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A Note on Gronwall-Bellman type Integral Inequalities

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Abstract

Integral inequality plays a big role in the study of differential, integral and partial differential equations. Gronwall inequality is an important tool to obtain various estimates in the theory of ordinary and stochastic differential equations. In this work, extension of Gronwall-Bellman type Inequalities are obtained.

1. Introduction

In Mathematics, an inequality is a relation that holds between two different values. The following are some properties of inequalities:
Transitivity
The transitive property of inequality states:
For any real number a, b, c :
if $a \geq b$ and $b \geq c$ then $a \geq c$.
if $a \leq b$ and $b \leq c$, then $a \leq c$.
If either of the premises is a strict inequality, then the conclusion is a strict inequality:
if $a \geq b$ and $b > c$, then $a > c$
if $a > b$ and $b \geq c$, then $a > c$
Since a=b implies a $\geq$ b, these imply:

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if $a = b$ and $b > c$, then $a > c$
if $a > b$ and $b = c$, then $a > c$
Converse
The relations $\leq$ and $\geq$ are each other's converse:
for any real numbers a and b:
if $a \leq b$, then $b \geq a$
if $a \geq b$, then $b \leq a$
Additive Inverse
The properties for the additive inverse states:
for any real numbers a and b, negation inverse, the inequality:
if $a \leq b$, then $-a \geq -b$
if $a \geq b$, then $-a \leq -b$
Multiplicative Inverse
The properties of the multiplicative inverse states:
for any non-zero real numbers a and b that are both positive or both negative:
if $a \leq b$, then $1/a \geq 1/b$
if $a \geq b$, then $1/a > 1/b$

These can also be written in chained notation as :
for any non-zero real numbers a and b:
if $0 < a \leq b$, then $1/a \geq 1/b > 0$
if $a \leq b < 0$, then $0 > 1/a \geq 1/b$
if $a < 0 < b$, then $1/a < 0 < 1/b$
if $0 > a \geq b$, then $1/a \leq 1/b < 0$
if $a \geq b > 0$, then $0 < 1/a \leq 1/b$
if $a > 0 > b$, then $1/a > 0 > 1/b$.

Gronwall inequality is an important tool to obtain various estimates in the theory of ordinary and stocastic differential equations.

Gronwall is now remembered for his remarkable inequality called Gronwall's inequality of 1919, he proved a remarkable inequality, sometimes also called Gronwall's lemma which has attracted, and continues to attract attention (Gronwall, 1919).

Pachpatte (1973) worked on Grownwall-Bellman inequality. He stated Grownwall inequality as
Let $u(t), f(t)$ and $g(t)$ be real valued non-negative continuous function defined on I for which the inequality

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g(r)u(r)dr\right)ds, \quad t \in I,$$

holds, where u_0 is a non-negative constant. Then

$$u(t) \leq u_0\left(1 + \int_0^t f(s)e^{\left(\int_0^t (f(r)+g(r))dr\right)}ds\right), \quad t \in I.$$

Oguntuase (2001) worked on an inequality of Grownwall. He stated Grownwall theorem as.
Let $u(t), f(t)$ be non-negative continuous functions in a real interval $I = [a, b]$. Suppose that $k(t, s)$ and it's partial derivatives $k_t(t, s)$ exist and are non-negative continuous functions for almost every $t, s \in I$. If the inequality

$$u(t) \leq c + \int_0^t f(s)u(s)ds + \int_a^t f(s)\left(k(s,t)u(r)dr\right)ds, a \leq r \leq s \leq t \leq b \tag{1}$$

holds, where c is a non-negative constant, then

$$u(t) \leq c\left[1 + \int_a^t f(s)exp\left(\int_a^s (f(r) + k(r,r))dr\right)ds\right]$$

2. Main Result

This section consists of extensions of Grownwall-Bellman type integral inequalities.

Theorem: Let $u(t), f(t),$ and $g(t)$ be non-negative continuous functions in a real interval $I = [a, b]$. Suppose that $k(t, s)$ and its partial derivatives $k_t(t, s)$ exists and are non-negative continuous functions for almost every $t, s \in I$. Then, the inequality

(2)
$$u(t) \leq c + \int_a^t f(s)\,u(s)\,ds + \int_a^t f(s)\left(\int_a^s k(s,r)u(r)dr\right)ds + \int_a^t f(s)\left(\int_a^s g(s,r)u(r)dr\right)ds,$$
$a \leq r \leq s \leq t \leq b$ holds.

where c is a non-negative constant. Then,

$$u(t) \leq c\left[1 + \int_a^t f(s)\,exp\left(\int_a^s f(r) + k(r,r)\,dr\right)ds\right] \tag{3}$$

Proof :

Let

(4)
$$v(t) = c + \int_a^t f(s)\,u(s)\,ds + \int_a^t f(s)\left(\int_a^s k(s,r)\,u(r)dr\right)ds + \int_a^t f(s)\left(\int_a^s g(s,r)u(r)dr\right)ds.$$

then,

(5)
$$u(t) \leq v(t)$$

Assume $g(s,r) = 0$, then, $v(t)$ becomes

(6)
$$v(t) = c + \int_a^t f(s)\,u(s)\,ds + \int_a^t f(s)\left(\int_a^s k(s,r)\,u(r)dr\right)ds.$$

After differentiate equation (6) and make

(7)
$$\int_a^t f(s)\,u(s)\,ds = k(s)$$

hence

(8)
$$\int_a^s k(s,r)\,u(r)dr = N(s,r) \Rightarrow \int_a^s M(s,r)dr$$

From (6)
$\int k(s)ds = K(s)$

(9)
$$\int_a^t k(s)ds = K(s)\Big|_a^t = K(t) - K(a)$$

Also from (8)

$$\int_a^s M(s,r)dr = N(s,r)\Big|_a^s = N(s,s) - N(s,a)$$

Taking the limit, we have

(10)
$$N(s,r)|_a^s \Rightarrow P(s).$$

Putting (10) in (6), we have

$$\int_a^t f(s)\cdot\ P(s)ds \Rightarrow\ g(s) \tag{11}$$

and

$$\int_a^t g(s)ds = G(t) - G(a)(12)$$

Substituting (9) and (12) into (6), we have

$$V = c + K(t) - K(a) + G(t) - G(a)$$
$$V^{'} = \frac{dV}{dt} = \frac{d}{dt}c + \frac{d}{dt}[K(t)] - \frac{d}{dt}[K(a)] + \frac{d}{dt}[G(t)] - \frac{d}{dt}[G(a)]$$
$$V^{'} = 0 + [K'(t) = k(t)] - 0 + [G^{'}(t) = g(t)] - 0$$

Since $K(s) = f(s)\,u(s)$ from (7), it implies

$$k(t) = f(t)\,u(t)$$
$$g(s) = f(s)\cdot\ P(s)$$

$$= f(s)\ \cdot \int_a^s M(s,r)dr$$
$$g(s) = f(s)\cdot \int_a^s k(s,r)\,u(r)dr$$

also from (4)

$$g(t) = f(t)\cdot \int_a^t k(t,r)\,u(r)dr,$$

therefore

$$V^{'}(t) = f(t)\,u(t) + f(t)\cdot \int_a^t k(t,r)\,u(r)dr \tag{13}$$

$$\begin{aligned} V^{'}(t) &= f(t)\,u(t) + f(t)\cdot \int_a^t k(t,r)\,u(r)dr \quad V(a) = c \\ &\leq\ f(t)\left(V(t) + \int_a^t k(t,r)\,V(r)dr\right) \end{aligned} \tag{14}$$

putting

$$m(t) = V(t) + \int_a^t k(t,r)\,V(r)dr \tag{15}$$

it is clear that

$$V(t) \leq m(t) \tag{16}$$

To obtain $m^{'}(t)$,

let

$k(t,r)V(r) = j(t,r)$

and

$\int j(t,r)\mathrm{d}r = J(t,r)$

$$\begin{aligned}
\Rightarrow \int_a^t k(t,r)v(r)\mathrm{d}r &= \int_a^t (j,r)\mathrm{d}r \\
&= J(t,r)dr|_a^t \\
&= J(t,t) - J(t,a) \\
\Rightarrow m(t) &= V(t) + J(t,t) - J(t,a) \\
\Rightarrow m^{'}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}[V(t)] + \frac{\mathrm{d}}{\mathrm{d}t}[J(t,t)] - \frac{\mathrm{d}}{\mathrm{d}t}[J(t,a)] \\
\Rightarrow m^{'}(t) &= V^{'}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\int j(t,t)\mathrm{d}t\right] - \frac{\mathrm{d}}{\mathrm{d}t}\left[\int j(t,a)\mathrm{d}a\right] \\
\Rightarrow m^{'}(t) &= V^{'}(t) + j(t,t) - \frac{\mathrm{d}}{\mathrm{d}t}\left[\int j(t,a)\mathrm{d}a\right] \\
\Rightarrow m^{'}(t) &= V^{'}(t) + k(t,t)v(t) \\
&\quad - \frac{\mathrm{d}}{\mathrm{d}t}\left[\int k(t,a)v(a)\mathrm{d}a\right]
\end{aligned}$$

Changing the order of differentiation and integration

$$\begin{aligned}
m^{'}(t) &= V^{'}(t) + k(t,t)v(t) \\
&\quad - \int \frac{\mathrm{d}}{\mathrm{d}t}\left[K(t,a)v(a)\mathrm{d}a\right] \\
\Rightarrow m^{'}(t) &= V^{'}(t) + k(t,t)v(t) - \int k_t(t,a)v(a)\mathrm{d}a
\end{aligned}$$

by change of variable in the last expression
$\Rightarrow m^{'}(t) = V^{'}(t) + k(t,t)v(t) - \int k_t(t,r)v(r)\mathrm{d}r$

therefore

$$m^{'} = V^{'} + k(t,t)\,V(t) - \int_a^t k_t(t,r)\,V(r)dr \tag{17}$$

$$m(a) = V(a) = c \tag{18}$$

hence

$$\begin{aligned} &\leq\ V'(t) + k(t,t)\,V(t) \\ &\leq\ f(t)\,m(t) + k(t,t)\,V(t) \\ &\leq\ (f(t) + k(t,t))\,m(t) \end{aligned}$$

therefore

$$m'(t) \leq\ (f(t) + k(t,t))\,m(t)$$
$$\Rightarrow\ \frac{m'(t)}{m(t)} \leq\ (f(t) + k(t,t))$$

integrating (17) from a to t, we obtain

$$\Rightarrow\ \int_a^t \frac{m'(s)}{m(s)}ds \leq\ \int_a^t (f(s) + k(s,s))ds$$

hence

$$\Rightarrow\ ln\left(\frac{m(t)}{m(a)}\right) \leq\ \int_a^t (f(s) + k(s,s)ds \tag{19}$$

which is

$$\frac{m(t)}{m(a)} \leq\ e^{\int_a^t (f(s)+k(s,s))ds}$$

$$\Rightarrow\ m(t)\ \leq\ c\cdot\ e^{\int_a^t (f(s)+k(s,s))ds} \tag{20}$$

But

$$V(t) + \int_a^t k(t,r)\, V(r) dr = m(t) \tag{21}$$

hence from (14)

$$V^{'}(t) \,\leq\, f(t) \left[c \cdot \, e^{\int_a^t (f(s)+k(s,s))ds} \right] \tag{22}$$

Integrating both sides from a to t

$$\Rightarrow \int_a^t V^{'}(r) ds \leq \int_a^t c \cdot \, f(r) \cdot \, W(r) dr \tag{23}$$

implying that

$$V(t) \leq \, V(a) + c \int_a^t f(r) W(r) dr \tag{24}$$

where

$$W(r) = \, e^{\int_a^t (f(s)+k(s,s))dr}$$

$$\Rightarrow \, V(t) \leq \, c \left[1 + \int_a^t f(r) e^{\int_a^t (f(s)+k(s,s))dr} \right]$$

and

$$V(t) \leq \, c \left[1 + \int_a^t f(s) e^{\int_a^s (f(s)+k(s,s))ds} \right]$$

hence the result.

Theorem: Let $u(t)$, $f(t)$ and $g(t)$ be real-valued non-negative continuous function defined on a real real interval $I[a,b]$, for which the inequality

$$u(t) \leq u_0 + \int_0^t f(s) u(s) ds + \int_0^t f(s) \left(\int_0^s \left(\prod_{r=1}^{2} g(r) + \prod_{r=1}^{2} u(r) \right) dr \right) ds \tag{25}$$

holds where u_0 is a non-negative constant. Then

$$u(t) \leq u_0 \left(1 + \int_0^s f(s) e^{\left(\int_0^t (f(r)+g(r)) dr \right)} ds \right) \; t \in I \tag{26}$$

Proof
Consider

$$
\begin{aligned}
u(t) &\leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s \left(\prod_{r=1}^{2} g(r) + \prod_{r=1}^{2} u(r)\right) dr\right) ds \\
&\leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s \prod_{r=1}^{2} g(r)dr + \int_0^s \prod_{r=1}^{2} u(r)dr\right) ds \\
&\leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s \prod_{r=1}^{2} g(r)dr\right) ds + \int_0^t f(s)\left(\int_0^s \prod_{r=1}^{2} u(r)dr\right) ds \\
&\leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g_1(r)g_2(r)dr\right) ds + \int_0^t f(s)\left(\int_0^s u_1(r)u_2(r)dr\right) ds
\end{aligned}
$$

Suppose u_1 or u_2 =0, we have

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g_1(r)g_2(r)dr\right) ds$$

where u_0 remains non-negative constant

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g_1(r)g_2(r)dr\right) ds \tag{27}$$

Define a function $v(t)$ by the right hand side (27)
then it follows that

$$u(t) \leq v(t) \tag{28}$$

that is

$$v(t) = u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g_1(r)g_2(r)dr\right) ds \tag{29}$$

To get $v^{'}(t)$ from (29), we need to remove the integration by differentiating it.
From (29)

$$
\begin{aligned}
&\int_0^t f(s)u(s)ds = m(s) \qquad (30) \\
&\int_0^s g_1(r)g_2(r)dr = k(r)
\end{aligned}
$$

$$\Longrightarrow \int_0^s N(r)dr \tag{31}$$

From equation (30)

$$\int_0^t m(s)ds = M(s)|_0^t$$
$$= M(t) - M(0) \tag{32}$$

From (31)

$$\int_0^s N(r)dr = k(r)|_0^s$$
$$= K(s) - K(0) \tag{33}$$

Let equation (33) $\Longrightarrow$ $p(s)$
Substituting (33) into the second term in (29)

$$\int_0^t f(s)\rho(s)ds \tag{34}$$

Let

$$(34) = \int_0^t g(s)ds$$
$$= G(s)|_0^t$$

$$G(t) - G(0) \tag{35}$$

Substitute equation (32) and (35) into (29)
$v(t) = U_0 + M(t) - M(0) + G(t) - G(0)$
$v^{'}(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}[u_0] + \dfrac{d}{dt}[M(t)] - \dfrac{d}{dt}[M(0)] + \dfrac{d}{dt}[G(t)] - \dfrac{d}{dt}[G(0)]$
$v^{'}(t) = 0 + [M^{'}(t) = m(t)] - 0 + [G^{'}(t) = g(t)] - 0$

$v^{'}(t) = m(t) + g(t)$
Since $m(s) = f(s)u(s)$ from (30), it implies

$m(t) = f(t)u(t)$

also

$$g(s) = f(s)\rho(s)$$
$$g(s) = f(s)\int_0^s N(r)dr$$
$$g(s) = f(s)\int_0^s g_1(r)g_2(r)dr$$

Also

$$g(t) = f(t)\int_0^t g_1(r)g_2(r)dr$$

therefore

$$v^{'}(t) = f(t)u(t) + f(t)\int_0^t g_1(r)g_2(r)dr, \quad m(O) = v(O) \tag{36}$$

since u_0 is a non-negative constant which in view of (25) implies

$$v^{'}(t) \leq f(t)\left(v(t) + \int_0^t g(r)V(r)dr\right) \tag{37}$$

since $u(t) \leq v(t)$ by (28)
Let $g_2(r) = V(r)$, leaving $g_1(r)$ to be $g(r)$.
Putting
$m(t) = v(t) + \int_0^t g(r)V(r)dr$
it is clear that

$$v(t) \leq m(t) \tag{38}$$

Next is to solve for $m^{'}(t)$

$$m(t) = v(t) + \int_0^t g(r)V(r)dr \tag{39}$$

Let

$$g(r)V(r) = j(r)$$

and

$$\int j(r)dr = J(r)$$

$$\Longrightarrow \int_0^t g(r)V(r)dr = \int_0^t j(r)dr \tag{40}$$

$$= J(r)|_0^t$$

$$= J(t) - J(0)$$

$$\begin{aligned}
m(t) &= v(t) + J(t) - J(0) \\
\frac{d}{dt}[m(t)] &= \frac{d}{dt}[v(t)] + \frac{d}{dt}[J(t)] - \frac{d}{dt}[J(0)] \\
m^{'}(t) &= v^{'}(t) + \frac{d}{dt}\left[\int j(t)dt\right] \\
m^{'}(t) &= v^{'}(t) + j(t)
\end{aligned}$$

$$m^{'}(t) = v^{'}(t) + g(t)V(t) \quad \text{from (40)}$$

$$\begin{aligned}
m^{'}(t) &\leq v^{'}(t) + g(t)V(t) \\
&\leq f(t)m(t) + g(t)V(t)
\end{aligned}$$

$$m^{'}(t) \leq (f(t) + g(t))m(t) \ \text{ from(38)}$$

On integrating from 0 to t

$$\begin{aligned}
m^{'}(t) &\leq (f(t) + g(t))m(t) \\
\frac{m^{'}(t)}{m(t)} &\leq (f(t) + g(t)) \\
\Longrightarrow \int_0^t \frac{m^{'}(s)}{m(s)} &\leq \int_0^t (f(s) + g(s))ds
\end{aligned}$$

Remark 1: By change of variable, the limit of the integral can be evaluated as

$$\ln(m(s))|_0^t \leq \int_0^t (f(s) + g(s))\, ds$$

Remark 2: Since

$$\int \frac{f(x)}{f(x)} dx = \ln f(x)$$

$$\implies \ln m(t) - \ln m(0) \leq \int_0^t (f(s) + g(s))ds$$

$$\implies \ln \left(\frac{m(t)}{m(0)} \right) \leq \int_0^t (f(s) + g(s))ds$$

Taking exponential of both sides

$$\implies \frac{m(t)}{m(0)}) \leq e^{\int_0^t (f(s)+g(s))ds}$$

$$\implies m(t) \leq m(0) \cdot e^{\int_0^t (f(s)+g(s))ds}$$

$$m(0) = u_0$$

$$\implies m(t) \leq u_0 \cdot e^{\int_0^t (f(s)+g(s))ds} \tag{41}$$

Remark 3: Since $m(0) = u_0$, it is possible to cross multiply without worrying about inequality since the functions are all positive (from the theorem stated above)
Substitute (41) into (37)

$$v^{'}(t) \leq f(t) \left(v(t) + \int_0^t g(r)v(r)dr \right) \tag{42}$$

but we know that

$$v(t) + \int_0^t g(r)v(r)dr = m(t)$$

$$v^{'}(t) \leq f(t) \left[u_0 \cdot e^{\int_0^t (f(s)+g(s))ds} \right] \tag{43}$$

integrating both sides from 0 to t

$$\implies \int_0^t v^{'}(r)dr \leq \int_0^t u_0 f(r)W(r)dr$$

where

$$W(r) = e^{\int_0^t (f(s)+g(s))ds}$$

Remark 4: By applying Remark 1 as earlier stated implies

$$v(t) - v(0) \leq u_0 \int_0^t f(r)W(r)dr$$
$$\implies v(t) \leq v(0) + u_0 \int_0^t f(r)W(r)dr$$
$$v(0) = u_0 = c$$
$$v(t) \leq c + c\int_0^t f(r)W(r)dr$$
$$v(t) \leq c\left[1 + \int_0^t f(r)W(r)dr\right]$$
$$v(t) \leq c\left[1 + \int_0^t f(s) \cdot e^{\int_0^t (f(s)+g(s))ds} ds\right]$$

Taking it back in form of (26)

$$u(t) \leq u_0\left[1 + \int_0^t f(s) \cdot e^{\int_0^t (f(r)+g(r))dr} ds\right]$$

Theorem: (Pachpatte 1973)
Let $u(t)$ and $f(t)$ be non-negative, continuous function on $I = [0, \infty]$ for which the inequality

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds \tag{44}$$

holds, where u_0 is a non negative constant. Then $u(t) \leq u_0 e^{\int_0^t f(s)ds}$ (Pachpatte, 1973)

Example 2.1
Consider the following equation

$$u(x) \leq 2 + \frac{1}{3}\int_0^x xt^3 u(t)dt \qquad \text{(Wazwaz, 2011)} \tag{45}$$

Solution

by Gronwall inequality, (46) is equivalent to

$$u(t) \leq 2 + \int_0^x \frac{1}{3} x t^3 u(t) dt \tag{46}$$

where $u_0 = 2, f(t) = \frac{1}{3} x t^3$ and

$$u(t) \leq 2e^{\frac{1}{3}\int_0^t x s^3 ds} \tag{47}$$

hence

$$u(t) \leq 2(e^{\frac{xt^4}{12}} - 1) \tag{48}$$

which is the solution of $u(t)$ gives a solution of the form $u(t) \leq u_0 e^{\int_0^t f(s)ds}$

Example 2.2
Consider the following equation

$$u(x) \leq 1 + 3\int_0^x u(t) dt \tag{49}$$

Solution
(49) can be rewritten as:

$$u(x) \leq 1 + \int_0^x 3u(t) dt \tag{50}$$

by Gronwall inequality, (53) is equivalent to

$$\begin{aligned} & u(t) \leq u_0 e^{\int_0^t f(s)ds} \\ & where \mathrm{u}_0 = 1, f(t) = 3 \\ & = u(t) \leq e^{\int_0^t 3ds} \end{aligned} \tag{51}$$

hence

$$u(t) = e^{3t} \tag{52}$$

See, Wazwaz (2011) for further explanation.

3. Conclusion

In this work, extension of results of Oguntuase and Pachpatte on Gronwall-Bellman integral inequalities were obtained with application to differential equations.

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