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Laplace Homotopy Perturbation Method for Solving Fourth Order Parabolic Partial Differential Equations with Variable Coefficients

O. O. Olubanwo and O. S. Odetunde

Abstract

In this paper, we present a reliable combination of Laplace Transform and Homotopy perturbation method to solve one dimensional fourth order parabolic linear partial differential equations with variable coefficients. Some cases of one dimensional fourth order parabolic linear partial differential equations are considered to illustrate the ability and reliability of mixture of Laplace Transform and Homotopy perturbation method. We have compared the analytical solution obtained with the available Laplace decomposition solution and homotopy perturbation method of solution which is found to be exactly same. The result revealed that the combination of the Laplace Transform and homotopy perturbation method is practically well appropriate for use in such problems.

1. Introduction

Several problems arising in science and engineering are described by linear or non-linear Partial Differential Equations. Initial and boundary value problems occur in diverse field such as mechanics, mathematical physics, astrophysics, biology, electromagnetism, etc. Partial Differential Equations can be broadly classified as

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Department of Mathematical Sciences, Olabisi Onabanjo University Ago Iwoye, Nigeria
E-mail: olubanwo.oludapo@oouagoiwoye.edu.ng

elliptic, parabolic and hyperbolic. In this paper our discussion will be basically based on fourth order parabolic Partial Differential Equations. These Partial Differential Equations describe various physical phenomenon including free vibration of beams [8], steady natural convection [14], etc.
In recent years, several researchers have proposed various methods for solving the fourth order parabolic PDEs. Some of these methods are: Adomian decomposition method [15, 16], Variational iteration method [13, 4], B-spline method [12, 2], Homotopy perturbation method [5, 6] Laplace decomposition algorithm [11], Aboodh Transform Homotopy perturbation method [7], Elzaki Transform Homotopy perturbation method [3]. In this paper, we use the combination of Laplace transform and homotopy perturbation method. This method is very helpful in solving linear and nonlinear differential equations. The major aim of this paper is to consider the applicability and effectiveness of Laplace homotopy perturbation method in solving fourth order parabolic partial differential equations with variable coefficients. Solutions obtained are presented in series of rapidly convergent terms.

2. Laplace Homotopy Perturbation method

Consider a one-dimensional linear nonhomogeneous fourth order parabolic partial differential equation with variable coefficients of the form **[1, 3, 7, 11]**

$$\frac{\partial^2 U}{\partial t^2} + \varphi(x)\frac{\partial^4 U}{\partial x^4} = \emptyset(x,t) \tag{1}$$

where $\varphi(x)$ is a variable coefficient, with initial conditions $U(x,0) = f(x)$ and

$$\frac{\partial U}{\partial t}(x,0) = h(x) \tag{2}$$

and boundary conditions:
$U(a,t) = \beta_1$, $\mathrm{U(b,t)} = \beta_2(\mathrm{t})$

$$\frac{\partial^2 U}{\partial x^2}(a,t) = \beta_3(\mathrm{t}), \frac{\partial^2 U}{\partial x^2} = \beta_4(t) \tag{3}$$

Applying Laplace Transform on both sides of (1),

$$L\left\{\frac{\partial^2 U}{\partial t^2} + \varphi(x)\frac{\partial^4 U}{\partial x^4}\right\} = L\{\emptyset(x,t)\} \tag{4}$$

Applying the linearity property of the Laplace Transform on (4)

$$L\left\{\frac{\partial^2 U}{\partial t^2}\right\} + L\left\{\varphi(x)\frac{\partial^4 U}{\partial x^4}\right\} = L\left\{\emptyset(x,t)\right\} \tag{5}$$

$$s^2 U(x,t) - sU(x,0) - \frac{\partial U}{\partial t}'(x,0) + L\{\varphi(x)\frac{\partial^4 U}{\partial x^4}\} = L\{\emptyset(x,t)\} \tag{6}$$

$$s^2U(x,t) - sU(x,0) - \frac{\partial U}{\partial t}(x,0) = L\{\emptyset(x,t)\} - L\{\varphi(x)\frac{\partial^4 U}{\partial x^4}\} \tag{7}$$

By using the initial condition in (7), we obtain

$$s^2U(x,s) - sf(x) - h(x) = L\{\emptyset(x,t)\} - L\{\varphi(x)\frac{\partial^4 U}{\partial x^4}\}$$

$$s^2U(x,s) = sf(x) + h(x) + L\{\emptyset(x,t)\} - L\{\varphi(x)\frac{\partial^4 U}{\partial x^4}\}$$

$$U(x,s) = \frac{1}{s}f(x) + \frac{1}{s^2}h(x) + L\{\emptyset(x,t)\} - \frac{1}{s^2}L\{\varphi(x)\frac{\partial^4 U}{\partial x^4}\} \tag{8}$$

Let $g(x,s) = \frac{1}{s}f(x) + \frac{1}{s^2}h(x) + L\{\emptyset(x,t)\}$

$$U(x,s) = g(x,s) - \frac{1}{s^2}L\{\varphi(x)\frac{\partial^4 U}{\partial x^4}\} \tag{9}$$

Applying Laplace inverse on both sides of (9), we obtain

$$U(x,t) = L^{-1}\{g(x,s)\} - L^{-1}\{\frac{1}{s^2}L\{\varphi(x)\frac{\partial^4 U}{\partial x^4}\}\} \tag{10}$$

$U(x,t) = K(x,t) - \{\frac{1}{s^2}L\{\varphi(x)\frac{\partial^4 U}{\partial x^4}\}\}$ where $K(x,t) = L^{-1}\{g(x,s)\}$ represent the terms arising from the source term and the prescribed initial conditions.

Now, we apply the homotopy perturbation method.

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t) \tag{11}$$

By substituting (11) into (10), we have

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = K(x,t) - pL^{-1}\{\frac{1}{s^2}L\{\varphi(x)\frac{\partial^4 U}{\partial x^4}\} \tag{12}$$

This is the coupling of the Laplace Transform and the homotopy perturbation method. Comparing the coefficients of like powers of p, we have

$$p^0 : U_0(x,t) = K(x,t)$$

$$p^1 : U_1(x,t) = -L^{-1}\{\frac{1}{s^2}L\{\varphi(x)\,U_{0\ xxxx}(x,t)\}\}$$

$$p^2 : U_2(x,t) = -\ L^{-1}\ \{\frac{1}{s^2}L\{\varphi(x)\,U_{1\ xxxx}(x,t)\}\}$$

p^3: $U_3(x,t) = -\ L^{-1}\ \{\frac{1}{s^2}L\{\varphi(x)\,U_{2xxxx}(x,t)\}\}$ and so on.

In general, the recursive relation is given by:

$$p^m : U_m(x,t) = -\ L^{-1}\ \{\frac{1}{s^2}L\{\varphi(x)\,U_{(m-1)\ xxxx}(x,t)\}\}$$

Then, the solution can be expressed as:

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \cdots \tag{13}$$

3. Application

In this section, in order to demonstrate the applicability and efficiency of the Laplace Homotopy perturbation method of solving fourth order parabolic partial differential equations.

Example 1

Consider the following one dimensional fourth order homogenous partial differential equation [1, 3] as:

$$\frac{\partial^2 U}{\partial t^2} + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 U}{\partial x^4} = 0, \frac{1}{2} < x < 1,\ t > 0 \tag{14}$$

with initial conditions:

$$U(x,0) = 0, \quad \frac{\partial U}{\partial t}(x,0) = \left(\frac{1}{x} + \frac{x^5}{120}\right) \tag{15}$$

$$U(0.5,t) = \left(1 + \frac{(0.5)^5}{120}\right) sint, \quad U(1,t) = \frac{121}{120} sint \tag{16}$$

$$\frac{\partial^2 U}{\partial x^2}(0.5,t) = 0.02084 sint, \quad \frac{\partial^2 U}{\partial x^2}(1,t) = \frac{1}{6} sint$$

Applying Laplace Transform to (14)

$$L\left\{\frac{\partial^2 U}{\partial t^2}\right\} + L\left\{\left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 U}{\partial x^4}\right\} = 0$$

$$s^2 U(x,s) - sU(x,0) - \frac{\partial U}{\partial t}(x,0) + L\left\{\left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 U}{\partial x^4}\right\} = 0$$

Using the initial conditions from (15), we have

$$s^2 U(x,s) - \left(1 + \frac{x^5}{120}\right) = -L\left\{\left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 U}{\partial x^4}\right\}$$

$$U\left(x,s\right) = \frac{1}{s^2}\left(1 + \frac{x^5}{120}\right) - \frac{1}{s^2} L\left\{\left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 U}{\partial x^4}\right\} \tag{17}$$

Now taking the inverse Laplace Transform on both sides

$$U(x,s) = L^{-1}\left\{\frac{1}{s^2}\left(1 + \frac{x^5}{120}\right)\right\} - L^{-1}\left\{\frac{1}{s^2} L\left\{\left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 U}{\partial x^4}\right\}\right\}$$

$$U(x,t) = \left(1 + \frac{x^5}{120}\right) t - L^{-1}\left\{\frac{1}{s^2} L\left\{\left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 U}{\partial x^4}\right\}\right\} \tag{18}$$

Now applying homotopy perturbation method,

(19) $$U\left(x,t\right)=\sum_{n=0}^{\infty}p^{n}U_{n}\left(x,t\right)$$

Substituting (18) into (19)

(20)
$$\sum_{n=0}^{\infty}p^{n}U_{n}\left(x,t\right)=\left(1+\frac{x^{5}}{120}\right)t-pL^{-1}\left\{\frac{1}{s^{2}}L\left\{\left(\frac{1}{x}+\frac{x^{4}}{120}\right)\left(\sum_{n=0}^{\infty}p^{n}U_{n}(x,t)\right)_{xxxx}\right\}\right\}$$

Comparing the coefficients of the corresponding powers of p in (20)

$$p^{0}\colon\quad U_{0}\left(x,t\right)=K\left(x,t\right)=\left(1+\frac{x^{5}}{120}\right)t$$

$$p^{1}\colon\quad U_{1}\left(x,t\right)=-L^{-1}\left\{\frac{1}{s^{2}}L\left\{\left(\frac{1}{x}+\frac{x^{4}}{120}\right)U_{0\ xxxx}\left(x,t\right)\right\}\right\}$$

$$=-L^{-1}\left\{\frac{1}{s^{2}}L\left\{\frac{1}{x}+\frac{x^{4}}{120}\right\}xt\right\}$$

$$U_{1}\left(x,t\right)=-\left(1+\frac{x^{5}}{120}\right)\frac{t^{3}}{3!}$$

$$p^{2}\colon\quad U_{2}\left(x,t\right)=-L^{-1}\left\{\frac{1}{s^{2}}L\left\{\left(\frac{1}{x}+\frac{x^{4}}{120}\right)U_{1\ xxxx}\left(x,t\right)\right\}\right\}$$

$$U_{2}\left(x,t\right)=-L^{-1}\left\{\frac{1}{s^{2}}L\left\{\left(\frac{1}{x}+\frac{x^{4}}{120}\right)\left(-x\frac{t^{3}}{3!}\right)\right\}\right\}$$

$$U_{2}\left(x,t\right)=\left(1+\frac{x^{5}}{120}\right)\frac{t^{5}}{5!}$$

$$p^{3}\colon\quad U_{3}\left(x,t\right)=-L^{-1}\left\{\frac{1}{s^{2}}L\left\{\left(\frac{1}{x}+\frac{x^{4}}{120}\right)U_{2\ xxxx}\left(x,t\right)\right\}\right\}$$

$$U_{3}\left(x,t\right)=-L^{-1}\left\{\frac{1}{s^{2}}L\left\{\left(\frac{1}{x}+\frac{x^{4}}{120}\right)\left(x\frac{t^{5}}{5!}\right)\right\}\right\}$$

$$U_{3}\left(x,t\right)=-\left(1+\frac{x^{5}}{120}\right)\frac{t^{7}}{7!}$$

$$p^{4}\colon\quad U_{4}\left(x,t\right)=-L^{-1}\left\{\frac{1}{s^{2}}L\left\{\left(\frac{1}{x}+\frac{x^{4}}{120}\right)U_{3\ xxxx}\left(x,t\right)\right\}\right\}$$

$$U_{4}\left(x,t\right)=-L^{-1}\left\{\frac{1}{s^{2}}L\left\{\left(\frac{1}{x}+\frac{x^{4}}{120}\right)\left(-x\frac{t^{7}}{7!}\right)\right\}\right\}$$

$$U_{4}\left(x,t\right)=\left(1+\frac{x^{5}}{120}\right)\frac{t^{9}}{9!}$$

$\vdots$

p^m: $U_m(x,t) = -L^{-1}\left\{\left\{\left\{\frac{1}{s^2}L\left(\frac{1}{x}+\frac{x^4}{120}\right)\right\}U_{(m-1)xxxx}(x,t)\right\}\right\}$

$U_m(x,t) = (-1)^m\left(1+\frac{x^5}{120}\right)\frac{t^m}{(2m+1)!},\ m = 1, 2,\ 3, 4$

The rest of the components of iteration formula can be obtained by following the same procedure.
Thus, the solution can be written in the closed form as:

$$U(x,t) = \left(1+\frac{x^5}{120}\right)t - \left(1+\frac{x^5}{120}\right)\frac{t^3}{3!} + \left(1+\frac{x^5}{120}\right)\frac{t^5}{5!} - \left(1+\frac{x^5}{120}\right)\frac{t^7}{7!} + \left(1+\frac{x^5}{120}\right)\frac{t^9}{9!}\cdots$$

$$U(x,t) = \left(1+\frac{x^5}{120}\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\frac{t^9}{9!}-\ \cdots\right)$$

(21)

$$U(x,t) = \left(1+\frac{x^5}{120}\right) sint$$

Equation (21) is the exact solution for (14) which is the same as the solution in [1, 3].
The solution holds at all the initial and boundary points on the stipulated interval of the problem.

Example 2

Consider the fourth order homogenous partial differential equation [1, 3]

$$\frac{\partial^2 U}{\partial t^2} + \left(\frac{x}{sinx}-1\right)\frac{\partial^4 U}{\partial x^4} = 0, 0 < x < 1,\quad t > 0 \tag{22}$$

with the following initial conditions:

$$U(x,0) = x - sinx \ and\ \frac{\partial U}{\partial t}(x,0) = -x + sinx \tag{23}$$

and boundary conditions

$$\frac{\partial^2 U}{\partial x^2}(0,t) = 0, \frac{\partial^2 U}{\partial x^2}(1,t) = e^{-t}sin1 \tag{24}$$

Applying the Laplace transform into (22)

$$L\left\{\frac{\partial^2 U}{\partial t^2} + \left(\frac{x}{sinx}-1\right)\frac{\partial^4 U}{\partial x^4}\right\} = 0,$$

Applying the linearity property of Laplace transformation

$$L\left\{\frac{\partial^2 U}{\partial t^2}\right\} + L\left\{\left(\frac{x}{sinx}-1\right)\frac{\partial^4 U}{\partial x^4}\right\} = 0$$

$$s^2U(x,s) - sU(x,0) - \frac{\partial U}{\partial t} + L\left\{\left(\frac{x}{sinx} - 1\right)\frac{\partial^4 U}{\partial x^4}\right\} = 0$$

Using the initial boundary conditions from (23)

(25) $$U(x,s) = \frac{1}{s}(x - sinx) + \frac{1}{s^2}(-x + sinx) - \frac{1}{s^2}L\left\{\left(\frac{x}{sinx} - 1\right)\frac{\partial^4 U}{\partial x^4}\right\}$$

Taking the Laplace inverse of both sides of (25)

(26)

$$U(x,t)(x,t) = (x - sinx) + (-x + sinx)t - L^{-1}\left\{\frac{1}{s^2}L\left\{\left(\frac{x}{sinx} - 1\right)\frac{\partial^4 U}{\partial x^4}\right\}\right\}$$

Applying the Homotopy perturbation method,

(27) $$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t)$$

Substituting (27) into (26), we obtain

(28)

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = (x - sinx) + (-x + sinx)t - pL^{-1}\left\{\frac{1}{s^2}L\left\{\left(\frac{x}{sinx} - 1\right)\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xxxx}\right\}\right\}$$

Collecting the coefficients of the corresponding powers of p in (28), the following results are obtained.

$$p^0: \quad U_0(x,t) = K(x,t) = (x - sinx) + (-x + sinx)t$$

$$p^1: \quad U_1(x,t)(x,t) = -L^{-1}\left\{\frac{1}{s^2}L\left\{\left(\frac{x}{sinx} - 1\right)U_{0\ xxxx}(x,t)\right\}\right\}$$

$$= -L^{-1}\left\{\frac{1}{s^2}L\left\{\left(\frac{x}{sinx} - 1\right)(-sinx + tsinx)\right\}\right\}$$

(29)

$$U_1(x,t) = (x - sinx)\left(\frac{t^2}{2!} - \frac{t^3}{3!}\right)$$

$$p^2: \quad U_2(x,t) = -L^{-1}\left\{\frac{1}{s^2}L\left\{\left(\frac{x}{sinx} - 1\right)U_{1\ xxxx}(x,t)\right\}\right\}$$

$$= -L^{-1}\left\{\frac{1}{s^2}L\left\{\left(\frac{x}{sinx} - 1\right)(-sinx)\left(\frac{t^2}{2!} - \frac{t^3}{3!}\right)\right\}\right\}$$

(30) $$U_2(x,t) = (x - sinx)\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right)$$

$$p^3: \quad U_3(x,t) = -L^{-1}\left\{\frac{1}{s^2}L\left\{\left(\frac{x}{sinx} - 1\right)U_{2\ xxxx}(x,t)\right\}\right\}$$

$$= -L^{-1}\ \left\{\frac{1}{s^2}L\left\{\left(\frac{x}{sinx}-1\right)(-sinx)\left(\frac{t^4}{4!}-\frac{t^5}{5!}\right)\right\}\right\}$$

$$= (x-sinx)\left(\frac{t^6}{6!}-\frac{t^7}{7!}\right) \tag{31}$$

$$p^4: \quad U_4(x,t) = -L^{-1}\ \left\{\frac{1}{s^2}L\left\{\left(\frac{x}{sinx}-1\right)U_{3\ xxxx}(x,t)\right\}\right\}$$

$$= -L^{-1}\ \left\{\frac{1}{s^2}L\left\{\left(\frac{x}{sinx}-1\right)(-sinx)\left(\frac{t^5}{6!}-\frac{t^7}{7!}\right)\right\}\right\}$$

$$= (x-sinx)\left(\frac{t^8}{8!}-\frac{t^9}{9!}\right)$$

In general

$$p^m: \quad U_m(x,t) = -L^{-1}\ \left\{\frac{1}{s^2}L\left\{\left(\frac{x}{sinx}-1\right)U_{(m-1)\ xxxx}(x,t)\right\}\right\}$$

$$= (x-sinx)\left(\frac{t^{2m}}{2m!}-\frac{t^{2m+1}}{(2m+1)!}\right),\ for\ m=1,2,3,4,\cdots \tag{32}$$

The rest of the components of iteration formula can be computed by following the same procedure. Thus, the solution can be written as:

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + U_3(x,t)$$

$$U(x,t) = (x-sinx)+(-x+sinx)\,t+(x-sinx)\left(\frac{t^2}{2!}-\frac{t^3}{3!}\right)+(x-sinx)\left(\frac{t^4}{4!}-\frac{t^5}{5!}\right)$$

$$+(x-sinx)\left(\frac{t^6}{6!}-\frac{t^7}{7!}\right)+(x-sinx)\left(\frac{t^8}{8!}-\frac{t^9}{9!}\right)+\ldots$$

(33)

$$U(x,t) = (x-sinx)\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\frac{t^5}{5!}+\frac{t^6}{6!}-\frac{t^7}{7!}+\frac{t^8}{8!}-\frac{t^9}{9!}+\ldots\right)$$

The solution $U(x,t)$ can be written closed form as:

$$U(x,t) = (x-sinx)\,e^{-t} \tag{34}$$

Equation (34) is the exact solution for (22) for which is the same as the solution in [1, 3].
It can also be verified by substituting (34) into (23) and (24).

Example 3

Consider the following one dimensional non-homogenous fourth order parabolic differential equation [1, 3, 10]

$$\frac{\partial^2 U}{\partial t^2}+(1+x)\frac{\partial^4 U}{\partial x^4} = \left(x^4+x^3-\frac{6}{7!}x^7\right)cost,\ 0<x<1,\ t>0 \tag{35}$$

Subject to the initial conditions:

$$U_0(x,t)=\frac{6}{7!}x^7,\ \frac{\partial U}{\partial t}(x,0)=0 \tag{36}$$

and boundary conditions: $U(0,t)=0$, $U(1,t)=\frac{6}{7!}\cos t$ and

$$\frac{\partial^2 U}{\partial x^2}(0,t)=0,\ \frac{\partial^2 U}{\partial x^2}(1,t)=\frac{1}{20}cost \tag{37}$$

whose exact solution is

$$U(x,t)=\frac{6}{7!}x^7cost \tag{38}$$

Applying the Laplace transform on (35)

$$L\left\{\frac{\partial^2 U}{\partial t^2}\right\}+L\left\{(1+x)\frac{\partial^4 U}{\partial x^4}\right\}=L\left\{\left(x^4+x^3-\frac{6}{7!}x^7\right)cost\right\}$$

$$s^2U(x,s)-sU(x,0)-\frac{\partial U}{\partial t}(x,0)=-L\left\{(1+x)\frac{\partial^4 U}{\partial x^4}\right\}\frac{s}{s^2+1}\left(x^4+x^3-\frac{6}{7!}x^7\right)$$

Using the initial conditions from (36)

$$s^2U(x,s)-\frac{6}{7!}sx^7=\frac{s}{s^2+1}\left(x^4+x^3-\frac{6}{7!}x^7\right)-L\left\{(1+x)\frac{\partial^4 U}{\partial x^4}\right\}$$

Taking the Laplace inverse of both sides, we obtain

(39)

$$U(x,t)=\frac{6}{7!}+(1-cost)\left(x^4+x^3-\frac{6}{7!}x^7\right)-L^{-1}\left\{\frac{1}{s^2}L\left\{(1+x)\frac{\partial^4 U}{\partial x^4}\right\}\right\}$$

Applying the Homotopy perturbation method

$$U(x,t)=\sum_{n=0}^{\infty}p^nU_n(x,t) \tag{40}$$

(41)

$$\sum_{n=0}^{\infty}p^nU_n(x,t)=\frac{6}{7!}x^7+(1-cost)\left(x^4+x^3-\frac{6}{7!}x^7\right)-\quad pL^{-1}\left\{\frac{1}{s^2}L\left\{(1+x)\left(\sum_{n=0}^{\infty}p^nU_n(x,0)\right)_{xxxx}\right\}\right\}$$

Comparing the coefficients of the corresponding powers of p in (41)

Here we made an assumption that

$$K(x,t)=\left(x^4+x^3-\frac{6}{7!}x^7\right)(1-cost) \tag{42}$$

can be splitted into the sum of two parts namely, $K_0(x,t)$ and $K_1(x,t)$).

Therefore, from [9], we have

$$K(x,t)=K_0(x,t)+K_1(x,t) \tag{43}$$

Under this assumption, a slight variation is proposed only in the components U_0 *and* U_1. The variation we proposed is that only the part $K_0(x,t)$ can be allocated to U_0 , whereas, the remaining part $K_1(x,t)$ can be included with the other terms to define U_1 .

$$K_0(x,t) = \frac{6}{7!}\ x^7 cost \tag{44}$$

$$K_1(x,t) = \left(x^3 + x^4\right)(1 - cost) \tag{45}$$

In view of these, a modified recursive algorithm is formulated as follows:

$$p^0 : \qquad U_0(x,t) = \frac{6}{7!}\ x^7 cost \tag{46}$$

$$p^1 : \qquad U_1(x,t) = \left(x^3 + x^4\right)(1 - cost) - L^{-1}\left\{\frac{1}{s^2} L\left\{(1+x)\, U_{0\ xxxx}(x,t)\right\}\right\}$$
$$= 0$$

$$p^2 : \qquad U_2(x,t) = -L^{-1}\left\{\frac{1}{s^2} L\left\{(1+x)\, U_{1\ xxxx}(x,t)\right\}\right\}$$
$$= 0$$

$$p^3 : \qquad U_3(x,t) = -L^{-1}\left\{\frac{1}{s^2} L\left\{(1+x)\, U_{2\ xxxx}(x,t)\right\}\right\}$$
$$= 0$$

$$p^4 : \qquad U_4(x,t) = -L^{-1}\left\{\frac{1}{s^2} L\left\{(1+x)\, U_{3\ xxxx}(x,t)\right\}\right\}$$
$$= 0$$

The rest of the components of the iteration formula can be computed by following the same procedure

$$U_m(x,t) = 0,\ for\ all\ values\ of\ m \geq 1. \tag{47}$$

The solutions can be written as:

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + U_3(x,t) + U_4(x,t) + \ldots$$
$$U(x,t) = \frac{6}{7!}\ x^7 cost + 0 + 0 + 0 + 0 + \ldots$$

The solution to (35) can be written in closed form as:

$$U(x,t) = \frac{6}{7!}\ x^7 cost \tag{48}$$

Equation (48) is the exact solution for (35) which is the same as (38). The result obtained is the same as [1, 3, 10]. It can be verified through substitution.

Conclusion

In this paper, we introduced the Laplace homotopy perturbation method which is the combination of Laplace transform and homotopy perturbation method to solve one-dimensional fourth order parabolic partial differential equations. The main advantage of this method is that, it provides the user an analytical approximation to the solution in series of rapidly convergent sequence with elegantly computed terms. Its rapid convergence shows that the method is trustworthy and introduces a significant advancement in solving linear partial differential equations over existing methods.

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