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Result on the Existence and Uniqueness of the Fixed Point with Mann Iteration in a Complete G-Metric Space

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Abstract

In this research, we prove the existence and uniqueness of the fixed point in a complete G-metric space using several generalised contractive conditions formulated in G-metric settings, taking Mann iteration as our iterative procedure. Our results improve and extend some well-known results in the literature. We have utilized these concepts to deduce that Mann iterative procedure is G-convergent, G-Cauchy and G-complete in a complete $G-$metric space.

1. Introduction

A fixed point theorem required that a function F will have at least one fixed point (a point x for which $F(x) = x$), under some conditions on F that can be stated in general terms.
Ozgu and Ismet (2015) [10] in their work "Banach fixed point theorem for digital images proved Banach fixed point theorem for digital images", they gave the proof of theorem which is a generalization of the Banach contraction principle. Finally, they deal with an application of Banach fixed point theorem to image processing.

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Gain and Nashed (1971) [4] in their research titled "fixed points and stability for a sum of two operators in locally convex spaces", some fixed point theorems for a sum of two operators are proved, generalizing to locally convex spaces a fixed point theorem of M. A. Krasnoselskii, for a sum of a completely continuous and a contraction mapping, as well as some of its recent variants.
A notion of stability of solutions of nonlinear operator equations in linear topological spaces is formulated in terms of specific topologies on the set of nonlinear operators, and a theorem on the stability of fixed points of a sum of two operators is given. As a by product, sufficient conditions for a mapping to be open or to be onto are obtained.
Mustafa and Sims (2006) [8] introduced a new notion of generalized metric space called G-metric space, after proving that most of the results concerning the topological properties of D-metric space were incorrect. To repair this setback, they gave a more appropriate notion of a generalized metrics, called G-metric space, where they defined new properties for G-metric space and prove the relationship between metric space and G-metric space.
Mustafa *et al.* (2008) [7] proved some fixed point results for mapping satisfying sufficient conditions on complete G-metric space, also they showed that if the G-metric space (X, G) is symmetric, then the existence and uniqueness of these fixed point results follow from well-known theorems in usual metric space (X, d_G), where (X, d_G), is the usual metric space which defined from the G-metric space (X, G).
Rauf *et al.* (2017) [11] in their research work titled "some fixed point theorems for contractive conditions in a G-metric space" proved some fixed point theorems to show the existence and uniqueness of a fixed point under some weaker contractive conditions in complete G-metric space settings. Moreover, they obtained the G-Cauchy sequence for the unique fixed point. Their results extend and refine some recent results in the literature.
Singh, (2014) [14] proved some best proximity point theorems under generalized cyclic contraction condition which is new for this setting in the frame work of G- metric spaces. Suitable examples are also presented which substantiate the genuineness of their investigations in their note.
Mustafa and Obiedat (2010) [9] in their research work proved some fixed point results for mapping satisfying sufficient contractive conditions on a complete G-metric space, also showed that if the G-metric space (X, G) is symmetric, then the existence and uniqueness of these fixed point results follows from Reich theorems in usual metric space (X, dG), where (X, dG) the metric induced by the G-metric space (X, G).
Abbas *et al.* (2011) [1] in their research titled fixed and related fixed point theorems for three maps in G-metric spaces, using the setting of G-metric spaces, unique common fixed points of three maps that satisfy a generalized (ϕ, ψ)-weak

contractive condition are obtained. It is noted that the existence of a fixed point of any one of the mappings implies that the three mappings have a common fixed point. Also, it should be noted that metric spaces are nonlinear in nature and hence addition and scalar multiplication have no meanings in such spaces. However, a linear structure on a G-metric space can be defined so that Mann iterative process becomes well defined. Although obtaining the existing of a fixed point of contractive mappings through Mann iterative process in G-metric spaces is more tidious than obtaining through Picard iterative process. Yet, the nature of Mann iterative process in obtaining fixed point is achievable. Their results extend and generalize various well known comparable results in the existing literature.

2. Preliminary

The following definitions, examples and lemmas were given by [2] and [8].

Definition 2.1. Let X be a nonempty set, and suppose G is a function with domain consisting of all pairs of points of X and with range in $\mathbb{R}$. That is, $G : X \times X \times X \to \mathbb{R}+$, be a function satisfying the following properties:
$(G1)$ $G(x, y, z) = 0 \quad if \quad x = y = z$;
$(G2)$ $0 < G(x, x, y); \quad for \quad all \quad x, y \in X$, with $x \neq y$;
$(G3)$ $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$;
$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$, (symmetry in all three variables); and
$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).
Then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is called a G-metric space.

Example 2.2. If X is a non-empty subset of $\mathbb{R}$, then the function$G : X \times X \times X \to [0, \infty)$, given by

$$G(x, y, z) = |x - y| + |x - z| + |y - z|$$

is a G-metric on X .

Example 2.3. Every non-empty set X can be provided with the discrete G-metric , which is defined, for all $x, y, z \in X$, by

$$G(x, y, z) = \{ {}^{0,}_{1,} \quad {}^{if x=y=z}_{otherwise}$$

Example 2.4. Let $X = [0, \infty)$ be the interval of nonnegative real numbers and let G be defined by:

$G(x,y,z) = \{^{0,}_{max(x,y,z),} \quad ^{if x=y=z}_{otherwise}$
Then G is a complete G-metric on X.

Definition 2.5. Let (X,G) be a G-metric space, and let (x_n) be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence (x_n) if $lim_{n,m} G(x,x_n,x_m) = 0$, and one say that the sequence (x_n) is G-convergent to x.
Thus, that if $x_n \to 0$ in a G-metric space (X,G), then for any ϵ>0, there exists $N \in N$ such that $G(x,x_n,x_m)<\epsilon$, for all $n,m \in N$.

Definition 2.6. Let (X,G) be a G-metric space, and let (x_n) be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence (x_n) if $lim_{n,m} G(x,x_n,x_m) = 0$, and one say that the sequence (x_n) is G-convergent to x.
Thus, that if $x_n \to 0$ in a G-metric space (X,G), then for any ϵ>0, there exists $N \in N$ such that $G(x,x_n,x_m)<\epsilon$, for all $n,m \in N$.

Let (X,G) be G-metric space. Then the following are equivalent.

(1) (x_n) is G-convergent to x. for a function $f : X \to X$
(2) $G(x_n,x_n,x) = 0, as\, n \to \infty$.
(3) $G(x_n,x,x) = 0, as \quad n \to \infty$.
(4) $G(x_m,x_n,x) = 0, as \quad m,n \to \infty$.

Definition 2.7. Let (X,G) be a G-metric space, a sequence (x_n) is called G-Cauchy if given ϵ>0, there is $N \in$ such that $G(x_n,x_m,x_l)<\epsilon$, for all $n,m,l \in N$. That is $G(x_n,x_m,x_l) = 0$ as $n,m,l \to \infty$.

In a G-metric space, (X,G), the following are equivalent.

(1) The sequence (x_n) is G-Cauchy.
(2) For every ϵ>0, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_m)<\epsilon$, for all $n,m \in \mathbb{N}$.

Definition 2.8. Let $X = (X,d)$ be a metric space. A mapping $T : X \to X$ is called a contraction on X if there is a positive real number a<1 such that for all $x,y \in X$

$$d(Tx,Ty) \le ad(x,y) \qquad \text{where } (a<1)$$

Contractive condition to consider.

(1) (Banach) There exists a number a, $0 \le a < 1$, such that for each $x,y,z \in X$,

$$G(T_x,T_y,T_z) \le aG(x,y,z) \tag{2.1}$$

(2) There exists a real number $L \ge 0$ and a strict comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $x,y,z \in X$ such that

$$G(T_x,T_y,T_z) \le LG(x,T_x,T_x) + \varphi G(x,y,z) \tag{2.2}$$

(3) There exist a monotonic increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and a strict comparison function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $x, y, z \in X$,

$$G(T_x, T_y, T_z) \leq \varphi G(x, T_x, T_x) + \psi G(x, y, z)$$

where $0 \leq \varphi + \psi < 1$

(4) There exist real numbers $a, b, c, 0 \leq a + b + c < 1$, such that for each $x, y, z \in X$.

$$G(T_x, T_y, T_z) \leq aG(x, T_x, T_x) + bG(y, T_y, T_y) + cG(x, y, z)$$

(5) There exist a number $h, 0 \leq h<1$, such that, for each $x, y \in X$,

$$G(T_x, T_y, T_z) \leq h \max\{G(x, T_x, T_x), G(y, T_y, T_y)\} \tag{2.3}$$

Lemma 1 [8][2] Let (X, G) be a G-metric space. Then, for any $x, y, z, a \in X$, the following properties hold:

(1) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$;
(2) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$;
(3) If $n \neq 2$ and $x_1, x_2 \cdots x_n \in X$, then
$G(x_1, x_n, x_n) \leq \sum_{i=1}^{n-1} G(x_i, x_{i+1}, x_{i+1})$
and
$G(x_1, x_1, x_n) \leq \sum_{i=1}^{n-1} G(x_i, x_i, x_{i+1})$.
(4) If $G(x, y, z) = 0$, then $x = y = z$;
(5) $G(x, y, z) \leq G(x, a, z) + G(a, Y, z)$;
(6) $G(x, y, z) \leq \frac{2}{3}[G(x, y, a) + G(x, a, z) + G(a, y, z)]$;
(7) If $x \in X[z, a]$, then $|G(x, y, z) - G(x, y, a)| \leq G(a, x, z)$; and
(8) $G(x, y, y) \leq 2G(x, y, z)$.

Lemma 2 [2] If (X, G) is a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables, that is, if $x, y, z \in X$ and $\{x_n\}$,$\{y_n\}$, $\{z_n\} \subseteq X$ are sequence in X such that $\{x_m\} \to x, \{y_m\} \to y and \{z_m\} \to z$ then $\{G(x_m, y_m, z_m)\} \to G(x, y, z)$.

3. Main Result

Theorem 3.1. A Let (X, G) be a complete $G-$ metric space and let $T : X \to X$ be a continuous mapping satisfying the contractive condition (2.1) , where the sequence $\{x_n\}$ defined the Mann iterative procedure. Then T has a unique fixed point p in T; $T_p = p$.

Proof. In G-metric space $G(x, y, z)$, if we define $x = x_n, y = y_n$ and $y = y_n$ also if we define $x_{n+1} \neq x_n$, and $x_{n+1} = y_n = z_n$ Then

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$$

Recall that we defined our sequence $\{x_n\}$ as a Mann iteration, Therefore from equation (2.1) we have

$$\begin{aligned}
G(x_n, x_{n+1}, x_{n+1}) &= G[(1-\gamma_n)x_n + \gamma_n T x_n, (1-\gamma_n)x_{n+1} \\
&+ \gamma_n T x_{n+1}, (1-\gamma_n)x_{n+1} + \gamma_n T x_{n+1}] \\
&\leq aG[(1-\gamma_n)x_{n-1} + \gamma_n x_n, (1-\gamma_n)x_n \\
&+ \gamma_n x_{n+1}, (1-\gamma_n)x_n + \gamma_n x_{n+1}] \\
&= aG[(1-\gamma_n)Tx_{n-2} + \gamma_n T x_{n-1}, (1-\gamma_n)Tx_{n-1} \\
&+ \gamma_n T x_n, (1-\gamma_n)Tx_{n-1} + \gamma_n T x_n] \\
&\leq a^2 G[(1-\gamma_n)x_{n-2} + \gamma_n x_{n-1}, (1-\gamma_n)x_{n-1} \\
&+ \gamma_n x_n, (1-\gamma_n)x_{n-1} + \gamma_n x_n]
\end{aligned} \tag{1}$$

iteratively to n gives

$$\begin{aligned}
&\leq a^n G[(1-\gamma_n)x_0 + \gamma_n x_1, (1-\gamma_n)x_1 \\
&+ \gamma_n x_2, (1-\gamma_n)x_1 + \gamma_n x_2]
\end{aligned} \tag{2}$$

(2) show the sequence $\{x_n\}$ defined by Mann iteration is $G-$convergent.
For every $m > n \in N$
(3)

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m) \\
&\leq a^n G[(1-\gamma_n)x_0 + \gamma_n x_1, (1-\gamma_n)x_1 + \gamma_n x_2, (1-\gamma_n)x_1 + \gamma_n x_2] \\
&+ a^{n+1} G[(1-\gamma_n)x_0 + \gamma_n x_1, (1-\gamma_n)x_1 + \gamma_n x_2, (1-\gamma_n)x_1 + \gamma_n x_2] \\
&+ \cdots + a^{m-1} G[(1-\gamma_n)x_0 + \gamma_n x_1, (1-\gamma_n)x_1 + \gamma_n x_2, (1-\gamma_n)x_1 + \gamma_n x_2] \\
&\leq [a^n + a^{n+1} + \cdots + a^{m-1}] G[(1-\gamma_n)x_0 + \gamma_n x_1, (1-\gamma_n)x_1 \\
&+ \gamma_n x_2, (1-\gamma_n)x_1 + \gamma_n x_2]
\end{aligned}$$

from equation (3) we have

$$\frac{a^n(1-a^{m-1})}{1-a} G[(1-\gamma_n)x_0 + \gamma_n x_1, (1-\gamma_n)x_1 + \gamma_n x_2, (1-\gamma_n)x_1 + \gamma_n x_2]$$

$$\text{but } (1-a^{m-1}) < 1$$

hence

$$\frac{a^n}{1-a} G[(1-\gamma_n)x_0 + \gamma_n x_1, (1-\gamma_n)x_1 + \gamma_n x_2, (1-\gamma_n)x_1 + \gamma_n x_2] \tag{4}$$

By (4) it shows the sequence $\{x_n\}$ is a $G-$complete. If we call the limit of the sequence p then, the contraction (2.1) with $x = x_n$, $y = z = p$. we have

$$G(x_{n+1}, T_p, T_p) \leq aG((1-\gamma_n)x_n + \gamma_n T x_n, (1-\gamma_n)p + \gamma_n T_p, (1-\gamma_n)p + \gamma_n T_p)$$

taking limit at $n \to \infty$, using the fact that the metric G is continuous and by lemma 2 we get that

$$G(p, T_p, T_p) \leq aG(p, p, p) = 0$$

Hence we conclude that $T_p = p$ which show the fixed point p in T Suppose that q is also a fixed point of T then from (2.1) with $x = p, y = z = q$

$$\begin{aligned} G(p, q, q) &\leq \gamma G[(1-\gamma_n)p + \gamma_n T_p, (1-\gamma_n)q + \gamma_n T_q, (1-\gamma_n)q + \gamma_n T_q] \\ &\leq \gamma G(p, q, q) \\ G(p, q, q) &\leq \gamma G(p, q, q) < G(p, q, q) \end{aligned} \tag{5}$$

which is a contradiction Hence $p = q$. Hence, we conclude that there exist a unique fixed point to Mann iteration under Banach contraction in a complete $G-$metric space. □

Theorem 3.2. Let (X, G) be a complete $G-$metric space and let $T : X \to X$ be a continuous mapping satisfying the contractive condition (2.2), where the sequence $\{x_n\}$ defined the Mann iterative procedure. Then T has a unique fixed point p in T; $T_p = p$.

$$G(T_x, T_y, T_z) \leq L\ G(x, T_x, T_x) + \varphi G(x, y, z)$$

Proof. In G-metric space $G(x, y, z)$ if we define $x = x_n, y = y_n andy = y_n$ also if we define $x_{n+1} \neq x_n$, and $x_{n+1} = y_n = z_n$
Then

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$$

Recall that the sequence $\{x_n\}$ is defined as a Mann iteration. Therefore

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G[(1-\gamma_n)x_n + \gamma_n Tx_n, (1-\gamma_n)x_{n+1} \\ &+ \gamma_n Tx_{n+1}, (1-\gamma_n)x_{n+1} + \gamma_n Tx_{n+1}] \\ &\leq L\ G[(1-\gamma_n)x_{n-1} + \gamma_n x_n, (1-\gamma_n)x_n + \gamma_n x_{n+1}, (1-\gamma_n)x_n \\ &+ \gamma_n x_{n+1}] + \varphi\ a[(1-\gamma_n)x_{n-1} + \gamma n Tx_n, (1-\gamma_n)x_n \\ &+ \gamma_n Tx_n, (1-\gamma_n)x_n + \gamma_n Tx_n] \\ &= L\ G[(1-\gamma_n)Tx_{n-2} + \gamma_n Tx_{n-1}, (1-\gamma_n)Tx_{n-1} \\ &+ \gamma_n Tx_n, (1-\gamma_n)Tx_{n-1} + \gamma_n Tx_n] + \varphi\ a[(1-\gamma_n)Tx_{n-2} \\ &+ \gamma_n Tx_{n-1}, (1-\gamma_n)Tx_{n-1} + \gamma_n Tx_n, (1-\gamma_n Tx_{n-1} + \gamma_n Tx_n] \\ &\leq L^2\ G[(1-\gamma_n)x_{n-2} + \gamma_n x_{n-1}, (1-\gamma_n)x_{n-1} + \gamma_n x_n, (1-\gamma_n)x_{n-1} \\ &+ \gamma_n x_n] + \varphi^2\ G[(1-\gamma_n)x_{n-2} + \gamma_n x_{n-1}, (1-\gamma_n)x_{n-1} + \gamma_n x_n, \\ &(1-\gamma_n)x_{n-1} + \gamma_n x_n] \end{aligned}$$

iteratively gives

$$\leq L^n\ G[(1-\gamma_n)x_0+\gamma_n x_1,(1-\gamma_n)x_1+\gamma_n x_2,(1-\gamma_n)x_1+\gamma x_2]+\varphi^n\ G[(1-\gamma_n)x_0 +\gamma_n x_1,(1-\gamma_n)x_1+\gamma x_2,(1-\gamma_n)x_1+\gamma_n x_2]$$

$$\leq [L^n+\varphi^n]\ G[(1-\gamma_n)x_0+\gamma_n x_1,(1-\gamma_n)x_1+\gamma_n x_2,(1-\gamma_n)x_1+\gamma_n x_2]$$

Let $L^n+\varphi^n = J$

$$\leq J\ G[(1-\gamma_n)x_0+\gamma_n x_1,(1-\gamma_n)x_1+\gamma_n x_2,(1-\gamma_n)x_1+\gamma_n x_2]$$

continue iteratively gives

$$\leq J^n\ G[(1-\gamma_n)x_0+\gamma_n x_1,(1-\gamma_n)x_1+\gamma_n x_2,(1-\gamma_n)x_1+\gamma_n x_2]$$

which shows the sequence is $G-$convergent. For every $m > n \in N$

$$\begin{aligned} G(x_n,x_m,x_m) &\leq G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},x_{n+2},x_{n+2})+\cdots+G(x_{m-1},x_m,x_m)\\ &\leq J^n\ G[(1-\gamma_n)x_0+\gamma_n x_1,(1-\gamma_n)x_1+\gamma_n x_2,(1-\gamma_n)x_1+\gamma_n x_2]\\ &+J^{n+1}\ G[(1-\gamma_n)x_0+\gamma_n x_1,(1-\gamma_n)x_1+\gamma_n x_2,(1-\gamma_n)x_1+\gamma_n x_2]+\cdots\\ &+J^{m-1}\ G[(1-\gamma_n)x_0+\gamma_n x_1,(1-\gamma_n)x_1+\gamma_n x_2,(1-\gamma_n)x_1+\gamma_n x_2]\\ &\leq [J^n+J^{n+1}+\cdots+J^{m-1}]G[(1-\gamma_n)x_0+\gamma_n x_1,(1-\gamma_n)x_1+\gamma_n x_2,\\ &(1-\gamma_n)x_1+\gamma_n x_2] \end{aligned}$$

$$(6)\qquad \leq \frac{J^n}{1-J}G[(1-\gamma_n)x_0+\gamma_n x_1,(1-\gamma_n)x_1+\gamma_n x_2,(1-\gamma_n)x_1+\gamma_n x_2]$$

(6) prove that $\{x_n\}$ is a $G-$Cauchy. Hence $\{x_n\}$ is a $G-$complete. If the limit of $\{x_n\}$ is defined as p then (2.2) becomes

$$\begin{aligned} G(x_{n+1},T_p,T_p) &\leq L\ G[(1-\gamma_n)x_n+\gamma_n Tx_n,(1-\gamma_n)x_{n+1}+\gamma_n Tx_{n+1},(1-\gamma_n)x_{n+1}\\ &+\gamma_n Tx_{n+1}]+\varphi\ G[(1-\gamma_n)x_n+\gamma_n Tx_n,(1-\gamma_n)p\\ &+\gamma_n Tp,(1-\gamma_n)p+\gamma_n Tp] \end{aligned}$$

taking limit at $n\to\infty$, using the fact that the metric G is continuous and by lemma 2 we get that

$$G(p,T_p,T_p)\leq \varphi\ G(p,T_p,T_p)$$

since $0 \leq\ \phi\ < 1$

Hence

$p = T_p$

suppose q is also a fixed point of T. Then (2.2) becomes

$$\begin{aligned} G(p,q,q) &\leq L\ G[(1-\gamma_n)x_n+\gamma_n Tx_n,(1-\gamma_n)p+\gamma_n Tp,(1-\gamma_n)p+\gamma_n Tp]\\ &+\varphi\ G[(1-\gamma_n)x_n+\gamma_n Tx_n,(1-\gamma_n)q+\gamma_n Tp,(1-\gamma_n)q+\gamma_n Tq] \end{aligned}$$

Taking limit at $n\to\infty$

$$G(p,q,q)\leq L\ G(p,p,p)+\varphi\ G(p,q,q)$$

$$G(p,q,q) \leq \varphi\ G(p,q,q) < G(p,p,p)$$

which is a contradiction.
Hence we say

$$p = q$$

□

Theorem 3.3. Let (X,G) be a complete $G-$ metric space and let $T : X \to X$ be a continuous mapping satisfying the contractive condition (2.3), where the sequence $\{x_n\}$ defined the Mann iterative procedure. Then T has a unique fixed point p in T; $T_p = p$.

$$G(T_x,T_y,T_z) \leq h \max[(x,T_x,T_x), G(y,T_y,T_y)]$$

.

Proof. In G-metric space $G(x,y,z)$ if we define $x = x_n, y = y_n$, $y = y_n$, $x_{n+1} \neq x_n$ and $x_{n+1} = y_n = z_n$
Then

$$G(x_n,x_{n+1},x_{n+1}) = G(Tx_{n-1},Tx_n,Tx_n)$$

The sequence $\{x_n\}$ as a Mann iteration, therefore

$$\begin{aligned} G(x_n,x_{n+1},x_{n+1}) &= G[(1-\gamma_n)x_n + \gamma_n Tx_n, (1-\gamma_n)x_{n+1} + \gamma_n Tx_{n+1}, \\ &(1-\gamma_n)x_{n+1} + \gamma_n Tx_{n+1}] \\ &\leq h \max[G((1-\gamma_n)x_{n-1} + \gamma_n Tx_n, (1-\gamma_n)x_n + \gamma_n Tx_{n+1}, \\ &(1-\gamma_n)x_n + \gamma_n Tx_{n+1}), G((1-\gamma_n)x_n + \gamma_n Tx_{n+1}, (1-\gamma_n)x_{n+1} \\ &+ \gamma_n Tx_{n+2}, (1-\gamma_n)x_{n+1} + \gamma_n Tx_{n+2})] \\ &= h \max[G((1-\gamma_n)Tx_{n-2} + \gamma_n Tx_{n-1}, (1-\gamma_n)Tx_{n-1} \\ &+ \gamma_n Tx_n, (1-\gamma_n)Tx_{n-1} + \gamma_n Tx_n), G((1-\gamma_n)Tx_{n-1} \\ &+ \gamma_n Tx_n, (1-\gamma_n)Tx_n + \gamma_n Tx_{n+1}, (1-\gamma_n)Tx_n + \gamma_n Tx_{n+1})] \\ &\leq h^2 \max[G((1-\gamma_n)x_{n-2} + \gamma_n x_{n-1}, (1-\gamma_n)x_{n-1} \\ &+ \gamma_n x_n, (1-\gamma_n)x_{n-1} + \gamma_n x_n), G((1-\gamma_n)x_{n-1} + \gamma_n x_n, (1-\gamma_n)x_n \\ &+ \gamma_n x_{n+1}, (1-\gamma_n)x_n + \gamma_n x_{n+1})] \\ &\leq h^n \max[G((1-\gamma_n)x_0 + \gamma_n x_1, (1-\gamma_n)x_1 + \gamma_n x_2, (1-\gamma_n)x_1 \\ &+ \gamma_n x_2)G((1-\gamma_n)x_1 + \gamma_n x_2, (1-\gamma_n)x_2 + \gamma_n x_3, (1-\gamma_n)x_2 + \gamma_n x_3)] \\ &\leq h^n\ G[(1-\gamma_n)x_1 + \gamma_n x_2, (1-\gamma_n)x_2 + \gamma_n x_3, (1-\gamma_n)x_2 + \gamma_n x_3] \end{aligned}$$

Hence the sequence is a $G-$convergent for all $m, n > N$

$$
\begin{aligned}
G(x_n, x_m, x_m) &\leq G(x_n, x_{m+1}, x_{m+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m) \\
&\leq h^n \; G[(1-\gamma_n)x_1 + \gamma_n x_2, (1-\gamma_n)x_2 + \gamma_n x_3, (1-\gamma_n)x_2 + \gamma_n x_3] \\
&+ h^{n+1} G[(1-\gamma_n)x_1 + \gamma_n x_2, (1-\gamma_n)x_2 + \gamma_n x_3, (1-\gamma_n)x_2 + \gamma_n x_3] + \cdots \\
&+ h^{m-1} G[(1-\gamma_n)x_1 + \gamma_n x_2, (1-\gamma_n)x_2 + \gamma_n x_3, (1-\gamma_n)x_2 + \gamma_n x_3] \\
&\leq [h^n + h^{n+1} + \cdots h^{m-1}] G[(1-\gamma_n)x_1 + \gamma_n x_2, (1-\gamma_n)x_2 + \gamma_n x_3, \\
&(1-\gamma_n)x_2 + \gamma_n x_3] \frac{h^n}{1-h} G[(1-\gamma_n)x_1 + \gamma_n x_2, (1-\gamma_n)x_2 \\
&+ \gamma_n x_3, (1-\gamma_n)x_2 + \gamma_n x_3]
\end{aligned}
$$

suppose the limit of the sequence is defined as p then our contractive becomes

$$
\begin{aligned}
G(x_{n+1}, T_p, T_p) \leq h \max[&G(1-\gamma_n)x_n + \gamma_n T x_n, (1-\gamma_n)x_{n+1} \\
&+ \gamma_n T x_{n+1}, (1-\gamma_n)x_{n+1} + \gamma_n T x_{n+1}, G(1-\gamma_n)p \\
&+ \gamma_n T p, (1-\gamma_n)T_p + \gamma_n T_p, (1-\gamma_n)x_{n+1} + \gamma_n T x_{n+1}]
\end{aligned}
$$

Taking limit at $n \to \infty$ and by lemma 2

$$G(p, T_p, T_p) \leq G(p, T_p, T_p)$$

which implies

$$p = T_p$$

Suppose q is also a fixed point of T
then the contraction becomes

$$
\begin{aligned}
G(p, q, q) &\leq h \max[G(1-\gamma_n)p + \gamma_n T_p, (1-\gamma_n)p + \gamma_n T_p, (1-\gamma_n)p + \gamma_n T_p, \\
&\quad (1-\gamma_n)q + \gamma_n T_q, (1-\gamma_n)q + \gamma_n T_q, (1-\gamma_n)q + \gamma_n T_q] \\
G(p, q, q) &\leq h \max[G(p, p, p), G(q, q, q)] = 0
\end{aligned}
$$

which implies

$$p = q$$

. □

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