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On the Numerical Solution of Linear and Non-linear Fredholm Integral Equations

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Abstract

In this research article, solutions of linear and nonlinear Fredholm Integral Equations of the second kind were obtained through the Variational Iteration Method (VIM). The accuracy rate acquired from the comparison of the obtained results with the exact solution uncovers that the method is efficient and computationally effective.

1. Introduction

The standard integral equation is of the form:

$$y(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)y(t)dt \tag{1}$$

where $g(x)$ and $f(x)$ are limits of integration, λ is a constant parameter and $k(x,t)$ is a function called the kernel or nucleus of the integral equation.

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Integral equation has many application in mathematical models, such as in conformal mapping, scattering in quantum mechanics, water waves and diffraction problems.
An integral equation may be Volterra, Fredholm or Volterra-Fredholm depending on the nature of the limits of integration.
The Fredholm integral equation containing the unknown function $y(x)$ is characterized by fixed limits of integration in the form:

$$y(x) = f(x) + \lambda \int_a^b k(x,t)y(t)dt, \tag{2}$$

where a and b are constants. For its first kind, the unknown function $y(x)$ occurs only under the integral sign of the form:

$$f(x) = \int_a^b k(x,t)y(t)dt. \tag{3}$$

However, for its second kind; the unknown function $y(x)$ appears both inside and outside the integral sign of the form:

$$y(x) = f(x) + \lambda \int_a^b k(x,t)y(t)dt, \tag{4}$$

the kernel and the function $f(x)$ are given real-valued functions.
The aim of this research is to apply the Variational Iteration Method for obtaining solutions of linear and nonlinear Fredholm integral Equations of the second kind and subsequently justify the conclusion that the Variational Iteration Method is an efficient tool for solving integral equations.
Many methods with reliable efficiency and accuracy have been implemented by many researchers. The Fredholm Integral Equation was investigated and solved with the aid of Adomian Decomposition Method in [1]. Integral Mean Value Theorem was also implemented in [2] to find the numerical solutions of Fredholm Integral Equations of the second kind.
Fuzzy Transform approach to find the numerical solution of Fredholm Integral Equation of second kind was employed by [3].
The variational Iteration Method was also used by [4] to find the solution of Fredholm Integral Equation.

Method

Variational Iteration Method

To illustrate the basic concept of the Variational Iteration Method, consider this differential equation in its general form:

$$Ly + Ny = g(x) \tag{5}$$

where L is a linear operator, N is a nonlinear operator and $g(x)$ is the inhomogeneous term. According to He's Variational Iteration [5-6, 8] one can construct a correctional function as follows:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda[Ly_n(\tau) + N\tilde{y}(\tau) - g(\tau)]d\tau \tag{6}$$

where λ is a Lagrange Multiplier which can be identified optionally by VIM. The subscript n stands for the nth approximation and $\tilde{y}_n$ is referred to as the restricted variation i.e. $\partial\tilde{y}_n = 0$. The successive approximation $y_{n+1}, n \geqslant 0$ can be readily obtained and consequently, the solution is given as

$$y = \lim_{n \longrightarrow \infty} y_n$$

Solution of Fredholm Integral Equation by Variational Iteration Method

In this section, we find the numerical solution of the Fredholm Integral Equation of the form:

$$y(x) = f(x) + \int_a^b K(x,t)y(t)dt,\ a \leq x \geq b \tag{7}$$

where $k(x,t)$ is the kernel of the equation, $y(x)$ is the unknown function to be found by the Variational Iteration Method, $f(x)$ is a known function.
Equation (7) is known as the Fredholm Integral Equation of the second kind.
This can apply to nonlinear type of the form:

$$y(x) = f(x) + \int_a^b k(x,t)y^2(t)\ dt \tag{8}$$

Applying the technique to (7), we first differentiate the equation to obtain

$$y'(x) = f'(x) + \int_a^b \frac{dk(x,t)}{dx} y(t)\ dt \tag{9}$$

and the iteration becomes:

$$y_{n+1}(x) = y_n(x) + \lambda(\tau) \int_0^x \left[y_n'(\tau) - f'(\tau) - \int_a^b \frac{\partial k(\tau,t)}{\partial \tau} \tilde{y}_n(\tau)dt \right] d\tau \tag{10}$$

where $\lambda(\tau)$ can be optimally identified by making equation (9) correction functional i.e.

$$\partial y_{n+1}(x) = \partial y_n(x) + \partial \int_0^x \lambda(\tau)[y_n'(\tau) - f'(\tau) - \int_a^b \frac{\partial k(\tau,t)}{\partial(\tau)} \tilde{y}_n(t)dt]d\tau \tag{11}$$

At max/min :

$$\partial y_{n+1}(x) = 0,\ \ \lambda(\tau) = -1 \tag{12}$$

Equation (12) is true for both equations (10) and (11).

Numerical Example 1: Consider the linear Fredholm equation:

$$y(x) = e^{mx} - \frac{x}{m^2}[e^m(m-1)+1] + \int_0^1 xty(t)dt,\ \ 0 \le x \le 1 \tag{13}$$

The exact solution is $y(x) = e^{mx},\ m > 0.$

The equivalent iterative formula is:

$$y_{n+1}(x) = y_n(x) - \int_0^x \left[y_n'(\tau) - f'(\tau) - \int_0^1 \frac{\partial k(\tau,t)}{\partial \tau} y_n(t)dt\right] d\tau \tag{14}$$

where

$$y_0(x) = e^{mx} - \frac{x}{m^2}\left(e^m(m-1)+1\right) \tag{15}$$

$$f(x) = e^{mx} - \frac{x}{m^2}\left[e^m(m-1)+1\right] \tag{16}$$

$$k(x,t) = xt \tag{17}$$

Implementing (10), then we have:

$$y_1(x) = e^{mx} - \frac{(e^m(m-1)+1)}{m^2}x + \frac{2}{3}\frac{(me^m - e^m + 1)}{m^2}x \tag{18}$$

$$y_2 = e^{mx} - \frac{(e^m(m-1)+1)}{m^2}x + \frac{8}{9}\frac{(me^m - e^{mx} + 1)}{m^2}x \tag{19}$$

$$y_3 = e^{mx} - \frac{(e^m(m-1)+1)}{m^2}x + \frac{26}{27}\frac{(me^m - e^m + 1)}{m^2}x \tag{20}$$

$$\vdots$$

$$y_1(x) = e^{mx} - \frac{1}{3m^2}\left[e^m(m-1)+1\right]x \tag{21}$$

$$y_2(x) = e^{mx} - \frac{1}{3^2m^2}\left[e^m(m-1)+1\right]x \tag{22}$$

$$y_3(x) = e^{mx} - \frac{1}{3^3m^2}\left[e^m(m-1)+1\right]x \tag{23}$$

$$\vdots$$

$$\lim_{n\to\infty} y_n(x) = e^{mx}$$

which is the exact solution.

Numerical Example 2. Consider the nonlinear Fredholm Integral Equation:

$$y(x) = \frac{x}{8}\left(8c - c^2\right) + \frac{1}{2}\int_0^1 xty^2(t)\ dt \tag{24}$$

with exact solution $y(x) = cx,\ c > 0$.

The iterative technique implore is:

$$y_{n+1}(x) = y_n(x) - \int_0^x \left[y_n'(\tau) - f'(\tau) - \frac{1}{2}\int_0^1 \frac{\partial k(\tau,t)}{\partial \tau} y_n^2(t)dt\right] d\tau \tag{25}$$

where

$$y_0 = \frac{1}{8}(8c - c^2)x \tag{26}$$

$$f(x) = \frac{1}{8}(8c - c^2)x \tag{27}$$

$$k(x,t) = xt \tag{28}$$

(29)

Then

$$y_1 = \frac{1}{8}x(8c - c^2)x + 0.001953125000c^2(c-8)^2x \tag{30}$$

$$(31)\quad y_2(x) = \frac{1}{8}(8c - c^2)x + 0.00195312500c^2(c-8)^2x - 0.00975662500c^4x + 0.03125000000c^5x + 4.768371582 \times 10^{-7}c^8x - 0.00001525878906c^7x + 0.0001220703125c^6x$$

$$(32)\quad y_3(x) = \frac{1}{8}(8c - c^2)x + 0.0019531250000c^2(c-8)^2x - 0.0019531250000c^4x + 0.03125000000c^3x - 0.001953125000c^5x + 0.0000038146972266c^8x + 0.00003051757812c^7x + 0.0001220703125c^6x - 8.344650270 \times 10^{-7}c^9x - 9.313225750 \times 10^{-10}c^{12}x + 4.470348358 \times 10^{-8}c^{11}x - 2.086162568 \times 10^{-7}c^{10}x - 4.074536264 \times 10^{-10}c^{13}x + 2.842170942 \times 10^{-14}c^{16}x - 1.818989404 \times 10^{-12}c^{15}x + 4.365574568 \times 10^{-11}c^{14}x$$

Table 1: Comparison of Results obtained from the Exact Solutions and numerical solutions for various m in Example 2 in eleven different forms.

	Exact Solution	VIM Solution			% Accuracy
m	$y(x)$	$y_1(x)$	$y_2(x)$	$y_3(x)$	
0	0	0	0	0	100
0.2	$0.2x$	$0.199753x$	$0.199987x$	$0.199999x$	99.9995
0.4	$0.4x$	$0.398050x$	$0.399805x$	$0.399980x$	99.9950
0.6	$0.6x$	$0.593503x$	$0.599031x$	$0.599847x$	99.9745
0.8	$0.8x$	$0.784800x$	$0.796989x$	$0.799398x$	99.9247
1.0	$1.0x$	$0.970703x$	$0.99278x$	$0.998202x$	99.8202
1.2	$1.2x$	$1.150050x$	$1.185327x$	$1.195625x$	99.6354
1.4	$1.4x$	$1.321753x$	$1.373379x$	$1.390771x$	99.3407
1.6	$1.6x$	$1.484800x$	$1.555579x$	$1.582478x$	98.9044
1.8	$1.8x$	$1.638253x$	$1.730484x$	$1.769322x$	98.2957
2.0	$2.0x$	$1.781250x$	$1.896601x$	$1.949639x$	97.4819

The accuracies of VIM at m for $y_1(x), y_2(x)$, and $y_3(x)$ are clearly established from Table 1

Numerical Example 3. Consider the nonlinear Fredholm Integral Equation

$$y(x) = dx + \frac{d^2}{4} - \int_0^1 t y^2(t) dt \tag{33}$$

with the exact solution $y(x) = dx$, d¿0.
The iteration becomes

$$y_{n+1}(x) = y_n(x) - \int_0^x \left[y_n'(\tau) - f'(\tau) + \int_0^1 \frac{\partial k(\tau,t)}{\partial \tau} y_n^2(t) dt \right] d\tau \tag{34}$$

where

$$y_0(x) = dx \tag{35}$$

$$f(x) = dx + \frac{d^2}{4} \tag{36}$$

$$k(x,t) = t \tag{37}$$

and the following results can easily be obtained:

$$y_1 = dx \tag{38}$$

$$y_2 = dx \tag{39}$$

$$y_3 = dx \tag{40}$$

$$\vdots$$

$$\lim_{n\to\infty} y_n(x) = y(x) = dx \tag{41}$$

Discussion of Results

In example 2, as presented in Table 1, the minimum accuracy is obtained to be 89.0625% when $n = 0$ and $m = 2$ while the minimum accuracy is 97.4819% when $n = 2$ and $m = 2$.
The method converged rapidly to the exact solutions of Examples 1 and 3 as shown in the results. The method converged to the exact solution in Example 3.

Conclusion. In this paper, the Variational Iteration Method was carefully applied to find the numerical solutions to both linear and nonlinear Fredholm Integral Equation of second kind. It is evident that the results obtained are in good agreement with the exact solution which also proved that the method was powerful, efficient and reliable.

References

[1] Kirtiwant P. Ghadle and Priyanka A. Pathade and Fawziah M. Al-Saar (2018). The approximate solutions of Fredholm Integral Equations by Adomian Decomposition Method and its Modification. Int. J. Math And Appl. *b.(2-A), 327 - 336.*

[2] Avazzadeh Z., Heydari M. and Loghmani G. B. (2011). Numerical Solution of Fredholm Integral Equations of the Second Kind by using Integral Mean Value Theorem. Applied Mathematical Modelling. *35, 2374 - 2383.*

[3] Ezzati R. and Mokhtari F. (2012). Numerical Solution of Fredholm Integral Equations of he Second Kind by using fuzzy transform. Int. J. Phy Sci *7: 1528 - 1537.*

[4] Biazar J. and Ebrahimi H. (2009). Variational Iteration Method for Fredholm Integral Equations of the Second Kind. Iranian Journal of Optimization *1, 13 - 23.*

[5] He Ji-Huan (1999). Variational Iteration Method a kind of Non-Linear Analytical Technique: Some Examples. International Journal of Non-Linear Mechanics. *34, 699 - 708.*

[6] He J. H. (1997). A New Approach to Non-Linear Partial Differential Equations. Commun. Non-Linear Sci. Numer. Simulation. *2(4): 230 - 235.*

[7] Inokuti M., Sekine H. and Mura T. (1978). General Use of The Lagrange Multiplier in Non-Linear Mathematical Physics. S. Nemat-Nassar (Ed.), Variational Method in the Mechanics of Solids. *Pergamon Press, Oxford, 1978 pp. 158 - 162.*

[8] Olayiwola M. O. (2016). Variational Iteration Method: A Computational Tool for Solving Coupled System of Non- linear Partial Differential Equations. Journal of Science and Arts. *16, 3(36): 243 - 248.*