Nigerian Journal of Mathematics and Applications
Volume **26**, (2017), 113 – 127. *Printed by Unilorin press*

http://www.njmaman.com

Numerical Integration of Seventh order Boundary Value Problems by Standard Collocation Method via Four Orthogonal Polynomials

K. A. Bello[1]* O. A. Taiwo[1], A. Abdulkareem[1], J. U. Abubakar[1] and F. A. Adebisi[2]

Abstract

Based on standard collocation technique, four (4) different orthogonal polynomials were used as basis functions in the numerical treatment of seventh (7^{th}) order boundary value problems in Ordinary Differential Equations. The performance of each of these polynomials as basis function in the trial solution was then compared. The results obtained from three examples showed that Chebyshev polynomial is the best in term of performance, and closely followed by Hermites polynomial, which was followed by Legendre polynomial while the least in performance is Laguerre polynomial.

1. Introduction

Seventh order boundary value problems appear in several branches of applied mathematics and engineering sciences. For example, problems that arise in Mathematics modeling of induction motors with two rotor circuits, induction motor model contains fifth order differential equation model, which includes two stator

Received: 14/03/2017, Accepted: 13/06/2017, Revised: 15/07/2017.
2015 *Mathematics Subject Classification.* 65D30 & 33C45. * Corresponding author.
Key words and phrases. Standard collocation technique, Orthogonal polynomials, Seventh order ordinary differential equations
Department of Mathematics University of Ilorin, Ilorin, Nigeria
Department of Mathematical and Physical Sciences, Osun State University, Osogbo, Nigeria
E-mail: bello.ak@unilorin.edu.ng

variables, two rotor state variables and shaft speed (Ghazala and Hamood, 2014). For the effect of a starting cage, deep bars or rotor distributed parameter, additional rotor circuit is required so that two more state variables and two equations may be added. Additional state variable create computational burden. To avoid this, models are often limited to the fifth order and rotor impendence is algebraically altered as function of rotor speed to account for torque discrepancies at startup. This is done by the assumption of frequency of rotot currents depends on rotor speed. Fifth order model running near full speed subjected to sudden voltage dip, would not reproduce the transient drag torque produced by stationary flux linkage trapped in the stator windings, although such behavior would show up in the seventh order model. Many researchers have solved seventh order BVPs using different methods. Siddiqi *et. al.* (2012) solved these using Variation of parameters method, Differential transformation method and Variation iteration technique. Reproducing kernel space method was used by Haar (2017). Siddiqi and Muzammal (2013) also also used Adomian Decomposition Method for solution of seventh order boundary Value problems. The pitfalls of the above mentioned methods is that they take more time in terms of computation even the Adomian package program is very expensive. The researcher is still using standard collocation method because of its simplicity in terms of computation and better CPU.

Orthogonal polynomials play an important role in mathematics and in physics, often as solution to differential equation or eigen functions of differential operators. The widely used orthogonal polynomials are the classical orthogonal polynomials, consisting of the Hermite polynomials, Laguerre polynomials, the Jacobi polynomials together with their special cases, the Gegenbauer polynomials, Chebyshev polynomials, Legendre polynomial (Yisa, 2015a).

The usefulness of various type of orthogonal polynomials like Hermites, Chebyshev, Legendre and Laguerre polynomials can never be over emphasized. This is evidenced in the works of many scientists like, Gauss in his popular integration method called Gauss Quadrature formula, where he derived the nodes for any given problem by equating to zero the polynomial of equivalent order, mostly Legendre. Lanczos (1938, 1956) [8] elucidated more of on the properties of these polynomials and came to the conclusion that chebyshev polynomial is the best, considering the mini-max property of the latter in the error propagation.

Here, we use four frequently encountered orthogonal polynomials namely, Hermite, Chebyshev, Legendre and Laguerre polynomial as basis functions. We compare their results to determine the order of their responsiveness.

Here, we consider the seventh order BVPs of the form

$$y^{(7)}(x) + f(x)y(x) = r(x),\ x \in (a, b)$$

subject to the following conditions

$$y(x_0)=\alpha,\ y(x_n)=\beta,\ y^{(1)}(x_0)=\alpha,$$
$$y^{(1)}(x_n)=\beta',\ y^{(2)}(x_0)=\alpha'$$
$$y^{(2)}(x_n)=\beta'',\ y^{(3)}(x_0)=\alpha''$$

This article is organized as follows: Section 2 deals with orthogonal polynomials, Section 3 is on the methodology, numerical examples are considered in section 4 while Section 5 concludes the work.

2. Orthogonal Polynomials

Consider the polynomial

$$Q_N(x)=a_0+a_1x+a_2x^2+\cdots+a_{n-1}x^{n-1}+a_Nx^N$$

where $N=0,1,2,\cdots$ and $a_n\in\mathbb{R}$
Two functions $Q_n(x)$ and $Q_m(x)$ are orthogonal if the inner product

$$<Q_n(x),Q_m(x)>=0,\ n\neq m$$

That is, if

$$\int_a^b w(x)Q_n(x)Q_m(x)dx=0$$

where $w(x)$ is the weight function and a and b, are constants.

2.1 Properties of Orthogonal Polynomials.

(1) Any polynomial $f(x)$ of degree n can be expanded in terms of $p_0(x)$, $p_1(x),\cdots,p_n(x)$, that is, there exist coefficients a_i such that

$$f(x)=\sum_{i=0}^{n}a_ip_i(x)$$

(2) Given an orthogonal set of polynomial $\{p_0(x),p_1(x),\cdots\}$, each polynomial, $p_k(x)$ is orthogonal to any polynomial of degree $<k$

(3) Any orthogonal set of poynomials $\{p_0(x),p_1(x),\cdots\}$, has a recurrence formula that relates any three consecutive polynomial in the sequence, that is , the relation $p_{n+1}=(a_nx+b_n)p_n-c_np_{n-1}$ exists, where the coefficients a, b and c depend on n. Such a recurrence formula is often used to generate higher order members in the set

(4) Each polynomial in $\{p_0(x),p_1(x),\cdots\}$ has all n of its roots real, distinct and strictly within the interval of orthogonality (i.e not on its ends). This is an extremely unusual property! It is particularly important when considering the classes of polynomials that arise as quantum mechanical solutions to a given Hamiltonian (or other Hermitian)operator

(5) Furthermore the roots of the n^{th} degree polynomial, p_n lie strictly inside the roots of the $(n+1)^{th}$ degree polynomial p_{n+1}
In what now follows, we shall briefly define the Orthogonal polynomials adopted in this work.

2.2 Some Orthogonal Polynomials

2.2.1 Hermite Polynomial [1]

Hermite polynomial was named in honour of Charles Hermite (1882 - 1901). The Hermite polynomials are orthogonal in the interval $(-\infty, \infty)$ with respect to the weight function $w(x) = e^{-x^2}$. They are defined by means of their Rodrigue formula

$$H_n(x) = (-1)^n e^{-x^2} \frac{d^n}{dx^n} e^{-x^2}$$

and the recurrence formula is given as

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \text{ where } n = 1, 2, \cdots$$

2.2.2 Laguerre Polynomial [2]

Laguerre polynomial was named in honour of Edmond Laguerre (1834 - 1885). Laguerre polynomials are orthogonal in the interval $(0, \infty)$ with respect to the weight fuction $w(x) = e^x$. It is given by Rodrigue formula

$$L_n(x) = e^x \frac{d^n}{dx^n} x^n e^{-x} \text{ where } n = 1, 2, \cdots$$

and the recureence formula is given as

$$L_{n+1}(x) = (x - 2n - 1)L_n(x) + n^2 L_{n-1}(x)$$

2.2.3 Chebyshev Polynomial [8]

The Chebyshev polynomials named after a Russian Mathematics, Pafnuty Lvovich Chebyshev (1821 - 1892). The Chebyshev polynomials of the first kind denoted by $T_n(x)$ are sets of polynomials degree n defined as the solution to the first kind Chebyshev differential equation

$$(1 - x^2)y'' - xy' + n^2 y = 0$$

A Chebyshev polynomial of degree n of the first kind is defined as

$$T_n(x) = \cos(n \cos^{-1} x, \; x \in [-1, 1]$$

and the recurrence relation is given by

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \; n \geq 1$$

Also, in $a \leq x \leq b$, the Chebyshev polynomial is defined as

$$T_n(x) = \cos\left[n\cos\left(\frac{2x-a-b}{b-a}\right)\right],\ x \in [a,b]$$

and satisfies the recurrence relation;

$$T_{n+1}(x) = 2\left(\frac{2x-a-b}{b-a}\right)T_n(x) - T_{n-1}(x),\ n \geq 0,\ x \in [a,b]$$

2.2.4 Legendre Polynomial [2]

The Legendre Polynomial was introduced in 1784 by a French Mathematician, Legendre A.M. (1752 - 1833).The Legendre Polynomial is defined and denoted by

$$p_{n+1}(x) = \frac{1}{n+1}\{(2n+1)xp_n(x) - np_{n-1}(x)\}$$

and $p_n(x)$ is expressed by Rodrigue's formula

$$p_n(x) = \frac{d^n}{2^n n! dx^n}(x^2-1)^n\ ;\ n = 0,1,2,\cdots$$

2.3 Co-ordinate Transformation and Shifted Orthogonal Polynomial

In order to shift from the interval $[-1,1]$ to the interval $[a,b]$, let

$$y = mx + n$$

$$a = -m + n$$

$$b = m + n$$

Add equation (18) and (19) we have

$$2n = a + b$$

$$n = \frac{a+b}{2}$$

Putting (21) in (19) and simplify to obtain

$$m = \frac{a-b}{2}$$

Substituting (21) and (22) in (17), we have

$$x = \frac{2y-(a+b)}{b-a}$$

2.3.1 Shifted Chebyshev Polynomial [9]

Shifed Chebyshev Polynomial are orthogonal on $[0,1]$. And it has rodrigue formula

$$T_n^*(x) = \cos\{n\cos^{-1}(2x-1)\}$$

And recurrence relation

$$T_{n+1}^*(x) = 2(2x-1)T_n(x) - T_{n-1}(x)$$

2.3.2 Shifted Legendre Polynomial [5]

Shifted Legendre Polynomial are orthogonal on interval $[0,1]$. They are define by means of their rodrigue formula

$$p_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-x)^n \text{ where } n = 1,2,\cdots$$

and the recurrence formula is given as

$$p_n(x) = \frac{(2x-1)(2n+1)}{n+1}p_n(x) - \frac{n}{n+1}p_{n-1}(x), \text{ where } n = 1,2,\cdots n$$

3 Methodology

Here, we assumed an approximate solution of the form

$$y(x) \approx y_{n(x)} = \sum_{i=0}^{n} a_i Q_i(x)$$

where $Q_i(x)$ is orthogonal polynomial.
Substituting (28) in (1) and (2) to obtain

$$\frac{d^7}{dx^7}\left\{\sum_{i=0}^{n} a_i Q_i(x)\right\} + f(x)\left\{\sum_{i=0}^{n} a_i Q_i(x)\right\} = r(x)$$

and

$$\sum_{i=0}^{n} a_i Q_i(x_0) = \alpha,\ \sum_{i=0}^{n} a_i Q_i(x_n) = \beta,\ \sum_{i=0}^{n} a_i Q_i'(x_0) = \alpha',\ \sum_{i=0}^{n} a_i Q_i'(x_n) = \beta'$$
$$\sum_{i=0}^{n} a_i Q_i''(x_0) = \alpha'',\ \sum_{i=0}^{n} a_i Q_i''(x_n) = \beta'',\ \sum_{i=0}^{n} a_i Q_i'''(x_0) = \alpha'''$$

Equation (29) is simplified to get

$$\frac{d^7}{dx^7}(a_0Q_0(x) + a_1Q_1(x) + a_2Q_2(x) + \cdots a_nQ_n(x))$$
$$+ f(x)(a_0Q_0(x) + a_1Q_1(x) + a_2Q_2(x) + \cdots a_nQ_n(x)) = r(x)$$

and

$$\left[\left(\frac{d^7}{dx^7}+f(x)\right)Q_0(x)\right]a_0+\left[\left(\frac{d^7}{dx^7}+f(x)\right)Q_1(x)\right]a_1+\left[\left(\frac{d^7}{dx^7}+f(x)\right)Q_2(x)\right]a_2$$
$$+\cdots+\left[\left(\frac{d^7}{dx^7}+f(x)\right)Q_n(x)\right]a_n=r(x)$$

Equation(32) is collocated at x_j as

$$\left[\left(\frac{d^7}{dx^7}+f(x_j)\right)Q_0(x_j)\right]a_0+\left[\left(\frac{d^7}{dx^7}+f(x_j)\right)Q_1(x_j)\right]a_1+\left[\left(\frac{d^7}{dx^7}+f(x_j)\right)Q_2(x_j)\right]a_2$$
$$+\cdots+\left[\left(\frac{d^7}{dx^7}+f(x_j)\right)Q_n(x_j)\right]a_n=r(x_j)$$

where

$$x_j=a_j+\frac{(b-a)j}{N+1},\ j=1,2,3,\cdots,N$$

Equation (33), after collocation, is put in matrix form as

$$\begin{pmatrix}(\frac{d^7}{dx^7}+f(x_1))Q_0(x_1) & (\frac{d^7}{dx^7}+f(x_1))Q_0(x_1) & \cdots & (\frac{d^7}{dx^7}+f(x_1))Q_0(x_1)\\ (\frac{d^7}{dx^7}+f(x_2))Q_0(x_2) & (\frac{d^7}{dx^7}+f(x_2))Q_0(x_2) & \cdots & (\frac{d^7}{dx^7}+f(x_2))Q_0(x_2)\\ \vdots & \vdots & & \vdots\\ (\frac{d^7}{dx^7}+f(x_n))Q_0(x_n) & (\frac{d^7}{dx^7}+f(x_n))Q_0(x_n) & \cdots & (\frac{d^7}{dx^7}+f(x_n))Q_0(x_n)\end{pmatrix}\begin{pmatrix}a_0\\a_1\\\vdots\\a_n\end{pmatrix}=\begin{pmatrix}r(x_1)\\r(x_2)\\\vdots\\r(x_n)\end{pmatrix}$$

Equation (35) will give $(N-(n-1))$ system of algebraic equations.
Also N equations are derived from the seven initial conditions, thus,

$$y_n(x_0)=a_0Q_0(x_0)+a_1Q_1(x_0)+a_2Q_2(x_0)+\cdots+a_nQ_n(x_0)=\alpha$$
$$y_n(x_n)=a_0Q_0(x_n)+a_1Q_1(x_n)+a_2Q_2(x_n)+\cdots+a_nQ_n(x_n)=\beta$$
$$y_n^{(1)}(x_0)=a_0Q_0^{(1)}(x_0)+a_1Q_1^{(1)}(x_0)+a_2Q_2^{(1)}(x_0)+\cdots+a_nQ_n^{(1)}(x_0)=\alpha'$$
$$y_n^{(1)}(x_n)=a_0Q_0^{(1)}(x_n)+a_1Q_1^{(1)}(x_n)+a_2Q_2^{(1)}(x_n)+\cdots+a_nQ_n^{(1)}(x_n)=\beta'$$
$$y_n^{(2)}(x_0)=a_0Q_0^{(2)}(x_0)+a_1Q_1^{(2)}(x_0)+a_2Q_2^{(2)}(x_0)+\cdots+a_nQ_n^{(2)}(x_0)=\alpha''$$
$$y_n^{(2)}(x_n)=a_0Q_0^{(2)}(x_n)+a_1Q_1^{(2)}(x_n)+a_2Q_2^{(2)}(x_n)+\cdots+a_nQ_n^{(2)}(x_n)=\beta''$$
$$y_n^{(3)}(x_0)=a_0Q_0^{(3)}(x_0)+a_1Q_1^{(3)}(x_0)+a_2Q_2^{(3)}(x_0)+\cdots+a_nQ_n^{(3)}(x_0)=\alpha'''$$

Equation (36) are also put in matrix form as

$$\begin{pmatrix} Q_0(x_0) & Q_1(x_0) & \cdots & Q_n(x_0) \\ Q_0(x_n) & Q_1(x_n) & \cdots & Q_n(x_n) \\ Q_0^{(1)}(x_0) & Q_1^{(1)}(x_0) & \cdots & Q_n^{(1)}(x_0) \\ Q_0^{(1)}(x_n) & Q_1^{(1)}(x_n) & \cdots & Q_n^{(1)}(x_n) \\ Q_0^{(2)}(x_0) & Q_1^{(2)}(x_0) & \cdots & Q_n^{(2)}(x_0) \\ Q_0^{(2)}(x_n) & Q_1^{(2)}(x_n) & \cdots & Q_n^{(2)}(x_n) \\ \vdots & \vdots & & \vdots \\ Q_0^{(n)}(x_0) & Q_1^{(n)}(x_0) & \cdots & Q_n^{(n)}(x_0) \\ Q_0^{(n)}(x_n) & Q_1^{(n)}(x_n) & \cdots & Q_n^{(n)}(x_n) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \\ \vdots \\ \vdots \\ \vdots \\ \\ a_n \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \alpha' \\ \beta' \\ \alpha'' \\ \beta'' \\ \vdots \\ \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix}$$

Equation (35) and (37) are combined to give $(N+1)$ system of equations. This system of $N+1$ equations are then solved simultaneously to obtained the unknown constants $a_j,\ j = 0, 1, 2, \cdots, n$ which are substituted back into the trial solution to obtain the required approximate solutions

4 Numerical Examples

Example 1

Consider the linear seventh order boundary value problem

$$\frac{d^7y}{dx^7} = xy(x) + e^x(x^2 - 2x - 6)$$

$$y(0) = 1,\ y'(0) = 0,\ y''(0) = -1,\ y'''(0) = -2,\ y(1) = 0,$$
$$y'(1) = -e,\ y''(1) = -2e$$

The exact solution of the problem is given as

$$y(x) = (1 - x)e^x$$

Solution. Here, we demonstrate the methodology using Hermite polynomial

Let

$$y \cong y_N(x) \sum_{n=0}^{N} a_n H_n(x)$$

where H_n are Hermite polynomials

Substituting (41) into (38) and (39) respectively, then we have;

$$\frac{d^7}{dx^7}\left(\sum_{n=0}^{N} a_n H_n(x)\right) = x\left(\sum_{n=0}^{N} a_n H_n(x)\right) + e^x(x^2 - 2x - 6)$$

$$y_N(0) = 1,\ y_N'(0) = 0,\ y_N''(0) = -1,\ y_N'''(0) = -2,\ y_N(1) = 0,$$
$$y_N'(1) = -e,\ y_N''(1) = -2e$$

Choosing $N = 7$, then (41) becomes

$$y_7(x) = a_0H_0(x) + a_1H_1(x) + a_2H_2(x) + a_3H_3(x) + a_4H_4(x) + a_5H_5(x) + a_6H_6(x) + a_7H_7(x)$$

i.e

$$y_7(x) = a_0 + 2xa_1 + a_2(4x^2 - 2) + a_3(8x^3 - 12x) + a_4(16x^4 - 48x^2 + 12) + a_5(32x^5 - 160x^3 + 120x) + a_6(64x^6 - 480x^4 + 720x^2 - 120) + a_7(128x^7 - 1344x^5 + 3360x^3 - 1680x)$$

Substituting (44) into (38) and after simplification we obtained

$$645120a_7 = x(a_0 + 2a_1(x) + a_2(4x^2 - 2x) + a_3(8x^3 - 12x) + a_4(16x^4 - 48x^2 + 12) + a_5(32x^5 - 160x^3 + 120x) + a_6(64x^6 - 480x^4 + 720x^2 - 120) + a_7(128x^7 - 1344x^5 + 3360x^3 - 1680x)) + 2.718281828^x(x^2 - 2x - 6)$$

now collocating (45) using

$$x_i = a + \frac{(b-a)i}{N+1}$$
$$i = 1, 2, ..., [N - (n-1)]$$

when i=1, $x_1 = \frac{1}{8}$ we obtain;

$$645120a_7 = 0.125a_0 + 0.031250a_1 - 0.023437500a_2 - 0.1855468750a_3 + 1.406738281a_4 + 1.836059570a_5 - 13.60836792a_6 - 25.43480682a_7 - 7.06447238$$

(46) is written in matrix form as

$$\begin{pmatrix} 0.125 & 0.031250 & -0.023437500 & -0.1855468750 & 1.406738281 & 1.836059570 & -13.60836792 & -645145.43480682 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix}$$

$$= 7.06447238 \qquad (1)$$

which is one algebraic equations with 8 unknowns. The remaining 7 equations are derived from the 7 initial conditions. i.e.,

$$y_7'(x) = 2a_1 + 8xa_2 + a_3(24x^2 - 12) + a_4(64x^3 - 96x) + a_5(160x^4 - 480x^2 + 120) + a_6(384x^5 - 1920x^3 + 1440x) + a_7(896x^6 - 6720x^4 + 10080x^2 - 1680)$$

$$y_7''(x) = 8a_2 + 48xa_3 + a_4(192x^2 - 96) + a_5(640x^3 - 960x) + a_6(1920x^4 - 5760x^2 + 1440) + a_7(5376x^5 - 26880x^3 + 20160x)$$

$$y_7'''(x) = 48a_3 + 384xa_4 + a_5(1280x^2 - 960) + a_6(7680x^3 - 11520x) + a_7(26880x^4 - 80640x^2 + 20160)$$

(2)
$$y_7^{iv}(x) = 384a_4 + 2560xa_5 + a_6(23040x^2 - 11520) + a_7(107520x^3 - 161280x)$$

(3) $y_7^{v}(x) = 2560a_5 + 46080xa_6 + a_7(322560x^2 - 161280)$

(4) $y_7^{vi}(x) = 46080a_6 + 645120xa_7$

(5) $y_7^{vii}(x) = 645120a_7$

Thus,

(6)
$y(0) = 1; \Rightarrow a_0 + 12a_4 - 120a_6 = 1$

(7)
$y'(0) = 0; \Rightarrow 2a_1 - 2a_2 - 12a_3 + 120a_5 - 1680a_7 = 0$

(8)
$y'(0) = -1; \Rightarrow 8a_2 - 96a_4 + 1440a_6 = -1$

(9)
$y'''(0) = 2; \Rightarrow 48a_3 - 960a_5 + 20160a_7 = -2$

(10)
$y(1) = 0; \Rightarrow a_0 + 2a_1 + 2a_2 - 4a_3 - 20a_4 - 8a_5 + 184a_6 + 464a_7 = 0$

(11)
$y'(1) = -e; \Rightarrow 2a_1 + 6a_2 + 12a_3 - 32a_4 - 200a_5 - 96a_6 + 2576a_7 = -2.718281828$

(12)
$y''(1) = -2e; \Rightarrow 8a_2 + 48a_3 + 96a_4 - 320a_5 - 2400a_6 - 1344a_7 = -5.436563657$

Also, putting (51) to (65) in matrix form, we obtained,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 120 & 0 & -120 & 0 \\ 0 & 2 & -2 & -12 & 0 & 120 & 0 & -1680 \\ 0 & 0 & 80 & 0 & -96 & 0 & 1440 & 0 \\ 0 & 0 & 0 & 48 & 0 & -960 & 0 & 20160 \\ 1 & 2 & 2 & -4 & -20 & -8 & 184 & 464 \\ 0 & 2 & 6 & 12 & -32 & -200 & -96 & 2576 \\ 0 & 0 & 2 & 48 & 96 & -320 & -2400 & 1344 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \\ 0 \\ -2.718281828 \\ -5.436563657 \end{pmatrix}$$

solving the system (62) and (47) simultaneously using MAPPLE software we get,

$$a_0 = 1.122958872\,, a_1 = -0.5599818551\,, a_2 = -0.2407495784\,,$$
$$a_3 = -0.06625028456\,, a_4 = -0.01144812150\,, a_5 = -0.001455109745\,,$$
$$a_6 = -0.0001201548869\,, a_7 = -0.00001075851669$$

substituting the values of (63) into (41), we obtain our approximate solution;

$$y(x) = 1 + 5\times 10^{-10}x - 0.5x^2 - 0.3333333334x^3 - 0.1254955983x^4$$
$$- 0.03210406541x^5 - 0.007689912762x^6 - 0.001377090136x^7$$

Following the procedure of the methodology, the approximate solution obtained for Laguerre, Chebyshev and Legendre Polynomial respectively are as follows:

$$y_L(x) = 0.9999999997 - 1.1\times 10^{-7}x - 0.50000034x^2 - 0.33333319x^3$$
$$- 0.12549582x^4 - 0.032103433x^5 - 0.00769054484x^6 - 0.001376879442x^7$$

$$y_C(x) = 0.9999999999 - 3.9154\times 10^{10}x - 0.5000000004x^2 - 0.3333333334x^3$$
$$- 0.1254958091x^4 - 0.03210343330x^5 - 0.00769054484x^6 - 0.001376879442x^7$$
$$y_P(x) = 0.9999999994 - 4.5596\times 10^{-10}x - 0.4999999996x^2 - 0.3333333333x^3$$
$$- 0.128298960x^4 - 0.02369397926x^5 - 0.01609999890x^6 + 0.001426271911x^7$$

Example 2

Consider the linear seventh order boundary value problem

$$\frac{d^7y}{dx^7} = -y(x) - e^x(2x^2 + 12x + 35)$$

$y(0) = 1,\ y'(0) = 1,\ y''(0) = 0,\ y'''(0) = -3,\ y(1) = 0,\ y'(1) = -e,\ y''(1) = -4e$

The exact solution of the problem is given as

$$y(x) = x(1-x)e^x$$

The approximate solution obtained are as follows:

$$y_h(x) = 1\times 10^{-10} + 1.0000000001x - 0.5x^3 - 0.336916565x^4 - 0.1161229914x^5$$
$$- 0.03872249365x^6 - 0.008237949783x^7$$
$$y_l(x) = -4\times 10^{-7} + 1.000x - 5\times 10^{7}x^2 - 0.500000x^3 - 0.33691656x^4 - 0.116122988x^5$$
$$- 0.0387224936x^6 - 0.008237949801x^7$$
$$y_c(x) = -4.4756\times 10^{-11} + 1.00x - 1.25\times 10^{-10}x^2 - 0.5000000002x^3 - 0.3369165652x^4$$
$$- 0.11612299914x^5 - 0.03872249365x^6 - 0.008237949780x^7$$
$$y_p(x) = -5.882\times 10^{-12} + 1.000x - 6.06\times 10^{-10}x^2 - 0.4999999990x^3 - 0.3369165654x^4$$
$$- 0.1161229913x^5 - 0.03872249365x^6 - 0.008237949782x^7$$

Example 3

Consider the linear seventh order boundary value problem

$$\frac{d^7y}{dx^7} = y(x) - 7e^x$$

$y(0) = 1,\ y'(0) = 0,\ y''(0) = -1,\ y'''(0) = -2,\ y(1) = 0,\ y'(1) = -e,\ y''(1) = -2e$

The exact solution of the problem is given as

$$y(x) = (1-x)e^x$$

The approximate solution obtained are as follows:

$y_h(x) = 0.9999999995 + 2\times 10^{-10}x - 0.5x^2 - 0.3333333332x^3 - 0.1254955983x^4 - 0.03210406545x^5 - 0.007689912717x^6 - 0.001377090150x^7$

$y_l(x) = 0.9999999914 - 7.4\times 10^{-7}x - 0.4999997x^2 - 0.33333330x^3 - 0.125538571x^4 - 0.031975154x^5 - 0.00781882387x^6 - 0.001334119769x^7$

$y_c(x) = 1.00 - 1.1088\times 10^{-10}x - 0.5x^2 - 0.3333333335x^3 - 0.1255385687x^4 - 0.03197515433x^5 - 0.007818823847x^6 - 0.001334119772x^7$

$y_p(x) = -2.6799102\times 10^{-9}x + 1.000000000x^2 + 129.8381176x^3 - 387.6411679x^4 + 386.4862070x^5 - 128.6830991x^6 - 0.00005765024021x^7$

6 Conclusion

In this paper, we solved seventh order BVPs arising from Mathematical modeling of induction motors with two rotor circuits to test the efficiency of four orthogonal polynomials using standard collocation method. We have taken three test problems of linear type only; the results obtained from the three examples show that Hermite and Chebyshev perform best followed by Legendre polynomial and then the least in performance is Laguerre polynomial.

References

[1] Bellman R. E. and Kabala R. E. (1965). Quasilinearization and nonlinear boundary value problems, Amer. New York: Elsevier.

[2] Chen C. F. and Hsiao C. H. (1997). Haar wavelet method for solving lumped and distributed-parameter systems, IEEE Proc. Control Theory Appl., **144**: 87-94 , Dol: 10.1049/ip-cta:19970720.

[3] Ghazala A. and Hamood R. (2014). Numerical solution of seventh order boundary value problems using the reproducing kernel space, Resea. J. Appl. Sci. Eng. Tech. [Online]. **7**: 892-896, http://www.maxwellsci.com/print/rjaset/v7-892-896.pdf.

[4] Haar A. (2017). Zur theoric der orthogonalen funktionsysteme, Math. Annal, Springer, **4**: 7-20.

[5] Lepik U. and Hein H. (2014). Haar wavelets, Haar wavelets with Applications, Springer **4**: 7-20.

[6] Siddiqi S. S. and Muzammal I. (2013). Solution of seventh order boundary value problems by variation of parameters method, Rasea. J. Appl. Sci. Eng. Tech. [Online]. **5**: 176-179.

[7] Siddiqi S. S., Ghazala A. and Muzammal I. (2012). Solution of seventh order boundary value problems by variation iteration technique. Appl. Math. Sci.hikari.com/ams/ams-93-96-2012/siddiqiAMS93-96-2012-2.pdf.

[8] Yisa, B. M. (2015, a). Comparative Analysis of the Numerical Effectiveness of the Four Kinds of Chebyshev Polynomials, J. of NAMP, **32**: 193-198.

[9] Yisa, B. M. (2015, b). Numerical Performances of Two Orthogonal Polynomials in the Tau Method for solutions of Ordinary Differential Equations, J. of NAMP, **30**: 15-20.

Table 1. Table of Results for Example 1

width=1.4center

x	Exact	$H_n(x)$	$L_n(x)$	$T_n(x)$	$P_n(x)$
0.00	1.0000	0.00000000000	3.870000×10^{-7}	1.00000×10^{-10}	6.000000×10^{-10}
0.10	0.9947	4.6217700×10^{-5}	4.399839×10^{-4}	4.621190×10^{-5}	4.641680×10^{-5}
0.20	0.9771	2.1757200×10^{-5}	2.752580×10^{-3}	2.17571×10^{-5}	1.946010×10^{-5}
0.30	0.9449	4.3430000×10^{-7}	6.173840×10^{-3}	4.34900×10^{-7}	8.223600×10^{-6}
0.40	0.8951	$8.52031000 \times 10^{-6}$	9.208010×10^{-3}	8.521600×10^{-6}	2.402250×10^{-5}
0.50	0.8244	4.430700×10^{-5}	1.037330×10^{-2}	4.430870×10^{-5}	6.620910×10^{-5}
0.60	0.7288	4.202810×10^{-5}	9.259490×10^{-3}	4.202630×10^{-5}	1.877510×10^{-5}
0.70	0.6041	2.132330×10^{-5}	6.197460×10^{-3}	2.132200×10^{-5}	3.149300×10^{-6}
0.80	0.4451	5.814800×10^{-6}	2.739390×10^{-3}	5.814100×10^{-6}	3.371800×10^{-6}
0.90	0.2459	5.981560×10^{-5}	5.496220×10^{-4}	5.981530×10^{-5}	5.797555×10^{-5}
1.00	0.0000	8.000000×10^{-12}	4.818000×10^{-6}	1.08460×10^{-10}	6.000000×10^{-10}

Table 2. Table of Results for Example 2

width=1.4center

x	Exact	$H_n(x)$	$L_n(x)$	$T_n(x)$	$P_n(x)$
0.00	1.0000	1.00000×10^{-10}	-4.00000×10^{-7}	-4.4756×10^{-11}	-5.88200×10^{-12}
0.10	0.9947	3.48922×10^{-5}	3.52878×10^{-5}	3.48925×10^{-5}	3.48925×10^{-5}
0.20	0.9771	2.11908×10^{-5}	2.08073×10^{-5}	2.11904×10^{-5}	2.110905×10^{-5}
0.30	0.9449	4.12330×10^{-5}	4.15991×10^{-5}	4.12334×10^{-5}	4.12335×10^{-5}
0.40	0.8951	1.37326×10^{-5}	1.33867×10^{-5}	1.37321×10^{-5}	1.37321×10^{-5}
0.50	0.8244	5.55262×10^{-5}	5.58514×10^{-5}	5.55268×10^{-5}	5.55269×10^{-5}
0.60	0.7288	3.13565×10^{-5}	3.16627×10^{-5}	3.13573×10^{-5}	3.13573×10^{-5}
0.70	0.6041	4.45510×10^{-5}	4.48421×10^{-5}	4.45518×10^{-5}	4.45519×10^{-5}
0.80	0.4451	3.06987×10^{-5}	3.09812×10^{-5}	3.06996×10^{-5}	3.06996×10^{-5}
0.90	0.2459	3.93292×10^{-5}	3.96114×10^{-5}	3.93301×10^{-5}	3.93300×10^{-5}
1.00	0.0000	1.06700×10^{-9}	-2.91400×10^{-7}	1.00000×10^{-10}	2.68000×10^{-10}

TABLE 3. Table of Results for Example 3

width=1.4center

x	Exact	$H_n(x)$	$L_n(x)$	$T_n(x)$	$P_n(x)$
0.00	1.0000	5.0000×10^{-5}	8.60000×10^{-9}	0.00000000	0.00000000
0.10	0.9947	4.62122×10^{-5}	4.63723×10^{-5}	4.62149×10^{-5}	4.62149×10^{-5}
0.20	0.9771	2.17567×10^{-5}	2.14988×10^{-5}	2.17221×10^{-5}	2.17221×10^{-5}
0.30	0.9449	4.34700×10^{-7}	8.37400×10^{-7}	5.53700×10^{-7}	5.53700×10^{-7}
0.40	0.8951	8.52070×10^{-6}	9.06300×10^{-6}	8.75800×10^{-6}	8.75790×10^{-6}
0.50	0.8244	4.43073×10^{-5}	5.50297×10^{-5}	4.46426×10^{-5}	4.46426×10^{-5}
0.60	0.7288	4.20277×10^{-5}	4.12422×10^{-5}	4.16718×10^{-5}	4.16718×10^{-5}
0.70	0.6041	2.13230×10^{-5}	2.05792×10^{-5}	2.10449×10^{-5}	2.10450×10^{-5}
0.80	0.4451	5.81460×10^{-6}	5.17860×10^{-6}	5.67400×10^{-6}	5.67420×10^{-6}
0.90	0.2459	5.98154×10^{-5}	5.98154×10^{-5}	5.97872×10^{-5}	5.97874×10^{-5}
1.00	0.0000	1.1700×10^{-10}	0.00000000	0.00000000	2.69000×10^{-10}