Nigerian Journal of Mathematics and Applications
Volume **27**, (2018), 28 − 41. *Printed by Unilorin press*
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Application of Variational Iteration Method to Linear and Non-Linear Stiff Differential Equations

M. O. Olayiwola

Abstract

In this paper, the variational iteration method (VIM) is used to solve linear, non-linear and system of stiff differential equations numerically. The numerical solutions obtained is in good agreement with the exact solutions of problems under investigations. The method is a promising tool for the solution of stiff differential equations for the three systems.

1. Introduction

A stiff differential equation is so called when the solution of the system contains components which change significantly at different rates for a particular change in the independent variable. These equations arise in electrical circuits, kinetics, vibrations, combustion, chemical reactions and theory of fluid mechanics.
In 1987, Inokuti et. al [1] proposed a general Lagrange multiplier method to solve nonlinear problems. This method was later modified by Ji-Huan He into an iteration method [2-4]. Different types of linear and non linear differential equations can be solved by the variational iteration method [5-6]. The method requires less computational time and converges rapidly to the exact solution. Many researchers have used other methods in [7-12]. The basic idea of the He's Variational Iteration Method (VIM) [2-4], can be explained by considering the

Received: 29/05/2018, Accepted: 12/12/2018, Revised: 29/12/2018.
2015 *Mathematics Subject Classification.* 26A18, & 34A34.
Key words and phrases. VIM, Stiff differential equations linear, nonlinear and systems of differential equations
Department of Mathematical Sciences, Faculty of Basic and Applied Sciences, College of Science, Engineering and Technology, Osun State University, Osogbo, Nigeria.
E-mail: olayiwola.oyedunsi@uniosun.edu.ng, +2348131109234

following algorithm.
Given a nonlinear equation:

$$Ly + Ny = g(x) \tag{1}$$

where L is the linear operator, N is the nonlinear operator and $g(x)$ is the inhomogeneous term. According to the method, the corresponding variational iteration method for solving (1) is given as:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau)\,[Ly_n(\tau) + Ny_n(\tau) - g(\tau)]d\tau] \tag{2}$$

where λ is a Lagrange multiplier which can be identified optimally by variational iteration method. The subscript n denote the nth approximation, y_n is considered as a restricted variation i.e $\delta y_n = 0$. The successive approximation $y_{n+1}, n \geq 0$ of the solution u can be easily obtained by determine the Lagrange multiplier and the initial guess y_0, consequently, the solution is given as

$$y(x) = \lim_{n\to\infty} y_n$$

.

APPLICATION OF VARIATIONAL ITERATION METHOD

Problem 1:
We shall consider the following stiff ODE equation

$$\frac{d^2y}{dx^2} + (k+1)\frac{dy}{dx} + ky = 0 \tag{3}$$

with the initial conditions: $y(0) = 1,\ y'(0) = -1$
The exact solution of the equation is $y = e^{-x}$
However, we also have unwanted solution e^{-kx} which also satisfies equation (3), where k is any arbitrary constant between 0 and 1.

The iteration formula based on (2) is given by:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda[y_n''(\tau) + (k+1)y_n'(\tau) + ky_n(\tau)]d\tau \tag{4}$$

We take $\lambda = \tau - x$ which optimises equation (4)
when $n = 0$

$$y_0 = 1 - x \tag{5}$$

$$y_1(x) = y_0(x) + \int_0^x (\tau - x)[y_0''(\tau) + (k+1)y_0'(\tau) + ky_0(\tau)]d\tau \tag{6}$$

$$where\ y_0(\tau) \quad = 1-\tau,\ y_0^{'}(\tau) = -1,\ y_0^{''} = 0 \tag{7}$$

$$y_1(x) \quad = 1 - x + \textstyle\int_0^x (\tau - x)[0 + (k+1)(-1) + k(1-\tau)]d\tau$$

$$= 1 - x + \frac{x^2}{2} + \frac{kx^3}{6} \tag{8}$$

when $n = 1$

$$y_1(\tau) \quad = 1 - \tau + \frac{\tau^2}{2} + \frac{k\tau^3}{6} \tag{9}$$

$$y_1'(\tau) \quad = -1 + \tau + \frac{k\tau^2}{2} \tag{10}$$

$$y_1''(\tau) \quad = 1 + k\tau \tag{11}$$

$$y_2(x) \quad = 1 - x + \frac{x^2}{2} + \frac{kx^3}{6} + \textstyle\int_0^x (\tau - x)\left[1 + k\tau + (k+1)\left(-1 + \tau + \frac{k\tau^2}{2}\right) + k\left(1 - \tau + \frac{\tau^2}{3} + \frac{k\tau^3}{6}\right)\right]d\tau$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{kx^4}{12} - \frac{k^2x^4}{24} - \frac{k^2x^5}{120} \tag{12}$$

when $n = 2$

$$y_2(\tau) \quad = 1 - \tau + \frac{\tau^2}{2} - \frac{\tau^3}{6} - \frac{k\tau^4}{12} - \frac{k^2\tau^4}{24} - \frac{k^2\tau^5}{120} \tag{13}$$

$$y_2'(\tau) \quad = -1 + \tau - \frac{\tau^2}{2} - \frac{k\tau^3}{3} - \frac{k^2\tau^3}{6} - \frac{k^2\tau^4}{24} \tag{14}$$

$$y_2''(\tau) \quad = 1 - \tau - k\tau^2 - \frac{k^2\tau^2}{2} - \frac{k^2\tau^3}{6} \tag{15}$$

$$\begin{aligned} y_3(x) \quad &= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{kx^4}{12} - \frac{k^2x^4}{24} - \frac{k^2x^5}{120} + \textstyle\int_0^x (\tau - x)\left[1 - \tau - k\tau^2 - \frac{k^2\tau^2}{2}\frac{k^2\tau^3}{6}\right]d\tau \\ &+ \textstyle\int_0^x (\tau - x)\left[(k+1)\left(-1 + \tau - \frac{\tau^2}{2} - \frac{k\tau^3}{3} - \frac{k^2\tau^3}{6} - \frac{k^2\tau^4}{24}\right)\right]d\tau \\ &+ \textstyle\int_0^x (\tau - x)\left[k\left(1 - \tau + \frac{\tau^2}{2} - \frac{\tau^3}{6}\frac{k\tau^4}{12} - \frac{k^2\tau 4}{24} - \frac{k^2\tau^5}{120}\right)\right]d\tau \end{aligned}$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \frac{kx^5}{40} + \frac{k^2x^5}{40} + \frac{k^2x^6}{240} + \frac{k^3x^5}{120} + \frac{k^3x^6}{360} + \frac{k^3x^7}{5040} \quad (16)$$

NUMERICAL RESULTS.

Table 1: Table of Comparison of Exact and VIM when $n = 0, k = 1$.

x	EXACT	VIM	ABSOLUTE ERROR
0	1	1	0.000000
0.01	0.990049834	0.99005	0.000000
0.02	0.980198673	0.980199	0.000000
0.03	0.970445534	0.970446	0.000000
0.04	0.960789439	0.960789	0.000001
0.05	0.951229425	0.951229	0.000003
0.06	0.941764534	0.941764	0.000005
0.07	0.932393819	0.932393	0.000010
0.08	0.923116346	0.923115	0.000017
0.09	0.913931185	0.913929	0.000027
0.1	0.904837418	0.904833	0.000041

In the Table 1 above, it is evident that the absolute error, obtained from the absolute difference between Exact and VIM, for the equation is quite moderate and reasonable. This proved that the method is simple, effective and straight forward.

Table 2: Table of Comparison of Exact and VIM when $n = 1, k = 1$.

x	EXACT	VIM	ABSOLUTE ERROR
0	1	1	0.000000
0.01	0.99005	0.99005	0.000000
0.02	0.980199	0.980119	0.000000
0.03	0.970446	0.970446	0.000000
0.04	0.960789	0.960789	0.000000
0.05	0.951229	0.951229	0.000000
0.06	0.941756	0.941756	0.000000
0.07	0.932394	0.932394	0.000000
0.08	0.923116	0.923116	0.000000
0.09	0.913931	0.913931	0.000000
0.1	0.904837	0.904838	0.000001

In table 2 above, we observe that the absolute error obtained from the absolute difference between Exact and VIM, for the equation is quite negligible. This confirms the effectiveness and efficiency of the method.

Table 3: Table of Comparison of Exact and VIM when $n = 2, k = 1$.

x	EXACT	VIM	ABSOLUTE ERROR
0	1	1	0.000000
0.01	0.99005	0.99005	0.000000
0.02	0.980199	0.980119	0.000000
0.03	0.970446	0.970446	0.000000
0.04	0.960789	0.960789	0.000000
0.05	0.951229	0.951229	0.000000
0.06	0.941756	0.941756	0.000000
0.07	0.932394	0.932394	0.000000
0.08	0.923116	0.923116	0.000000
0.09	0.913931	0.913931	0.000000
0.1	0.904837	0.904837	0.000000

In the table 3 above, the method converges to exact solution.

Problem 2. Consider the linear stiff initial value problem [12]:

$$\frac{dy_1}{dt} = -0.1y_1 - 49.9y_1 \tag{17}$$

$$\frac{dy_2}{dt} = 50y_2 \tag{18}$$

$$\frac{dy_3}{dt} = 70y_1 - 120y_3 \tag{19}$$

subject to the initial conditions:

$$y_1(0) = 2, y_2(0) = 1, y_3(0) = 2.$$

the exact solution of the system (17), (18), (19) are given by

$$y_1(t) = e^{-50t} + e^{-0.1t}, \tag{20}$$

$$y_2(t) = e^{50t}, \tag{21}$$

$$y_3(t) = e^{-50t} - e^{-120t} \tag{22}$$

The iterative formula based on (2) becomes;

$$y_{1,n+1} = y_{1,n}(t) + \int_0^t \lambda_1(y_{1,n}'(\tau) + 0.1y_{1,n}(\tau) + 49.9y_{2,n}(\tau))d\tau \tag{23}$$

$$y_{2,n+1} = y_{2,n}(t) + \int_0^t \lambda_2(y_{2,n}'(\tau) + 50y_{2,n}(\tau))d\tau \tag{24}$$

$$y_{3,n+1} = y_{3,n}(t) + \int_0^t \lambda_3(y_{3,n}'(\tau) - 70y_{2,n}(\tau) + 120y_{3,n}(\tau))d\tau \tag{25}$$

$$where\ \lambda_1 = \lambda_2 = \lambda_3 = -1\ and\ n = 0, 1, 2, 3, ...$$

at $n = 0$,

$$y_{1,0} = 2, y_{2,0} = 1, y_{3,0} = 2$$

$$\begin{aligned} y_{1,1} &= y_{1,0}(t) + \int_0^t -1(0 + 0.1y_{1,0}(\tau) + 49.9y_{2,0}(\tau))d\tau \\ &= 2 + \int_0^t -1(0.2 + 49.9)d\tau \\ &= 2 + \int_0^t (-50.1)d\tau \end{aligned}$$

$$= 2 - 50.1t \tag{26}$$

$$\begin{aligned} y_{2,1} &= y_{2,0} - \int_0^t (0 + 50y_{2,0}(\tau))d\tau \\ &= 1 + \int_0^t -1(50(1))d\tau \end{aligned}$$

$$= 1 - 50t \tag{27}$$

$$\begin{aligned} y_{3,1} &= y_{3,0}(t) + \int_0^t -1(0 - 70y_{2,0}(\tau) + 120y_{3,0}(\tau))d\tau \\ &= 2 + \int_0^t -1(-70(1) + 120(2))d\tau \\ &= 2 + \int_0^t (70 - 240)d\tau \end{aligned}$$

$$= 2 - 170t \tag{28}$$

at $n = 1$

$$\begin{aligned} y_{1,2} &= y1,1(t) + \int_0^t -1(-50.1 + 0.1y_{1,1}(\tau) + 49.9y_{2,1}(\tau))d\tau \\ &= 2 - 50.1t + \int_0^t -1(-50.1 + 0.1(2 - 50.1\tau) + 49.9(1 - 50\tau))d\tau \\ &= 2 - 50.1t + \int_0^t (-50.1 + 0.2 - 50.1\tau + 49.9 - 2495\tau)d\tau) \end{aligned}$$

$$= 2 - 50.1t + 1250.005t^2 \tag{29}$$

$$\begin{aligned} y_{2,2} &= y_{2,1}(t) + \int_0^t -1(-50 + 50y_{2,1}(\tau))d\tau \\ &= 1 - 50tu + \int_0^t -1(-50 + 50(1 - 50\tau))d\tau \end{aligned}$$

$$= 1 - 50t + 1250t^2 \tag{30}$$

$$\begin{aligned} y_{3,2} &= y_{3,1}(t) + \int_0^t -1(-170 - 70y2,1(\tau) + 120y_{3,1}(\tau))d\tau \\ &= 2 - 170t + \int_0^t -1(-170 - 70(1-50\tau) + 120(2-170\tau))d\tau \\ &= 2 - 170t + \int_0^t -1(0 + 3500\tau - 20400\tau)d\tau \end{aligned}$$

$$= 2 - 170t - 8450t^2 \tag{31}$$

at $n = 2$

$$\begin{aligned} y_{1,3} &= y_{1,2}(t) + \int_0^t -1(-50.1 + 2500.01\tau + 0.1y_{1,2}(\tau) + 49.9y_{2,2}(\tau))d\tau \\ &= 2 - 50.1t + 1250.005t^2 + \int_0^t -1(-50.1 + 2500.01\tau + 0.1(2 - 50.1\tau + 1250.005\tau^2) \\ &+ 49.9(1 - 50\tau + 1250\tau^2))d\tau \\ &= 2 - 50.1t + 1250.005t^2 + \int_0^t (-50.1 + 2500.01\tau + 0.2 - 0.51\tau + 125.0005\tau^2 \\ &+ 49.9 - 2495\tau + 62375\tau^2)d\tau \end{aligned}$$

$$= 2 - 50.1t + 1250.005t^2 - 20833.33350t^3 \tag{32}$$

$$\begin{aligned} y_{2,3} &= y_{2,2}(t) + \int_0^t -1(-50 + 2500\tau + 50y_{2,2}(\tau))d\tau \\ &= 1 - 50t + 1250t^2 + \int_0^t -1(-50 + 2500 + 50(1 - 50\tau + 1250\tau^2))d\tau \end{aligned}$$

$$= 1 - 50t + 1250t^2 - 20833.33333t^2 \tag{33}$$

$$\begin{aligned} y_{3,3} &= y_{3,2}(t) + \int_0^t -1(-170 + 16900\tau - 70y_{2,2}(\tau) + 120y_{3,2}(\tau))d\tau \\ &= 2 - 170t + 8450t^2 + \int_0^t -1(-10 + 16900\tau - 70(1 - 50\tau + 1250\tau^2) \\ &+ 120(2 - 170t + 8450\tau^2))d\tau \\ &= 2 - 170t + \int_0^t -1(-170 + 16900\tau - 70 + 3500\tau + 87500\tau^2 \\ &+ 240 - 20400\tau + 1014000\tau^2)d\tau \end{aligned}$$

$$= \quad 2 - 170t + 8450t^2 \tag{34}$$

at $n = 3$

$$\begin{aligned}
y_{1,4} &= y_{1,3}(t) + \int_0^t -1(50.1 + 2500.01\tau - 62500.0005\tau^2 + 0.1y_{1,3}(\tau) + 49.9y_{2,3}(\tau))d\tau \\
&= 2 - 50.1t + 1250.005t^2 - 20833.33350t^3 + \int_0^t -1(-50.1 + 2500.01\tau - 62500.0005\tau^2 \\
&+ 0.1(2 - 50.1\tau + 1250.005\tau^2 - 20833.33350\tau^3) + 49.9(1 - 50\tau + 1250\tau^2 - 20833.33333\tau^3))d\tau
\end{aligned}$$

$$= \quad 2 - 50.1t + 1250.005t^2 - 20833.33200t^3 + 260416.6665t^4 \tag{35}$$

$$\begin{aligned}
y_{2,4} &= y_{2,3}(t) + \int_0^t -1(-50 + 2500\tau - 62499.99999\tau^2 + 50y_{2,3}(\tau))d\tau \\
&= 1 - 50t + 1250t^2 - 20833.33333t^3 + \int_0^t -1(-50 + 2500\tau - 62499.99999\tau^2 \\
&+ 50(1 - 50\tau + 1250\tau^2 - 20833.33333\tau^3))d\tau
\end{aligned}$$

$$= \quad 1 - 50t + 1250t^2 - 20833.33333t^3 + 260416.6665t^4 \tag{36}$$

$$\begin{aligned}
y_{3,4} &= y_{3,3}(t) + \int_0^t -1(-170 + 16900\tau - 926500\tau^2 - 70y_{2,3}(\tau) + 120y_{3,3}(\tau))d\tau \\
&= 2 - 170t + 8450t^2 + \int_0^t -1(-170 + 16900\tau - 926500\tau^2 - 70(1 - 50\tau + 1250\tau^2 \\
&- 20833.33333\tau^3) + 120(2 - 170\tau + 8450\tau 2 - 30833.33333\tau^3))d\tau
\end{aligned}$$

$$y_{3,4} \quad = \quad 2 - 170t + 8450t^2 - 30833.33333t^3 + 8900416.668t^4 \tag{37}$$

at $n = 4$

$$\begin{aligned}
y_{1,5} &= y_{1,4}(t) + \int_0^t -1(-50.1 + 2500.01\tau - 62500.0005\tau^2 + 1041666.666\tau^3 \\
&+ 0.1y_{1,4}(\tau) + 49.9y_{2,4}(\tau))d\tau \\
&= 2 - 50.1t + 1250.005t^2 - 20833.33350t^3 + \int_0^t -1(-50.1 + 2500.01\tau - 62500.0005\tau^2 \\
&+ 1041666.666\tau^3 + 0.1(2 - 50.1\tau + 1250.005\tau^2 - 20833.33200\tau^3 + 260416.6665\tau^4)
\end{aligned}$$

$$+ \quad 49.9(1 - 50\tau + 1250\tau^2 - 20833.33333\tau^3 + 260416.6665\tau^4))d\tau \tag{38}$$

$$= \quad 2 - 50.1t + 1250.005t^2 - 20833.33350t^3 + 260416.6666t^4 - 2604166.666t^5$$

$$\lim_{n\to\infty} y_{1,5}(t) \approx e^{-50t}+e^{-0.1t} \tag{39}$$

$$\begin{aligned} y_{2,5} &= y_{2,4}(t)+\int_0^t -1(-50+2500\tau-62499.99999\tau^2+104166.666\tau^3+50y_{2,4}(\tau))d\tau \\ &= 1-50t+1250t^2-20833.33333t^3+260416.6665t^4+\int_0^t -1(-50+2500\tau-62499.99999\tau^2 \\ &\quad +104166.666\tau^3+50(1-50\tau+1250\tau^2-20833.33333\tau^3 \\ &\quad +260416.6665\tau^4)d\tau \end{aligned} \tag{40}$$

$$y_{2,5} = 1-50t+1250t^2-20833.33333t^3+260416.6665t^4-260416.6664t^5$$

$$\lim_{n\to\infty} y_{2,5}(t) \approx e^{-50t} \tag{41}$$

$$\begin{aligned} y_{3,5} &= y_{3,4}(t)+\int_0^t -1(-170+16900\tau-92499.99999\tau^2+35601666.67\tau^3-70y_{2,4}(\tau) \\ &+ 120y_{3,4}(\tau))d\tau \\ &= 2-170t+8450t^2-30833.33333t^3+8900416.668t^4+\int_0^t -1(-170+16900\tau \\ &- 92499.99999\tau^2+35601666.67\tau^3-70(1-50\tau+1250\tau^2-20833.33333\tau^3 \\ &\quad +260416.6665\tau^4)+120(2-170\tau+8450\tau^2-30833.33333\tau^3 \\ &\quad +8900416.668\tau^4))d\tau \end{aligned} \tag{42}$$

$$y_{3,5} = 2-170t+8450t^2-30833.33333t^3+8900416.666t^4-209964166.6t^5$$

$$\lim_{n\to\infty} y_{3,5}(t) \approx e^{-50t}+e^{-120t} \tag{43}$$

Problem 3. Consider the non-linear initial value problem [13]:

$$y_1' = -10002y_1+1000y_2^2 \tag{44}$$

$$y_2' = y_1-y_2-y_2^2 \tag{45}$$

subject to the initial conditions:

$$y_1(0) = 1, \tag{46}$$

$$y_2(0) = 1 \tag{47}$$

The exact solution of the system (44) and (45) are given by:

$$y_1(x) = e^{-2x} \tag{48}$$
$$y_2(x) = e^{-x} \tag{49}$$

The iterative formula based on (4) becomes;

$$y_{1,n+1}(x) = y_{1,n}(x) + \int_0^x \lambda(y'_{1,n}(\tau) + 1002y_{1,n}(\tau) - 1000y^2_{2,n}(\tau))d\tau \tag{50}$$
$$y_{2,n+1}(x) = y_{2,n}(x) + \int_0^x \lambda_2(y'_{1,n}(\tau) - y_{1,n}(\tau) + y_{2,n}(\tau) + y^2_{2,n}(\tau))d\tau \tag{51}$$

where $\lambda_1 = -\lambda_2 = -1$, when $n = 0$

$$y_{1,1} = y_{1,0}(x) + \int_0^x -1(0 + 1002(1) - 1000(1)^2)d\tau \tag{52}$$

$$= 1 - \int_0^x (1000 - 1002)d\tau$$
$$= 1 + \int_0^x 2d\tau$$

$$= 1 - 2x \tag{53}$$

$$y_{2,1}(x) = 1 + \int_0^x -1(0 - 1 + 1 + 1^2)d\tau$$
$$= 1 + \int_0^x (-1)d\tau$$
$$= 1 - \int_0^x d\tau$$

$$= 1 - x \tag{54}$$

Therefore, we have

$$y_{1,1}(x) = 1 - 2x \tag{55}$$
$$y_{2,1}(x) = 1 - x \tag{56}$$

when $n = 1$

$$y_{1,n+1}(x) = y_{1,n}(x) + \int_0^x \lambda(y'_{1,n}(\tau) + 1002y_{1,n}(\tau) - 1000y^2_{2,n}(\tau))d\tau \tag{57}$$

$$y_{1,2} = y_{1,1}(x) + \int_0^x -1(-2 + 1002(1-2T) - 1000(1-T)^2)d\tau$$
$$= 1 - 2x + \int_0^x -1(-2 + 1002 - 2004T - 1000 - 2000T - 1000T^2)d\tau$$

(58) $$= \quad 1 - 2x + 2x^2 + \frac{1000x^3}{3}$$

(59) $$y_{2,n+1}(x) \quad = y_{2,n}(x) + \int_0^x \lambda_2(y'_{1,n}(\tau) - y_{1,n}(\tau) + y_{2,n}(\tau) + y^2_{2,n}(\tau))d\tau$$

$$y_{2,2}(x) = 1 - x + \int_0^x -1(-(1-2\tau) + 1 - \tau + (1+\tau)^2)d\tau$$

(60) $$= \quad 1 - x + \frac{x^2}{2} - \frac{x^3}{3}$$

when $n = 2$

$$y_{1,3} = y_{1,2}(x) + \int_0^x -1\left[-2 + 4T + 1000T^2 + 1002\left(1 - 2\tau + 2\tau^2 + \frac{1000\tau^3}{3}\right)\right]d\tau$$

(61) $$+ \quad \int_0^x -1\left[-1000\left(1 - \tau + \frac{\tau^2}{2} - \frac{\tau^3}{3}\right)\right]d\tau$$

$$y_{1,3} = \left(1 - 2x + 2x^2 + \frac{1000x^3}{3}\right) + \int_0^x \left(-2 + 4\tau + 1000\tau^2 + 1002 - 2004\tau + 2004\tau^2\right)d\tau$$
$$+ \int_0^x \left(334000T^3 - 1000\left(1 - \tau + \frac{\tau^2}{2} - \frac{\tau^3}{3}\right)\right)d\tau$$

(62) $$= 1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{1000x^7}{63} - \frac{500x^6}{9} + \frac{550x^5}{3} - \frac{251750x^4}{3}$$

$$y_{2,3} \quad = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \int_0^x -1\left(-1 + \tau - \tau^2 - \left(1 - 2\tau + 2\tau^2 + \frac{1000\tau^3}{3}\right)\right)d\tau$$

(63) $$+ \int_0^x -1\left(\left(1 - \tau + \frac{\tau^2}{2} - \frac{\tau^3}{3}\right) + \left(1 - \tau + \frac{\tau^2}{2} - \frac{\tau^3}{3}\right)^2\right)d\tau$$

$$
\begin{aligned}
y_{2,3} &= 1-x+\frac{x^2}{2}-\frac{x^3}{3}+\int_0^x -1\left(-1+\tau-\tau^2-\left(1-2\tau+2\tau^2+\frac{1000\tau^3}{3}\right)\right)d\tau \\
&+ \int_0^x -1\left(\left(1-\tau+\frac{\tau^2}{2}-\frac{\tau^3}{3}\right)+1-2\tau+2\tau^2-\frac{5\tau^3}{3}+\frac{11\tau^4}{12}-\frac{\tau^5}{3}+\frac{\tau^6}{9}\right)d\tau \\
y_{2,3} &= 1-x+\frac{x^2}{2}-\frac{x^3}{3}+\left[\frac{\tau^3}{6}+\frac{503\tau^4}{6}-\frac{11\tau^5}{60}+\frac{\tau^6}{18}-\frac{\tau^7}{63}\right]_0^x
\end{aligned}
$$

$$
y_{2,3}=1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{503x^4}{6}-\frac{11x^5}{60}+\frac{x^6}{18}-\frac{x^7}{63} \tag{64}
$$

$$\vdots$$

$$
\begin{aligned}
y_{1,10}(x) &= 1-2x+2x^2-\frac{4x^3}{3}+\frac{2x^4}{3}+16850550x^5-58650x^6+\frac{1260050x^7}{63}-\frac{1391525x^8}{252} \\
&+ \frac{147591700x^9}{189}+O(x^{10})
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
y_{2,10}(x) &= 1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{67335x^5}{4}+\frac{21103x^6}{360}-\frac{2799x^7}{140}+\frac{18547x^8}{3360}-\frac{1475917x^9}{1890} \\
&+ O(x^{10})
\end{aligned} \tag{66}
$$

$$
\lim_{n\to\infty} y_1(x)=e^{-2x} \tag{67}
$$

$$
\lim_{n\to\infty} y_2(x)=e^{-x} \tag{68}
$$

Discussion of Results

In the Tables 1-3 above, it could be observed that the absolute error obtained from the absolute difference between Exact and VIM for Example 1 is quite moderate and reasonable. This proved that the method is simple, effective and straight forward for values of $n = 1, 2, 3$ and $k = 1$. Application of the method on Example 2 and 3 also shows the great computational strength and high convergence rate.

Conclusion

The effectiveness of the variational iteration method has been established through comparison of the obtained solutions with the exact solutions of linear, non-linear and systems of stiff differential equations. It is evident that the numerical solutions converge to the exact solution which proved the suitability of the method to more different types of stiff differential equations.

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