intersection (x_i, t_j) or simply (i, j) where $x_i = \Omega_1 + ih$, $i = 0, \ldots, N$ and $t_j = j\Delta t$, $j = 0, 1, \ldots, m$. The figure below shows some nodal points on the domain Ω.

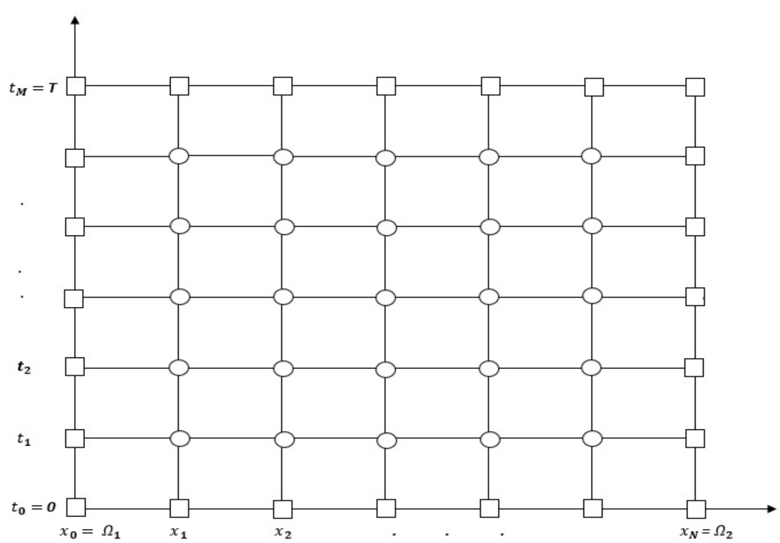

Figure 2: N grid points

2.3. **The Explicit Finite Difference Method.** The explicit FDM assume that the values of the derivative at the point (i, j) on the lattice to be the same as $(i, j+1)$ and it makes use of the backward approximation for the time derivative; see (Hull 2006). Thus (1) can then be approximated as

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t} \tag{2}$$

whereas, for the first derivative of the R.H.S of (1), we have from the central approximation as

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j+1} - u_{i-1,j+1}}{2\Delta x} \tag{3}$$

For the second derivative, we have from the standard approximation

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{(\Delta x)^2} \tag{4}$$

where (i, j) denotes the nodal points on the lattice. Substituting (2) and (4) into (1), we have

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \alpha \left[\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{(\Delta x)^2} \right] \tag{5}$$

If we set $\phi = \dfrac{\alpha \Delta t}{(\Delta x)^2}$ in (5), we have

$$-\phi u_{i+1,j+1} + (1 + 2\phi) u_{i+1,j} - \phi u_{i-1,j+1} = u_{i,j} \tag{6}$$

From (6), we notice that the unknown $u_{i,j}$ is written explicitly in terms of the unknowns. Furthermore, it relates the unknown value at $j\Delta t$ to the known values at $(j+1)\Delta t$. In obtaining the values of $u(x, 0)$, we approximate backward until $t = 0$ using the explicit scheme (6).

Applying (6) on the grid in Figure (2), we can obtain a system of equation of the form

$$u^{j+1} = Bu^j + d^j \tag{7}$$

$$u^j = (u_1^j, u_2^j, \ldots, u_{N-1}^j)^T \ \in \ \mathbb{R}^{N-1} \tag{7a}$$

and

$$B = \begin{pmatrix} 1-2\phi & \phi & \ldots & 0 \\ \phi & 1-2\phi & \ldots & 0 \\ \vdots & \ddots & & \\ & \ddots & \ddots & 0 \\ & \ddots & \ddots & \phi \\ \ldots & \phi & 1-2\phi & \end{pmatrix} \ \in \ \mathbb{R}^{(N-1)\times(N-1)} \tag{7b}$$

(7c)
$$d^n = \begin{pmatrix} \phi u_0^j \\ 0 \\ \vdots \\ 0 \\ \phi u_N^j \end{pmatrix} \in \mathbb{R}^{N-1}$$

2.4. **The Implicit Finite Difference Method.** Contrary to the explicit finite difference method, the implicit finite difference method makes use of the forward approximation for the time derivatives (Hull (2006), Nyachwaya *et al.* (2014)). Thus, the approximation for the time derivatives in (1) is defined as

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t} \text{ as in (2)}$$

where $u_{i,j} = u(i\Delta x, j\Delta t)$, $i = 1, 2, \ldots, M-1$, $j = N-1, N-2, \ldots, 1, 0$
For the first derivative of the L.H.S of (1), we use the central approximation

(8)
$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}$$

Similarly, for the second derivative, we use the standard approximation

(9)
$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2}$$

Substituting (2) and (9) into (1), we get

(10)
$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \alpha \left[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} \right]$$

Furthermore, simplification yields

(11)
$$u_{i,j+1} = u_{i,j} + \phi(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

where $\phi = \dfrac{\alpha \Delta t}{(\Delta x)^2}$
From (11), we see that the unknown function $u_{i,j+1}$ is written in terms of the known values $u_{i-1,j}$, $u_{i,j}$ and $u_{i+1,j}$. Moreover, it relates the unknown values at $(j+1)\Delta t$ to the known values at $j\Delta t$. Therefore, if we are given the values of $u_{i,j+1}$ for all i, at time step j, we can compute the value of $u_{i,j}$ implicitly upon evaluation of (11) using the corresponding grid points, we obtain a linear algebraic equation of the form

(12)
$$u^{j+1} = Au^j + b^j$$

(12a)
$$u^j = (u_1^j, u_2^j, \ldots, u_{N-1}^j)^T \in \mathbb{R}^{N-1}$$

and

$$\text{(12b)} \quad A = \begin{pmatrix} 1+2\phi & -\phi & 0 & 0 & \dots & 0 \\ -\phi & 1+2\phi & -\phi & 0 & \dots & 0 \\ 0 & -2\phi & 1+2\phi & -\phi & \dots & 0 \\ & \ddots & \ddots & \ddots & \ddots & -\phi \\ 0 & \dots & 0 & -\phi & 1+2\phi & \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}$$

$$\text{(12c)} \qquad d^n = \begin{pmatrix} \phi u_0^j \\ 0 \\ \vdots \\ 0 \\ \phi u_N^j \end{pmatrix} \in \mathbb{R}^{N-1}$$

using any known method for the solution of equations of the form (12), it is easy to compute the solutions $u^j \in \mathbb{R}^{N-1}$.

2.5. **The Crank Nicholson Method (C.N.M).** The C.N.M uses the average of the explicit scheme at (i, j) and the implicit scheme at $(i, j+1)$. Thus, from (5) and (10), we have

$$\text{(13)} \quad \frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{\alpha}{2}\left[\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{(\Delta x)^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2}\right]$$

On simplification, we get

$$\text{(14)} \quad -\frac{\phi}{2}u_{i-1,j+1} + (1+\phi)u_{i,j+1} - \frac{\phi}{2}u_{i+1,j+1} = \frac{\phi}{2}u_{i-1,j} + (1-\phi)u_{i,j} + \frac{\phi}{2}u_{i+1,j}$$

with $\phi = \dfrac{\alpha \Delta t}{(\Delta x)^2}$

Hence, equation (14) is the C.N.M for the solution (1).

Consequently, the C.N.M enjoys stronger stability and accuracy. The C.N.M is well known to be unconditionally stable with the local truncation error $O((\Delta t)^2, (\Delta x)^2, (\Delta y)^2)$.

2.6. **The Lax-Friedrichs Method ($\mathbf{L_x}$FM).** The L_xFM which is our main method in this article is described as a FTCS (forward in time centred in space) scheme. In this case, the L.H.S of equation (2) is approximated as

$$\text{(15)} \qquad \frac{\partial u}{\partial t} = u_{i,j+1} - \frac{1}{2}(u_{i+1,j} + u_{i-1,j})$$

so that (1) becomes

$$\text{(16)} \qquad u_{i,j+1} - \frac{1}{2}(u_{i+1,j} + u_{i-1,j}) = \phi[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

Other useful finite difference methods include the leapfrog method which is known to be marginally stable, the Geolnuv method and others, see Riley (2006), Hull (2006), Nyachwaya *et al.* (2014) and the references therein.

2.7. **Analysis of Finite Difference Method.** To analyse the FDMs, the Lax-Richmer theorem or the Lax equivalence theorem is used to show the essential factors that describes such numerical methods. These factors include the consistency, the stability and convergence of the method.
A FDM is said to be consistent if there exists an infinitesimal difference between the PDE and the finite difference equation as the step size get smaller

$$\lim_{\Delta t \to 0} [\text{PDE} - \text{FDE}] = 0$$

A FDM is said to be stable if the errors arising from the discrete numerical solution do not explode as time increases.
A FDM is said to be convergent if the discrete solution of the finite difference equation tends to the exact solution of the PDE, as the step sizes of the time and $u(x,t)$ are reduced (Fadugba and Nwozo, 2013).

Theorem 2.1. *(Lax Equivalence Theorem) (Morton and Mayers, 2005)*
For a consistent difference approximation to a well-posed linear evolutionary problem, the stability of the scheme is necessary and sufficient for convergence.

Well-posed linear evolutionary problems in this regard refer to problems whose solutions exist, whose solutions are unique and whose solutions continuously depend on the boundary conditions.
However, various methods exist to satisfying the stability of FDM, one of which is the eigenvalue approach, see (Hull, 2006). In this manuscript however, we shall compute the eigenvalue of the proposed method and confirm that $\lambda_{\max} < 1$.

3. Numerical Solution of a Seepage Problem

In this section, we develop a consistent Lax-Friedrichs method (L_xFM) for the solution of the seepage problem. We will equally carry out the stability analysis and convergence of our scheme. Finally, we present some numerical implementations of the proposed method in order to study the efficiency of the scheme for the solution of fluid flow seepage problem.

3.1. **The Seepage Problem.** In this article, we consider a two-dimensional time dependent diffusion equation subject to a constant coefficient

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le L,\ 0 \le y \le B,\ 0 \le t \le \infty \tag{17}$$

where α and β are constant coefficients, subject to Dirichlet boundary conditions.

For the purpose of this study, we assume that the fluid flow is laminar, the fluid particles are irrational, the fluid flow is steady, perfect and incompressible, the variation of density of the fluid flow is negligible, see Nyachwaya *et al.* (2014).
To study the seepage problem, we intend to obtain the fundamental solution of the time-dependent diffusion equation (17) via the Lax-Friedrichs method (L_xFM). Thus, we are concerned with the model (17) subject to:

$$\begin{array}{rcl} u(0,t) &=& 0 \\ u(1,t) &=& 0 \\ u(x,t) &=& 1-e^{-t}, \quad 0 \le t \le \infty \\ u(x,0) &=& 0 \end{array}$$

Here, $(x,t) \in \mathbb{R}$, we denote the solution at (x_i, t_i) by $u_{i,j}$.
The L_xFM is known to be FTCS scheme, thus, th approximation for (17) becomes (18)

$$u_{i,j+1} - \frac{1}{2}(u_{i+1,j}+u_{i-1,j}) - \frac{\alpha}{h^2}(u_{i+1,j}-2u_{i,j}+u_{i-1,j}) - \frac{\beta}{h^2}(u_{i,j+1}-2u_{i,j}+u_{i,j-1}) = 0$$

If we let $\phi = \frac{\beta}{h^2} = \frac{\alpha}{h^2}$, then (18) becomes

$$(19) \quad (1-\phi)u_{i,j+1} - \left(\frac{1}{2}+\phi\right)u_{i-1,j} - \left(\frac{1}{2}+\phi\right)u_{i+1,j} + 4\phi u_{i,j} - \phi u_{i,j-1} = 0$$

where $\phi \le \frac{1}{2}$, for stability and consistency of (19).
Next, we define a grid on the domain Ω. Let $N+1$ be given, and define $h = \frac{1}{N+1}$, then we define the grid as the collection of points

$$\mathrm{T}_h := \{(nh, mh) : \; n, m = 0, 1, \dots, N+1\}$$

We denote the interior points as

$$\mathrm{T}_I := \{(nh, mh) : \; n, m = 1, 2, \dots, N\}$$

and the boundary points as $\mathrm{T}_b = \mathrm{T}_h \setminus \mathrm{T}_I$.
In particular, we develop the L_xFM (19) in the spirit of Nyachwaya *et al.* (2014).
Fixing $j = 1$, and varying $i = 1, 2, 3, \dots, 15$ as follows;
For $i = 1$;

$$(1-\phi)u_{1,2} - \left(\frac{1}{2}+\phi\right)u_{0,1} - \left(\frac{1}{2}+\phi\right)u_{2,1} + 4\phi u_{1,1} - \phi u_{1,0} = 0$$

using the boundary conditions, we have

$$(1) \qquad 4\phi u_{1,1} - \left(\frac{1}{2}+\phi\right)u_{2,1} = \left(\frac{1}{2}+\phi\right)u_{0,1}$$

For $i = 2$;

$$(1-\phi)u_{2,2} - \left(\frac{1}{2}+\phi\right)u_{1,1} - \left(\frac{1}{2}+\phi\right)u_{3,1} + 4\phi u_{2,1} - \phi u_{2,0} = 0$$

using the boundary conditions, we have

$$-\left(\frac{1}{2}+\phi\right)u_{1,1} + 4\phi u_{2,1} - \left(\frac{1}{2}+\phi\right)u_{3,1} = 0 \tag{2}$$

For $i = 3$;

$$(1-\phi)u_{3,2} - \left(\frac{1}{2}+\phi\right)u_{2,1} - \left(\frac{1}{2}+\phi\right)u_{4,1} + 4\phi u_{3,1} - \phi u_{3,0} = 0$$

using the boundary conditions, we have

$$-\left(\frac{1}{2}+\phi\right)u_{2,1} - \left(\frac{1}{2}+\phi\right)u_{4,1} + 4\phi u_{3,1} = 0 \tag{3}$$

For $i = 4$;

$$(1-\phi)u_{4,2} - \left(\frac{1}{2}+\phi\right)u_{3,1} - \left(\frac{1}{2}+\phi\right)u_{5,1} + 4\phi u_{4,1} - \phi u_{4,0} = 0$$

using the boundary conditions, we have

$$-\left(\frac{1}{2}+\phi\right)u_{3,1} - \left(\frac{1}{2}+\phi\right)u_{5,1} + 4\phi u_{4,1} = 0 \tag{4}$$

For $i = 5$;

$$(1-\phi)u_{5,2} - \left(\frac{1}{2}+\phi\right)u_{4,1} - \left(\frac{1}{2}+\phi\right)u_{6,1} + 4\phi u_{5,1} - \phi u_{5,0} = 0$$

using the boundary conditions, we have

$$-\left(\frac{1}{2}+\phi\right)u_{4,1} - \left(\frac{1}{2}+\phi\right)u_{6,1} + 4\phi u_{5,1} = 0 \tag{5}$$

For $i = 6$;

$$(1-\phi)u_{6,2} - \left(\frac{1}{2}+\phi\right)u_{5,1} - \left(\frac{1}{2}+\phi\right)u_{7,1} + 4\phi u_{6,1} - \phi u_{6,0} = 0$$

using the boundary conditions, we have

$$-\left(\frac{1}{2}+\phi\right)u_{5,1} - \left(\frac{1}{2}+\phi\right)u_{7,1} + 4\phi u_{6,1} = 0 \tag{6}$$

For $i = 7$;

$$(1-\phi)u_{7,2} - \left(\frac{1}{2}+\phi\right)u_{6,1} - \left(\frac{1}{2}+\phi\right)u_{8,1} + 4\phi u_{7,1} - \phi u_{7,0} = 0$$

using the boundary conditions, we have

$$-\left(\frac{1}{2}+\phi\right)u_{6,1}-\left(\frac{1}{2}+\phi\right)u_{8,1}+4\phi u_{7,1}=0 \tag{7}$$

For $i = 8$;

$$(1-\phi)u_{8,2}-\left(\frac{1}{2}+\phi\right)u_{7,1}-\left(\frac{1}{2}+\phi\right)u_{9,1}+4\phi u_{8,1}-\phi u_{8,0}=0$$

using the boundary conditions, we have

$$-\left(\frac{1}{2}+\phi\right)u_{7,1}-\left(\frac{1}{2}+\phi\right)u_{9,1}+4\phi u_{8,1}=0 \tag{8}$$

For $i = 9$;

$$(1-\phi)u_{9,2}-\left(\frac{1}{2}+\phi\right)u_{8,1}-\left(\frac{1}{2}+\phi\right)u_{10,1}+4\phi u_{9,1}-\phi u_{9,0}=0$$

using the boundary conditions, we have

$$-\left(\frac{1}{2}+\phi\right)u_{8,1}-\left(\frac{1}{2}+\phi\right)u_{10,1}+4\phi u_{9,1}=0 \tag{9}$$

For $i = 10$;

$$(1-\phi)u_{10,2}-\left(\frac{1}{2}+\phi\right)u_{9,1}-\left(\frac{1}{2}+\phi\right)u_{11,1}+4\phi u_{10,1}-\phi u_{10,0}=0$$

using the boundary conditions, we have

$$-\left(\frac{1}{2}+\phi\right)u_{9,1}-\left(\frac{1}{2}+\phi\right)u_{11,1}+4\phi u_{10,1}=0 \tag{10}$$

For $i = 11$;

$$(1-\phi)u_{11,2}-\left(\frac{1}{2}+\phi\right)u_{10,1}-\left(\frac{1}{2}+\phi\right)u_{12,1}+4\phi u_{11,1}-\phi u_{11,0}=0$$

using the boundary conditions, we have

$$-\left(\frac{1}{2}+\phi\right)u_{10,1}-\left(\frac{1}{2}+\phi\right)u_{12,1}+4\phi u_{11,1}=0 \tag{11}$$

For $i = 12$;

$$(1-\phi)u_{12,2}-\left(\frac{1}{2}+\phi\right)u_{11,1}-\left(\frac{1}{2}+\phi\right)u_{13,1}+4\phi u_{12,1}-\phi u_{12,0}=0$$

using the boundary conditions, we have

$$-\left(\frac{1}{2}+\phi\right)u_{11,1}-\left(\frac{1}{2}+\phi\right)u_{13,1}+4\phi u_{12,1}=0 \tag{12}$$

For $i = 13$;

$$(1-\phi)u_{13,2} - \left(\frac{1}{2}+\phi\right)u_{12,1} - \left(\frac{1}{2}+\phi\right)u_{14,1} + 4\phi u_{13,1} - \phi u_{13,0} = 0$$

using the boundary conditions, we have

$$-\left(\frac{1}{2}+\phi\right)u_{12,1} - \left(\frac{1}{2}+\phi\right)u_{14,1} + 4\phi u_{13,1} = 0 \tag{13}$$

For $i = 14$;

$$(1-\phi)u_{14,2} - \left(\frac{1}{2}+\phi\right)u_{13,1} - \left(\frac{1}{2}+\phi\right)u_{15,1} + 4\phi u_{14,1} - \phi u_{14,0} = 0$$

using the boundary conditions, we have

$$-\left(\frac{1}{2}+\phi\right)u_{13,1} - \left(\frac{1}{2}+\phi\right)u_{15,1} + 4\phi u_{14,1} = 0 \tag{14}$$

For $i = 15$;

$$(1-\phi)u_{15,2} - \left(\frac{1}{2}+\phi\right)u_{14,1} - \left(\frac{1}{2}+\phi\right)u_{16,1} + 4\phi u_{15,1} - \phi u_{15,0} = 0$$

using the boundary conditions, we have

$$-\left(\frac{1}{2}+\phi\right)u_{14,1} - \left(\frac{1}{2}+\phi\right)u_{16,1} + 4\phi u_{15,1} = 0 \tag{15}$$

We summarise equations (1) - (15) as

$$\begin{array}{rc}
i=1: & 4\phi u_{1,1} - \left(\frac{1}{2}+\phi\right)u_{2,1} = \left(\frac{1}{2}+\phi\right)u_{0,1} \\
i=2: & 4\phi u_{2,1} - \left(\frac{1}{2}+\phi\right)u_{1,1} - \left(\frac{1}{2}+\phi\right)u_{3,1} = 0 \\
i=3: & 4\phi u_{3,1} - \left(\frac{1}{2}+\phi\right)u_{2,1} - \left(\frac{1}{2}+\phi\right)u_{4,1} = 0 \\
i=4: & 4\phi u_{4,1} - \left(\frac{1}{2}+\phi\right)u_{3,1} - \left(\frac{1}{2}+\phi\right)u_{5,1} = 0 \\
i=5: & 4\phi u_{5,1} - \left(\frac{1}{2}+\phi\right)u_{4,1} - \left(\frac{1}{2}+\phi\right)u_{6,1} = 0 \\
i=6: & 4\phi u_{6,1} - \left(\frac{1}{2}+\phi\right)u_{5,1} - \left(\frac{1}{2}+\phi\right)u_{7,1} = 0 \\
i=7: & 4\phi u_{7,1} - \left(\frac{1}{2}+\phi\right)u_{6,1} - \left(\frac{1}{2}+\phi\right)u_{8,1} = 0 \\
i=8: & 4\phi u_{8,1} - \left(\frac{1}{2}+\phi\right)u_{7,1} - \left(\frac{1}{2}+\phi\right)u_{9,1} = 0 \\
i=9: & 4\phi u_{9,1} - \left(\frac{1}{2}+\phi\right)u_{8,1} - \left(\frac{1}{2}+\phi\right)u_{10,1} = 0 \\
i=10: & 4\phi u_{10,1} - \left(\frac{1}{2}+\phi\right)u_{9,1} - \left(\frac{1}{2}+\phi\right)u_{11,1} = 0 \\
i=11: & 4\phi u_{11,1} - \left(\frac{1}{2}+\phi\right)u_{10,1} - \left(\frac{1}{2}+\phi\right)u_{12,1} = 0 \\
i=12: & 4\phi u_{12,1} - \left(\frac{1}{2}+\phi\right)u_{11,1} - \left(\frac{1}{2}+\phi\right)u_{13,1} = 0 \\
i=13: & 4\phi u_{13,1} - \left(\frac{1}{2}+\phi\right)u_{12,1} - \left(\frac{1}{2}+\phi\right)u_{14,1} = 0 \\
i=14: & 4\phi u_{14,1} - \left(\frac{1}{2}+\phi\right)u_{13,1} - \left(\frac{1}{2}+\phi\right)u_{15,1} = 0 \\
i=15: & 4\phi u_{15,1} - \left(\frac{1}{2}+\phi\right)u_{14,1} - \left(\frac{1}{2}+\phi\right)u_{16,1} = 0
\end{array} \tag{20}$$

In matrix form, we have

(21)

$$\begin{bmatrix} 4\phi & -(\frac{1}{2}+\phi) & 0 & 0 & 0 & \dots & 0 \\ -(\frac{1}{2}+\phi) & 4\phi & -(\frac{1}{2}+\phi) & 0 & 0 & \dots & 0 \\ 0 & -(\frac{1}{2}+\phi) & 4\phi & -(\frac{1}{2}+\phi) & 0 & \dots & 0 \\ 0 & 0 & -(\frac{1}{2}+\phi) & 4\phi & -(\frac{1}{2}+\phi) & & 0 \\ \vdots & \vdots & & & & \vdots & \\ 0 & 0 & & & & & -(\frac{1}{2}+\phi) \\ 0 & 0 & 0 & \dots & -(\frac{1}{2}+\phi) & & 4\phi \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \\ \vdots \\ u_{15,1} \end{bmatrix} = \begin{bmatrix} (\frac{1}{2}+\phi)u_{0,1} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

3.2. **Numerical Implementation and Stability Analysis.** In this subsection, we carry out the implementation of the method (21). We will equally investigate the stability and consistency of the method for $\phi < 1$.

3.3. **Numerical Implementation.** We consider the following cases: case 1; $\phi = \frac{1}{2}$, case 2; $\phi = \frac{1}{4}$, case 3; $\phi = \frac{1}{8}$ and case 4; $\phi = \frac{1}{16}$ and assemble the following systems:

For $\phi = \frac{1}{2}$, we have; (Case 1)

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 2 & -1 & & 0 \\ \vdots & \vdots & & & & \vdots & \\ & & & & & & -1 \\ 0 & 0 & 0 & \dots & -1 & & 2 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \\ \vdots \\ u_{15,1} \end{bmatrix} = \begin{bmatrix} u_0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{22}$$

For $\phi = \frac{1}{4}$, we have; (Case 2)

$$\begin{bmatrix} 1 & -\frac{3}{4} & 0 & 0 & 0 & \dots & 0 \\ -\frac{3}{4} & 1 & -\frac{3}{4} & 0 & 0 & \dots & 0 \\ 0 & -\frac{3}{4} & 1 & -\frac{3}{4} & 0 & \dots & 0 \\ 0 & 0 & -\frac{3}{4} & 1 & -\frac{3}{4} & & 0 \\ \vdots & \vdots & & & & \vdots & \\ & & & & & & -\frac{3}{4} \\ 0 & 0 & 0 & \dots & -\frac{3}{4} & & 1 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \\ \vdots \\ u_{15,1} \end{bmatrix} = \begin{bmatrix} \frac{3}{4}u_0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{23}$$

For $\phi = \frac{1}{8}$, we have; (Case 3)

$$
\begin{bmatrix}
\frac{1}{2} & -\frac{5}{8} & 0 & 0 & 0 & \dots & 0 \\
-\frac{5}{8} & \frac{1}{2} & -\frac{5}{8} & 0 & 0 & \dots & 0 \\
0 & -\frac{5}{8} & \frac{1}{2} & -\frac{5}{8} & 0 & \dots & 0 \\
0 & 0 & -\frac{5}{8} & \frac{1}{2} & -\frac{5}{8} & & 0 \\
\vdots & \vdots & & & & \vdots & \\
 & & & & & & -\frac{5}{8} \\
0 & 0 & 0 & \dots & -\frac{5}{8} & & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \\ \vdots \\ u_{15,1} \end{bmatrix}
=
\begin{bmatrix} \frac{5}{8}u_0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\tag{24}
$$

For $\phi = \frac{1}{16}$, we have; (Case 4)

$$
\begin{bmatrix}
\frac{1}{4} & -\frac{9}{16} & 0 & 0 & 0 & \dots & 0 \\
-\frac{9}{16} & \frac{1}{4} & -\frac{9}{16} & 0 & 0 & \dots & 0 \\
0 & -\frac{9}{16} & \frac{1}{4} & -\frac{9}{16} & 0 & \dots & 0 \\
0 & 0 & -\frac{9}{16} & \frac{1}{4} & -\frac{9}{16} & & 0 \\
\vdots & \vdots & & & & \vdots & \\
 & & & & & & -\frac{9}{16} \\
0 & 0 & 0 & \dots & -\frac{9}{16} & & \frac{1}{4}
\end{bmatrix}
\begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \\ \vdots \\ u_{15,1} \end{bmatrix}
=
\begin{bmatrix} \frac{9}{16}u_0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\tag{25}
$$

However, the proposed method is stable for $\phi < 1$ while unstable when $\phi \geq 1$. In what follows, we consider the solution of the seepage problem for the case where $\phi = \frac{1}{2}$ only, similar approach follows for $\phi < 1$. Using MATLAB program, we obtained:

$u_{1,1} = 0.810623171965676$
$u_{2,1} = 0.7565816271627167964$
$u_{3,1} = 0.702540082370252$
$u_{4,1} = 0.648490537572541$
$u_{5,1} = 0.594456992774829$
$u_{6,1} = 0.540415447977117$
$u_{7,1} = 0.486373903179406$
$u_{8,1} = 0.432332358381694$
$u_{9,1} = 0.378290813583982$
$u_{10,1} = 0.32424926878621$
$u_{11,1} = 0.270207723988559$
$u_{12,1} = 0.216166179190847$
$u_{13,1} = 0.162124634393135$
$u_{14,1} = 0.108083089595424$
$u_{15,1} = 0.054041544797712$

Similarly, fixing $j = 2, 3$ and varying $i = 1, \dots, 15$ respectively, we have

Table 1: Seepage Problem

Node	**j=1**	**j=2**	**j=3**
i=1	0.810623171965676	0.890824623405127	0.920329088541812
i=2	0.756581627167964	0.831436315178119	0.858973815972358
i=3	0.702540082370252	0.772048006951111	0.797618543402904
i=4	0.648498537572541	0.712659698724102	0.736263270833449
i=5	0.594456992774829	0.653271390497094	0.674907998263995
i=6	0.540415447977117	0.593883082270085	0.613552725694541
i=7	0.486373903179406	0.534494774043077	0.552197453125087
i=8	0.432332358381694	0.475106465816069	0.490842180555634
i=9	0.378290813583982	0.415718157589060	0.429486907986179
i=10	0.324249268786271	0.356329849362052	0.368131635416725
i=11	0.270207723988559	0.296941541135043	0.306776362847271
i=12	0.216166179190847	0.237553232908034	0.245421090277817
i=13	0.162124634393135	0.178164924681026	0.184065817708363
i=14	0.108083089595424	0.118776616454017	0.122710545138908
i=15	0.054041544797712	0.059388308227009	0.061355272569454

From Table 1, we observe that the seepage flow decreases as i increases from $i = 1, 2, \ldots, 15$. Also, for each value of i, $i = 1, 2, \ldots, 15$, the seepage flow increased as j increases from $j = 1, 2, 3$, and the seepage flow decreases as the value of i increases steadily. However, this behaviour is due to the large volume of water at the surface and the source. The flow of the fluid also experiences friction due to viscosity and medium particles. The solutions $u(x, t)$ of the model problem (17) decreases with increase in length of the fluid seepage from the source. However, the method developed in this article is conditionally stable for $\phi = \frac{1}{2}$ and thus the fluid flow decreases too slowly.

Next, we plot the behaviour of our method in (19) for $\phi = \frac{1}{2}$ in order to study the seepage problem. Using MATLAB program, we obtained figures 3(a) to 3(d) respectively. Figure 3(a) shows the surface plot of the solution, Figure 3(b) depicts the contour plot of the solution, Figure 3(c) represents the contour plot with coloured lines, and Figure 3(d) shows the mesh plot of the solution. The Graphical representation is presented as follows:

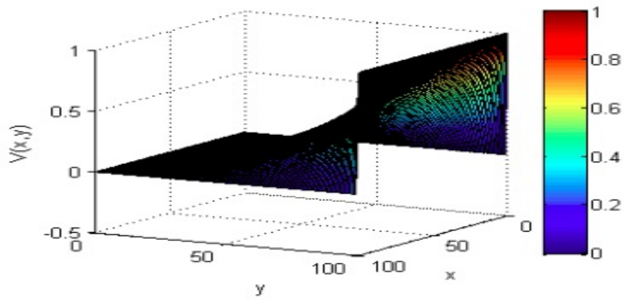

Figure 3(a): Surface plot of the solution

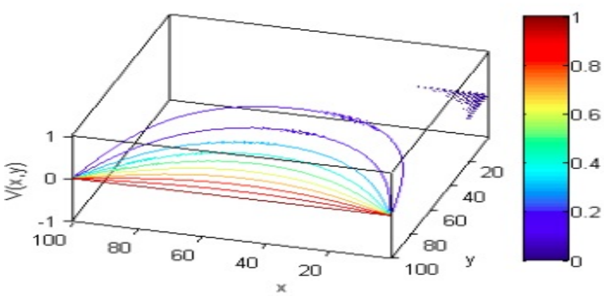

Figure 3(b): Surface plot of the solution

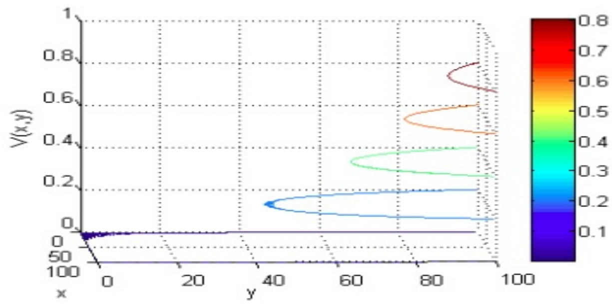

Figure 3(c): Surface plot of the solution

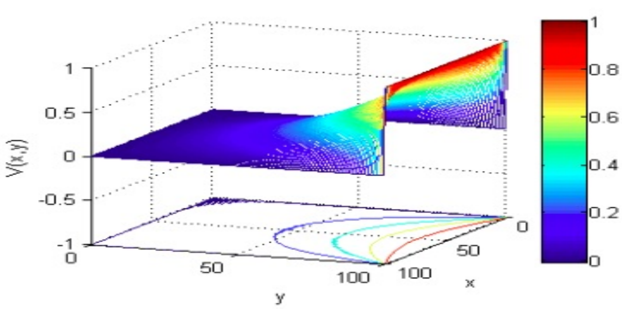

Figure 3(d): Surface plot of the solution

The scheme developed in this manuscript was found to be stable for $\phi < 1$. We also observed that the smaller rhe mesh sizes, the better the results. From figures 3(a) to 3(d), we observed that the surface of the plot is naturally not smooth. This is due to the fact that the governing differential equation is satisfied at only a selected discrete points within the region of computation. Also, given any value of x, the solution $u(x,t)$ increases to almost one as $t \to \infty$. Furthermore, for a given value of t, the solution $u(x,t)$ decreases indefinitely as $x \to \infty$. And the seepage gradually tends to zero as the the mesh size become smaller; see Table 1 which compares favourably well with the seepage solution in Nyachwaya *et al.* (2014).

3.4. **Stability Result.** We use matrix method to investigate the stability of our method. Let us consider (19)

$$(1-\phi)u_{i,j+1} - \left(\frac{1}{2}+\phi\right)u_{i-1,j} - \left(\frac{1}{2}+\phi\right)u_{i+1,j} + 4\phi u_{i,j} - \phi u_{i,j-1} = 0$$

Fixing j and varying $i = 1, 2, \ldots, N$, we have

$$\begin{array}{lcl} (1-\phi)u_{1,j+1} - \left(\frac{1}{2}+\phi\right)u_{0,j} - \left(\frac{1}{2}+\phi\right)u_{2,j} & = & \phi u_{1,j-1} - 4\phi u_{1,j} \\ (1-\phi)u_{2,j+1} - \left(\frac{1}{2}+\phi\right)u_{1,j} - \left(\frac{1}{2}+\phi\right)u_{3,j} & = & \phi u_{2,j-1} - 4\phi u_{2,j} \\ (1-\phi)u_{3,j+1} - \left(\frac{1}{2}+\phi\right)u_{2,j} - \left(\frac{1}{2}+\phi\right)u_{4,j} & = & \phi u_{3,j-1} - 4\phi u_{3,j} \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots & = & \vdots \\ (1-\phi)u_{N,j+1} - \left(\frac{1}{2}+\phi\right)u_{N-1,j} - \left(\frac{1}{2}+\phi\right)u_{N+1,j} & = & \phi u_{N,j-1} - 4\phi u_{N,j} \end{array} \tag{26}$$

For stability, we fix $\phi < 1$. Hence, in compact form, (26) becomes

$$Au = B \tag{27}$$

In what follows, we compute the eigenvalues of A using $|a - \lambda I| = 0$. Using MATLAB program, we obtain λ_i, $i = 1, \ldots, N+1$ such that $\lambda_{\max} < 1$ which guarantees the stability of our scheme (19).

3.5. **Comparison of Results.** In thus subsection, we compare our result with the seepage solution in Nyachwaya *et al.* (2014) for ADIS results case 1 and case 2.

Table 2: ADIS Results Case 1 (Nyawachaya* et al. *2014)

Node	**j= 1, 2,** $\cdots$, 15	**j=2**	**j=3**
i=1	0.6045336	0.6939972	0.7444681
i=2	0.4217462	0.5042542	0.5599788
i=3	0.2936422	0.3647182	0.4182483
i=4	0.2040735	0.2627184	0.31046636
i=5	0.1415805	0.188541	0.2291846
i=6	0.0980604	0.1348389	0.1683299
i=7	0.06780154	0.09610735	0.1230373
i=8	0.04678452	0.06825468	0.08948562
i=9	0.03219675	0.04826654	0.06471545
i=10	0.02206475	0.03392722	0.04645266
i=11	0.0150059	0.02362218	0.03296696
i=12	0.001004907	0.01615454	0.02293032
i=13	0.006506991	0.01064764	0.01533013
i=14	0.003884804	0.006444104	0.009379275
i=15	0.00181355	0.003034412	0.004446842

Table 3: ADIS Results Case 2 (Nyawachaya* et al. *2014)

Node	**j=1, 2,** $\cdots$, 15	**j=2**	**j=3**
i=1	0.7025233	0.7845406	0.8229078
i=2	0.5704533	0.6470262	0.6885759
i=3	0.462865	0.5329258	0.5750527
i=4	0.3751907	0.4382598	0.4791834
i=5	0.303697	0.3596916	0.3982389
i=6	0.2453305	0.2944249	0.3298622
i=7	0.1975912	0.2401146	0.2720229
i=8	0.1584287	0.1947920	0.2229692
i=9	0.1261562	0.1567996	0.1811884
i=10	0.09937964	0.1247361	0.1453703
i=11	0.07693952	0.09740956	0.1143734
i=12	0.0578613	0.07379606	0.08719519
i=13	0.04131437	0.0530043	0.06294429
i=14	0.02657653	0.03424338	0.04081506
i=15	0.01300325	0.01679745	0.0200664

Table 4: The Proposed L_xFM

Node	**j=1**	**j=2**	**j=3**
i=1	0.810623171965676	0.890824623405127	0.920329088541812
i=2	0.756581627167964	0.831436315178119	0.858973815972358
i=3	0.702540082370252	0.772048006951111	0.797618543402904
i=4	0.648498537572541	0.712659698724102	0.736263270833449
i=5	0.594456992774829	0.653271390497094	0.674907998263995
i=6	0.540415447977117	0.593883082270085	0.613552725694541
i=7	0.486373903179406	0.534494774043077	0.552197453125087
i=8	0.432332358381694	0.475106465816069	0.490842180555634
i=9	0.378290813583982	0.415718157589060	0.429486907986179
i=10	0.324249268786271	0.356329849362052	0.368131635416725
i=11	0.270207723988559	0.296941541135043	0.306776362847271
i=12	0.216166179190847	0.237553232908034	0.245421090277817
i=13	0.162124634393135	0.178164924681026	0.184065817708363
i=14	0.108083089595424	0.118776616454017	0.122710545138908
i=15	0.054041544797712	0.059388308227009	0.061355272569454

Tables 2 and 3 are the seepage solution obtained by ADIS methods in Nyachwaya *et al.* (2014). Table 4 depicts the solution of the seepage problem obtain by Lax-Friedrichs method. In both cases, we observe that the seepage flow decreases as i increases, and the seepage flow increases as j increases from $j = 1, 2, 3$. However, the slight discrepancy between our results in Table 1 and the results in Tables 2 and 3 are due to the conditional stability of the approach used.

Conclusion

In this paper, we developed a consistent Lax-Friedrichs method for the numerical solution of seepage problem. The scheme developed was found to be stable for $\phi < 1$, whereas instability sets in when $\phi \geq 1$. Since the method used in this work was evaluated at selected discrete points within the region of integration, the surface of the plot become naturally not smooth.

From our results, we deduced that for a given value of x, the solution $u(x,t)$ increases to almost one as $t \to \infty$. Also, for a given value of t, the solution $u(x,t)$ decreases indefinitely as $x \to \infty$. And we observed that the seepage gradually goes to null as the mesh size becomes smaller, which shows that the fluid flow seepage problem may eventually stop, see Table 1 and Figures 3(a) to 3(d).

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