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Numerical Solution of Two Dimensional Poisson and Laplace Equations

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Abstract

The study identifies the use of Finite Element Method with both rectangular and Triangular elements in solving two dimensional problems with Dirichlet boundary condition in a rectangular domain. We use the product functions of linear polynomial basis constructed by (Aliu & Bamigbola, 2014) on the rectangular elements. Attempt was made at constructing the basis functions for Triangular elements. The Finite Difference scheme was also used for comparison. The results obtained using both the rectangular and triangular elements via the finite element method and the Central Finite Difference Scheme compared favourably with the exact solution. The rectangular element however gives a better approximation.

1 Introduction

Several researchers have worked on numerical solution of partial differential equations using different computational approaches. (Mittal & Gahlaut, 1991) used the Finite Difference scheme to solve Poisson's equation in polar coordinates. (Jeon, 2001) employed an indirect scalar boundary integral formulaae for the biharmonic equation; (Li & Zhu, 2009) approached the solution to biharmonic problem via the Galerkin boundary method; (Mai-Day & Tanner, 2007) used a spectral collocation approach based on Chebyshev polynomial for biharmonic problems in irregular domains. They also adopted the

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collocation method based on one-dimensional radial Basis Function interpolation scheme for solving Partial Differential Equations; (Perrey & Morsche, 2002) used β-spline approximation and fast wavelet transform for a better evaluation of particular solutions of the poisson equation; (Young & Bang, 1997) employed the MATLAB programming method to solve PDE problems via the Finite Element method; (Salvador & Dina, 2002) had an experimental study of the Neumann and Dirichlet boundary conditions in two-dimensional electrostatic problems; (Shabbir, Rafiq, & Ahmed, 2012) considered the finite element solution for two-dimensional Laplace Equation with Dirichlet Boundary Conditions; (Jochen, 1999) Albert et al(1999) adopted a simple and open-box MATLAB implementation of combined Courant's triangles and elements on parallelograms for the numerical solution of elliptic problems with mixed Dirichlet and Neumann boundary conditions; (Parag & Krishna, 2013) considered the numerical solution for two-dimensional Laplace Equation with Dirichlet boundary condition. Working on the Exact and Numerical Solution of Poisson Equation for Electrostatic potential problems, (Selcuk, 2008) observed that the main advantage of the boundary element method is its replacement of the original problem with an integral equation defined on the boundary of the solution domain; (Slavyanov & Zenger., 2005) however asserted that, a popular and widely used approach to the solution of partial differential equation is the Finite Element Method, which as a numerical method, reduces the initial problem to the task of solving a system of linear equations. Thus, making it easily treated by computer assisted programs. The Laplace and and Poisson Equations were according to (Peter, 2014) observed as the basic equilibrium equations in a remarkable variety of physical system. Thus, our solution Approach to Partial Differential Equation shall be directed to solving Poisson and Laplace Equation related problems.

2 Finite Element Method

The finite element method hinges on the sub-domain principle, that is it involves division or partitioning of the domain of the problem under investigation into a finite number of sub- domains.

2.1 Basis functions for Triangular Elements

In this subsection, we wish to demonstrate the rigours involved by constructing the basis for triangular elements, whereas the product basis is simply obtained as discussed above. The use of triangular element in approximating a given region is important, especially when the region R is not rectangular. (Davis, 1984) asserted that if the Galerkin method can be formulated with triangular elements, irregular regions can be well handled though the use of triangulation. see fig below:

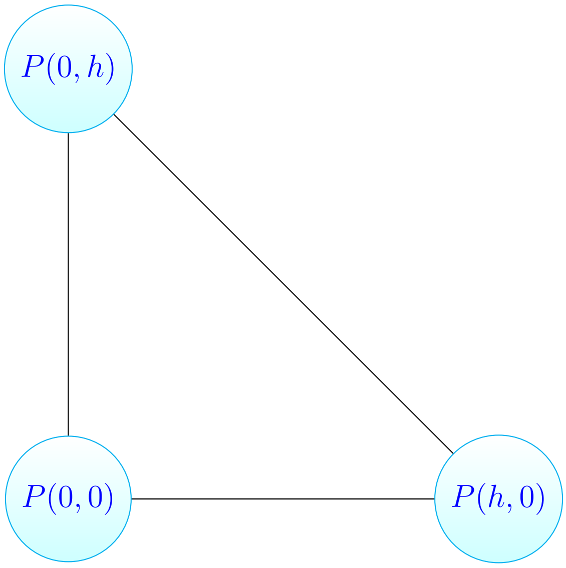

Fig.1 Reference Triangular Element

Construction of Linear Basis for the Triangular Elements
We note here that any triangle with vertices $p_i(x,y)$ $i=1,2,3$ can be transformed into a rectangular equilateral triangle with

$$x = x_1 + (x_2 - x_1)t + (x_3 - x_1)r,\ y = y_1 + (y_2 - y_1)t + (y_3 - y_1)r \tag{2.1}$$

We now wish to approximate the function $u(x,y)$ by the linear transformation

$$p(t,r) = a_1 + a_2 t + a_3 r \tag{2.2}$$

Let $t = \frac{x}{h},\ r = \frac{y}{h}$ with the nodal points defined at $(t,r) = (0,0),\ (h,0)\ (0,h)$ $p(x,y) = a_1 + a_2\frac{x}{h} + a_3\frac{y}{h}$
Thus $p_1(0,0) = a_1$
$p_2(h,0) = a_1 + a_2$
$p_3(0,h) = a_1 + a_3$
$\Rightarrow$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{2.3}$$

$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$
The basis functions are

$$\phi_i = \begin{pmatrix} 1 & x/h & y/h \end{pmatrix} A^{-1} \tag{2.4}$$

i.e.,
$\phi_1 = 1 - \frac{y}{h} - \frac{x}{h}$
$\phi_2 = \frac{x}{h}$
$\phi_3 = \frac{y}{h}$.
The above C_1^0 triangular elements also have local support and are complete.

2.2 Basis Function for Quadrilateral Elements

We make use of the product of already constructed one dimensional basis functions in part one of the thesis to obtain approximate solution to two dimensional problem. Consider a rectangular field defined on the coordinate axis.

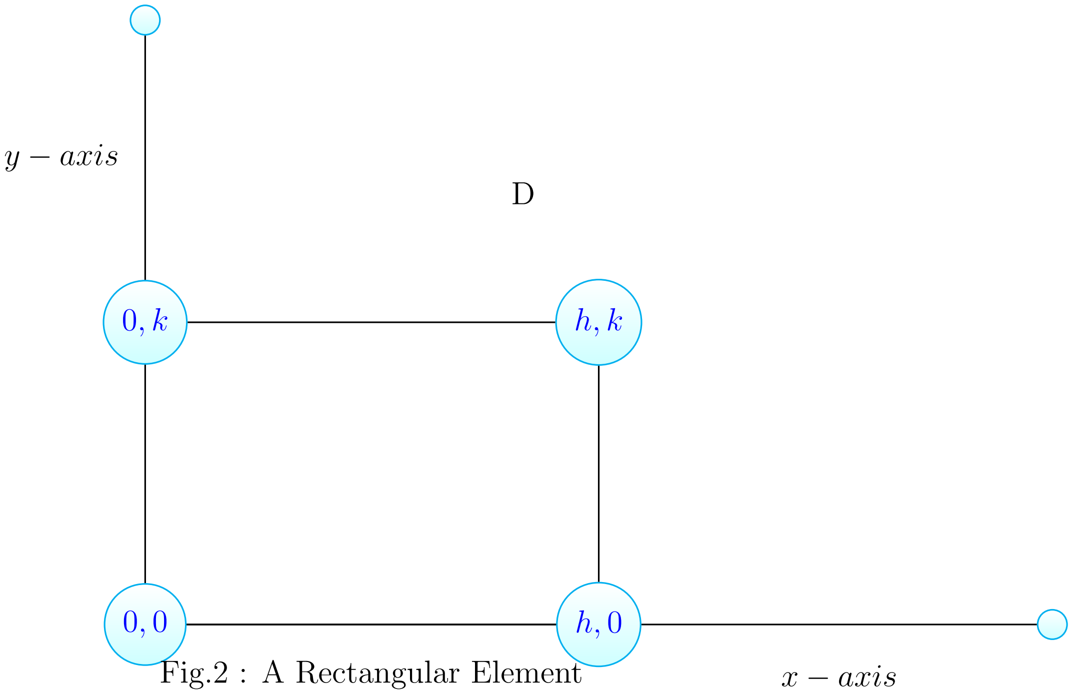

Fig.2 : A Rectangular Element

We recall the linear basis functions constructed for one dimensional elements;

$$\phi_1(x) = 1 - \frac{x}{h}, \phi_2(x) = \frac{x}{h} \tag{2.5}$$

Here, $\phi_1(y) = 1 - \frac{y}{k}$,
$\phi_2(y) = \frac{y}{k}$
We define the rectangular element as the product of the two linear basis, taking into account the anticlockwise rotation of the four cardinal points(vertices) of the quadrilateral.

Let $\alpha_i(x, y)$ $i = 1, 2, 3, 4$ denote the product basis function at the corners of the rectangular elements.

$$\alpha_1(x, y) = \phi_1(x)\phi_1(y) = 1 - \frac{x}{h} - \frac{y}{k} + \frac{xy}{hk}$$

$$\alpha_2(x, y) = \phi_2(x)\phi_1(y) = \frac{x}{h} - \frac{xy}{hk}$$

$$\alpha_3(x, y) = \phi_2(x)\phi_2(y) = \frac{xy}{hk}$$

$$\alpha_4(x, y) = \phi_1(x)\phi_2(y) = \frac{y}{k} - \frac{xy}{hk}$$

Note that the above basis satisfies requirements for local support and completeness.

2.3 Finite Element Solution

The FEM as an expansion scheme, prescribes

$$U(x_1, x_2) = \sum_{i=1}^{s^m} \alpha_i(x_1, x_2) U_i \tag{2.6}$$

and

$$f(x_1, x_2) = \sum_{i=1}^{s^m} \alpha_i(x_1, x_2) f_i \tag{2.7}$$

where U_i and f_i are constants to be determined, and α_i is the ith component in (3.3). Now, let us consider the equation:

$$Lu(x_1, x_2) = f(x_1, x_2) \tag{2.8}$$

$$au + b\frac{\partial u}{\partial n} = g \tag{2.9}$$

where for simplicity, L is a linear operator, f is the source (or load) function. $\frac{\partial u}{\partial n}$ is the directional derivative in the outward normal to the boundary $\partial\Omega$.
Let $U \approx u$. Then,

$$LU - f = R \tag{2.10}$$

where R is the residual. Note that R can be expressed in terms of the basis α.
From the study of analysis, linear independence and orthogonality are equivalent properties. Thus, for two-dimensional problems,

$$< R, \alpha_j >= 0 \quad j = 1,\ 2,\ \cdots,\ s^2 \tag{2.11}$$

$$\iint (LU - f)\alpha_j dx_1 dx_2 = 0 \quad j = 1,\ 2,\ \cdots,\ s^2 \tag{2.12}$$

$$\iint LU\alpha_j dx_1 dx_2 = \iint (f\alpha_j) dx_1 dx_2 \quad j = 1,\ 2,\ \cdots,\ s^2 \tag{2.13}$$

Applying the principle of integration by parts

$$\int udv = uv - \int vdu \tag{2.14}$$

to the LHS of (4.12) and noting that LU contains partial derivatives of α_j, the LHS becomes

$$\iint \left(\int \alpha_j d(LU) \right) dx_1 dx_2 = \iint LU\alpha_j dx_1 dx_2 - \iint \left(\int LUd(\alpha_j) \right) dx_1 dx_2 \tag{2.15}$$

Note that

$$d\alpha_j = \frac{\partial(\alpha_j)}{\partial(x_1)} dx_1 + \frac{\partial(\alpha_j)}{\partial(x_2)} dx_2$$

and that

$$\alpha_j = 0\, at\ the\ boundary$$

(). Thus, by Euler-Lagrange method ().

$$\iint LU_j dx_1 dx_2 = 0.$$

Hence, variational (weak) formulation for (4.7) is

$$\iint \left(\int LUd(\alpha_j) \right) dx_1 dx_2 = \iint (f\alpha_j) dx_1 dx_2 \qquad j = 1,\ 2,\ \cdots,\ s^2 \tag{2.16}$$

which on substituting (4.5) and (4.6) gives

$$\iint \left(\int \sum_{i=1}^{s^2} L(\alpha_i U_i) d(\alpha_j) \right) dx_1 dx_2 = \iint (\sum_{i=1}^{s^2} L(\alpha_i f_i \alpha_j) dx_1 dx_2 \qquad j = 1,\ 2,\ \cdots,\ s^2. \tag{2.17}$$

(4.14) yields a system of linear equations of the form

$$AU = b$$

where

$$A = (a_{i,j}) = \iint \left(\int L(\alpha_i) d(\alpha_j) \right) dx_1 dx_2$$

and

$$b = (b_i) \ = \iint L(\alpha_i f_i \alpha_j dx_1 dx_2.$$

3 Numerical Consideration

The following are the numerical problems considered as test cases of the product basis via the finite element method:

3.1 Problem One

Obtain the temperature distribution $u(x, y)$ over a square plate for which specific source provides a uniform heating of 5 units to sustain the heating but the edges of the plate are kept at an ice-cold temperature.
The above problem can be modeled as the Poisson equation

$$-(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = 5, 0 < x, y < 1, \tag{3.1}$$

$u = 0$ on the boundary of the square $0 \leq x \leq 1,\ 0 \leq y \leq 1$
Here:

$$L =\equiv -(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$$

, $f = 5$ and the domain is

$$D = \{(x, y)/0 \leq x, y \leq 1\}$$

being a square plate.
The analytical solution of the problem reads:

$$u(x,y) = \frac{40}{\pi^4} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \sin(2k-1)\pi x \sin(2k-1)\pi y \;\; k = 1.2.3, \cdot \tag{3.2}$$

For the basis:
$\alpha_1(x,y) = 1 - \frac{x}{h} - \frac{y}{h} + \frac{xy}{h^2}$
$\alpha_2(x,y) = \frac{x}{h} - \frac{xy}{h^2}$
$\alpha_3(x,y) = \frac{xy}{h^2}$
$\alpha_4(x,y) = \frac{y}{h} - \frac{xy}{h^2}$
The Local finite element equation for the problem reads;

$$\begin{pmatrix} 1/3 & -1/6 & -1/6 & 0 \\ -1/6 & 1/2 & -1/6 & -1/6 \\ -1/6 & -1/6 & 1/3 & 0 \\ 0 & -1/6 & 0 & 1/6 \end{pmatrix} \begin{pmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \end{pmatrix} = \begin{pmatrix} \frac{5}{8}h^2 \\ \frac{25}{24}h^2 \\ \frac{5}{8}h^2 \\ \frac{5}{24}h^2 \end{pmatrix} \tag{3.3}$$

With the domain of the problem divided into four square elements, $h = \frac{1}{2}$

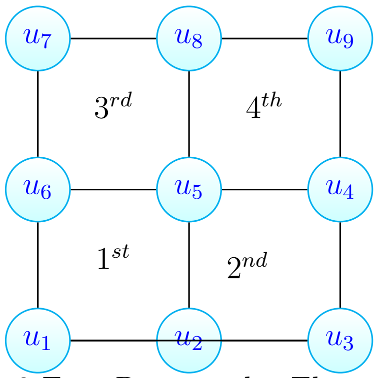

Fig.3 Four Rectangular Elements

If $u_1, u_2, u_3, u_4, ..., u_9$ are respectively the global nodal points (see Fig.2). To build the stiffness matrix for each element, we use the local finite element equation in (5.2) together with its transformed form for the two elements 3 and 4 whose nodal points are not in counter clockwise order.
The transformation reads

$$\begin{pmatrix} 1/2 & -1/6 & -1/6 & -1/6 \\ -1/6 & 1/3 & 0 & -1/6 \\ -1/6 & 0 & 1/6 & 0 \\ -1/6 & -1/6 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \end{pmatrix} = \begin{pmatrix} \frac{25}{96} \\ \frac{5}{32} \\ \frac{5}{96} \\ \frac{5}{32} \end{pmatrix} \tag{3.4}$$

the governing global system of linear equations is

$$\begin{pmatrix} 1/3 & -1/6 & 0 & 0 & -1/6 & 0 & 0 & 0 & 0 \\ -1/6 & 5/6 & -1/6 & -1/6 & -1/6 & -1/6 & 0 & 0 & 0 \\ 0 & -1/6 & 1/2 & -1/6 & -1/6 & 0 & 0 & 0 & 0 \\ 0 & -1/6 & -1/6 & 5/6 & -1/6 & 0 & 0 & -1/6 & -1/6 \\ -1/6 & -1/6 & -1/6 & -1/6 & 4/3 & -1/6 & -1/6 & -1/6 & -1/6 \\ 0 & -1/6 & 0 & 0 & -1/6 & 1/2 & 0 & -1/6 & 0 \\ 0 & 0 & 0 & 0 & -1/6 & 0 & 1/6 & 0 & 0 \\ 0 & 0 & 0 & -1/6 & -1/6 & -1/6 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -1/6 & -1/6 & 0 & 0 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 5/32 \\ 5/6 \\ 25/96 \\ 5/6 \\ 5/8 \\ 5/24 \\ 5/96 \\ 5/24 \\ 5/32 \end{pmatrix} \tag{3.5}$$

Imposing the boundary conditions

$$u_1 = u_2 = u_3 = u_4 = u_6 = u_7 = u_8 = u_9 = 0$$

gives the solution as
$\frac{4}{3}u_5 = \frac{5}{8}$
$\Rightarrow u_5 = \frac{15}{32}$ *or* 0.46875

With the domain of the problem divided into 9 rectangular elements

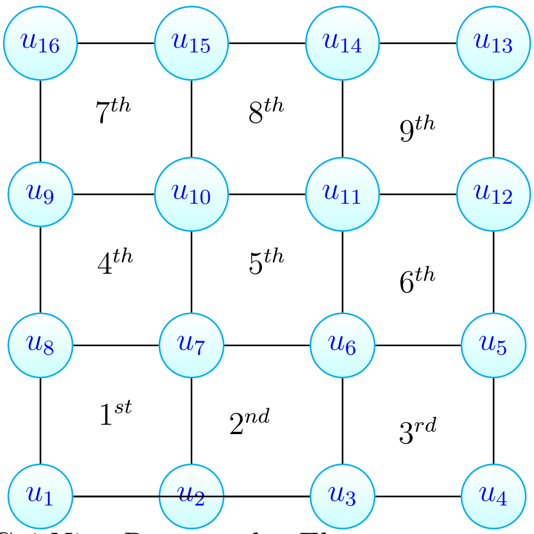

FIG.4 Nine Rectangular Elements

Assembling the stiffness matrices for the elements, together with the corresponding load vectors, we have the following global finite element equation:

$$\begin{pmatrix}
\frac{1}{3} & \frac{-1}{6} & 0 & 0 & 0 & 0 & \frac{-1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-1}{6} & 5/6 & \frac{-1}{6} & 0 & 0 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-1}{6} & 5/6 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-1}{6} & 1/2 & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-1}{6} & \frac{-1}{6} & 5/6 & \frac{-1}{6} & 0 & 0 & 0 & 0 & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & 0 & 0 \\
0 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 4/3 & \frac{-1}{6} & 0 & 0 & -1/6 & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & 0 & 0 \\
\frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & \frac{-1}{6} & 4/3 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-1}{6} & 0 & 0 & 0 & 0 & 0\frac{-1}{6} & 1/2 & 0 & \frac{-1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{6} & 0 & 1/2 & \frac{-1}{6} & 0 & 0 & 0 & 0 & \frac{-1}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 4/3 & \frac{-1}{6} & 0 & 0 & -1/6 & -1/6 & -1/6 \\
0 & 0 & 0 & 0 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & \frac{-1}{6} & 4/3 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 0 \\
0 & 0 & 0 & 0 & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & 0 & 0 & \frac{-1}{6} & 5/6 & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{6} & \frac{-1}{6} & 1/3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 0 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{6}
\end{pmatrix}
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \\ u_{16} \end{pmatrix}
=
\begin{pmatrix} 5/72 \\ 5/27 \\ 5/27 \\ 25/216 \\ 5/27 \\ 5/18 \\ 5/18 \\ 5/54 \\ 5/54 \\ 5/18 \\ 5/36 \\ 5/54 \\ 5/72 \\ 5/54 \\ 5/54 \\ 5/216 \end{pmatrix} \tag{3.6}$$

Imposing the boundary conditions $u_1 = u_2 = u_3 = u_4 = u_5 = u_8 = u_9 = u_{12} = u_{13} = u_{14} = u_{16} = 0$

We set the values of the rows and columns corresponding to the nodes to zero with their diagonal element to 1. We have

$$\Rightarrow \begin{pmatrix} 4/3 & -1/6 & -1/6 & -1/6 \\ -1/6 & 4/3 & -1/6 & -1/6 \\ -1/6 & -1/6 & 4/3 & -1/6 \\ -1/6 & -1/6 & -1/6 & 4/3 \end{pmatrix} \begin{pmatrix} u_6 \\ u_7 \\ u_{10} \\ u_{11} \end{pmatrix} = \begin{pmatrix} 5/18 \\ 5/18 \\ 5/18 \\ 5/18 \end{pmatrix}. \tag{3.7}$$

Hence,

$$\begin{pmatrix} u_6 \\ u_7 \\ u_{10} \\ u_{11} \end{pmatrix} = \begin{pmatrix} 0.33333 \\ 0.33333 \\ 0.33333 \\ 0.33333 \end{pmatrix} \tag{3.8}$$

i.e
$u(2/3, 1/3) = 0.33333$
$u(1/3, 1/3) = 0.33333$
$u(1/3, 2/3) = 0.33333$
$u(2/3, 2/3) = 0.33333$

3.2 PROBLEM 2

Consider a steady state heat condition in a rectangular plate with all its faces insulated, such that heat travels only along X,Y direction. The boundary conditions are as depicted in the fig bellow;

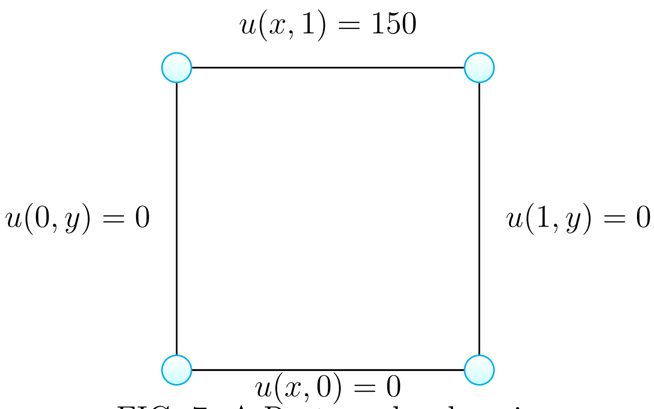

FIG. 7: A Rectangular domain

The problem is formulated as a two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = 0 \tag{3.9}$$

The Dirichlet boundary conditions read
$u(x,0) = 0,\ u(0,y) = 0,\ u(x,1) = 150,\ u(1,y) = 0$ The exact solution to the problem using separation of variable reads:

$$u(x,y) = \frac{600}{\pi} \sum_{k=1}^{\infty} \frac{sin(2k-1)\pi x sinh(2k-1)\pi y}{(2k-1)sinh(2k-1)\pi}; k = 1, 2, 3, ... \tag{3.10}$$

Dividing the domain into 9 square elements with 16 nodes, we have:

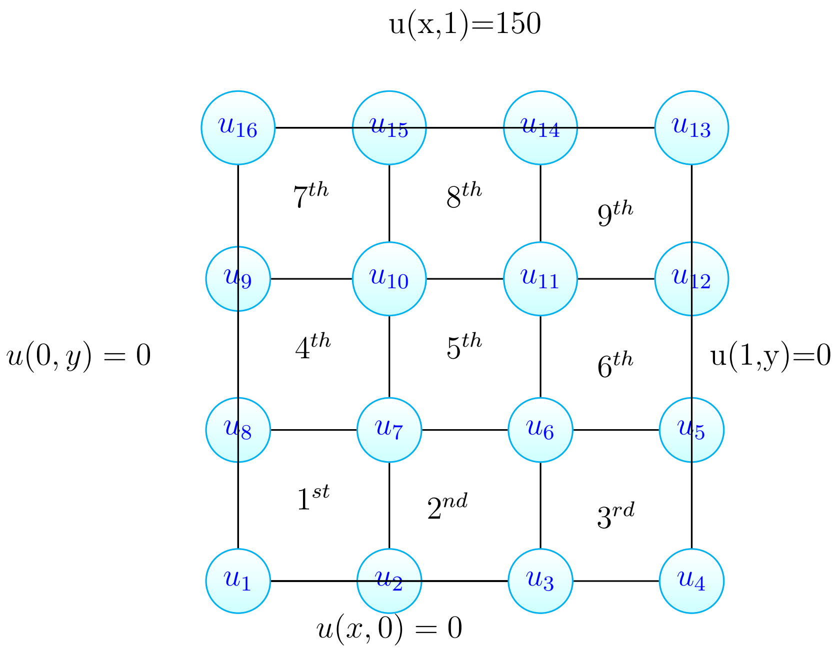

FIG.8: 9 Sure Elements

The boundary conditions read
$u_1 = u_2 = u_3 = u_4 = u_5 = u_8 = u_9 = u_{12} = 0; u_{13} = 75; u_{14} = u_{15} = 150, u_{16} = 75$

Using the usual weak formulation of the problem; The local finite element equation for a rectangular element is given as

$$\begin{pmatrix} 1/3 & -1/6 & -1/6 & 0 \\ -1/6 & 1/2 & -1/6 & -1/6 \\ -1/6 & -1/6 & -1/3 & 0 \\ 0 & -1/6 & 0 & 1/6 \end{pmatrix} \begin{pmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.11}$$

Assembling the elements according to the global nodal numbers, we have the following global finite element equation;

$$\begin{pmatrix}
\frac{1}{3} & \frac{-1}{6} & 0 & 0 & 0 & 0 & \frac{-1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-1}{6} & 5/6 & \frac{-1}{6} & 0 & 0 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-1}{6} & 5/6 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-1}{6} & 1/2 & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-1}{6} & \frac{-1}{6} & 5/6 & \frac{-1}{6} & 0 & 0 & 0 & 0 & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & 0 & 0 \\
0 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 4/3 & \frac{-1}{6} & 0 & 0 & -1/6 & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & 0 & 0 \\
\frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & \frac{-1}{6} & 4/3 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-1}{6} & 0 & 0 & 0 & 0 & 0\,\frac{-1}{6} & 1/2 & 0 & \frac{-1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{6} & 0 & 1/2 & \frac{-1}{6} & 0 & 0 & 0 & 0 & \frac{-1}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 4/3 & \frac{-1}{6} & 0 & 0 & -1/6 & -1/6 & -1/6 \\
0 & 0 & 0 & 0 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & \frac{-1}{6} & 4/3 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 0 \\
0 & 0 & 0 & 0 & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & 0 & 0 & \frac{-1}{6} & 5/6 & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{6} & \frac{-1}{6} & 1/3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 0 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & 0 & 0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{6}
\end{pmatrix}
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \\ u_{16} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.12}$$

Imposing the boundary conditions
$u_1 = 0, u_2 = 0, u_3 = 0, u_4 = 0, u_5 = 0, u_8 = 0, u_9 = 0, u_(12) = 0, u_{13} = 75, u_{14} = 150, u_{15} = 150, u_{16} = 75$
we have

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4/3 & -1/6 & 0 & 0 & -1/6 & -1/6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & & 0 & 0 & -1/6 & 4/3 & 0 & 0 & -1/6 & -1/6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1/6 & -1/6 & 0 & 0 & 4/3 & -1/6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1/6 & -1/6 & 0 & 0 & -1/6 & 4/3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \\ u_{16} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 125/2 \\ 125/2 \\ 0 \\ 75 \\ 150 \\ 150 \\ 75 \end{pmatrix} \tag{3.13}$$

Which reduces to

$$\begin{pmatrix} 4/3 & -1/6 & -1/6 & -1/6 \\ -1/6 & 4/3 & -1/6 & -1/6 \\ -1/6 & -1/6 & 4/3 & -1/6 \\ -1/6 & -1/6 & -1/6 & 4/3 \end{pmatrix} \begin{pmatrix} u_6 \\ u_7 \\ u_{10} \\ u_{11} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 125/2 \\ 125/2 \end{pmatrix} \Rightarrow \tag{3.14}$$

$$\Rightarrow \begin{pmatrix} u_6 \\ u_7 \\ u_{10} \\ u_{11} \end{pmatrix} = \begin{pmatrix} 16.6667 \\ 16.6667 \\ 58.3333 \\ 58.3333 \end{pmatrix}$$

4 ANALYSIS OF RESULTS

The results obtained when the rectangular elements and the Triangular elements are used in solving the problem are hereby compared, with the exact solution at some selected nodal points:

Table 1: Summary of Results for Problem 1

Comparison of Results							
Nodal point	4 Rec. FE	4 Tri. FE	9 Rec. FE	9 Tri. FE	18 Tri. FEM	16 Grid Points FD	Exact
u(1/2,1/2)	0.46875	0.31250					0.43165
u(2/3,1/3)			0.33333	0.27778	0.27778	0.27778	0.31186
u(1/3,1/3)			0.33333	0.27778	0.27778	0.27778	0.31186
u(1/3,2/3)			0.33333	0.27778	0.27778	0.27778	0.31186
u(2/3,2/3)			0.33333		0.27778	0.27778	0.31186

Table 2: Absolute Error in the computation

Absolute Error						
Nodal point	4 Rec. FE	4 Tri. FE	9 Rec. FE	9 Tri. FE	18 Tri. FE	16 Grids FD
u(1/2,1/2)	0.03710	0.11915				
u(2/3,1/3)			0.02147	0.03408	0.03408	0.03408
u(1/3,1/3)		0.02147	0.03408	0.03408	0.03408	
u(1/3,2/3)			0.02147	0.03408	0.03408	
u(2/32/3)			0.02147	0.03408	0.0.03408	0.03408

We can observe from the above analysis that the rectangular elements using the constructed basis function, better approximates the solution to the elliptic problem considered. The results obtained with rectangular elements give better approximation than that of triangular elements regardless of the number of elements used. It needs be noted however that the higher the number of elements employed in the construction, the more accurate the result. A uniform temperature was observed throughout the solution region.

Table 3: Summary of Results for Problem 2

Comparison of Results				
Nodal point	9 Rectangular Elements	18 Triangular Elements	9 Finite Diff. Grid point	Exact
u(2/3,1/3)	16.6667	18.7500	18.7500	17.8923
u(1/3,1/3)	16.6667	18.7500	18.7500	17.8923
u(1/3,2/3)	58.3333	56.2500	56.2500	57.1079
u(2/3,2/3)	58.3333	56.2500	56.2500	57.107

Table 4: Absolute Error in the computation

Absolute Error			
Nodal point	9 Rectangular Elements	18 Triangular Elements	9 Finite Diff. Grid point
u(2/3,1/3)	1.2256	0.8577	0.8577
u(1/3,1/3)	1.2256	0.8577	0.8577
u(1/3,2/3)	1.2254	0.8579	0.8579
u(2/32/3)	1,2254	0.8579	0.8579

5 CONCLUSION

We presented the application of Finite Element Method in solving two dimensional elliptic problems using the product function of one dimensional linear basis for rectangular elements, and basis functions constructed with the aid of rectangular equilateral triangular transformation for the triangular elements. The results obtained on application compared favourably with the exact solution where they are available and the Finite Difference solutions at given grid points. It could however be observed that a relatively better results was obtained when the present finite element approach is used. A more better approximation is assured if the elements are further discritized into smaller rectangular elements with an increased number of nodal points.

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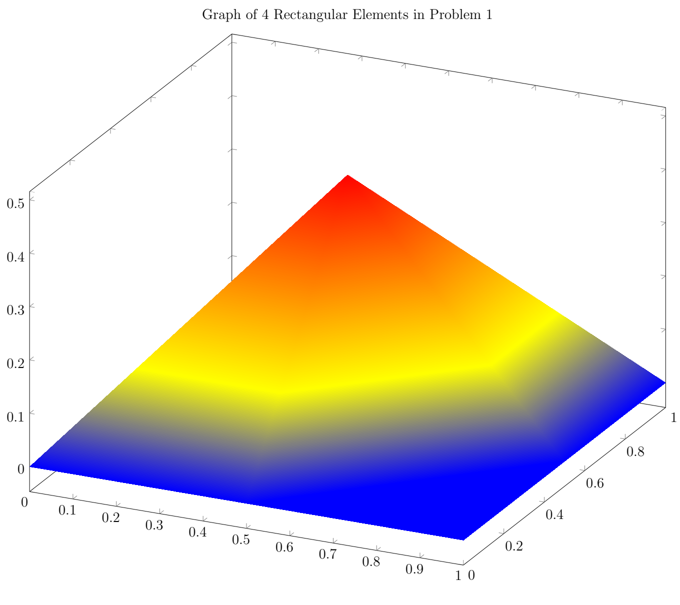
6 Appendices

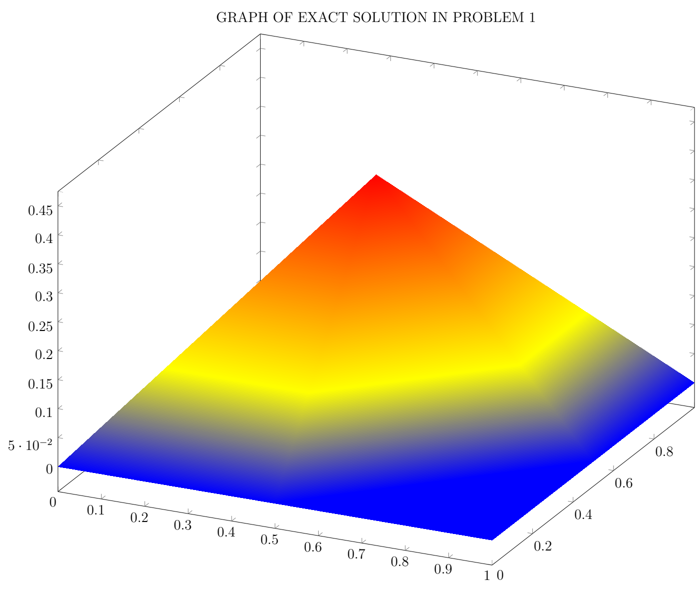
Appendix A

PLOTS 1 and 2 give the graphical solution to problem one for different number of elements compared for FEM, FDM and the analytical method.

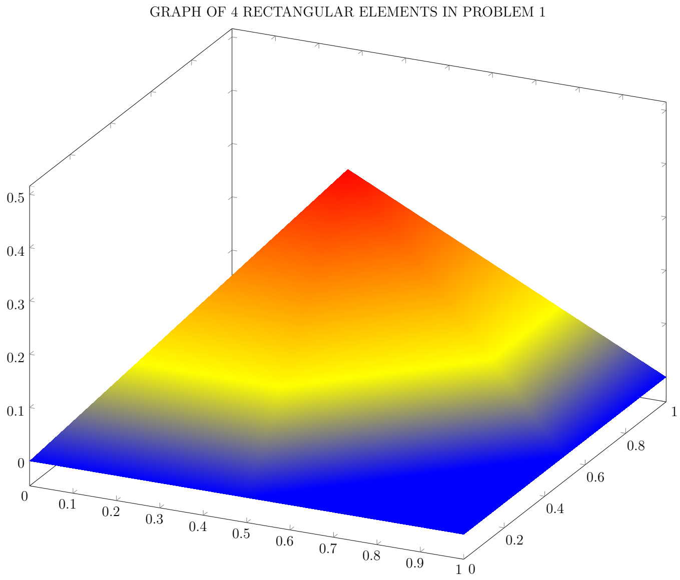
PLOT 1: (SOLUTION TO PROBLEM ONE (4 ELEMENTS) WITH BOUNDARY CONDITION)

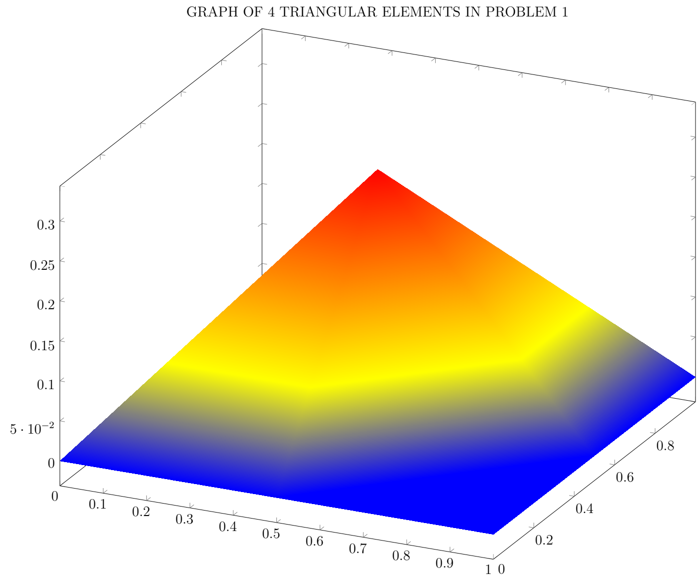
4 Rectangular Elements and Exact Solution to Problem 1 plotted side to side





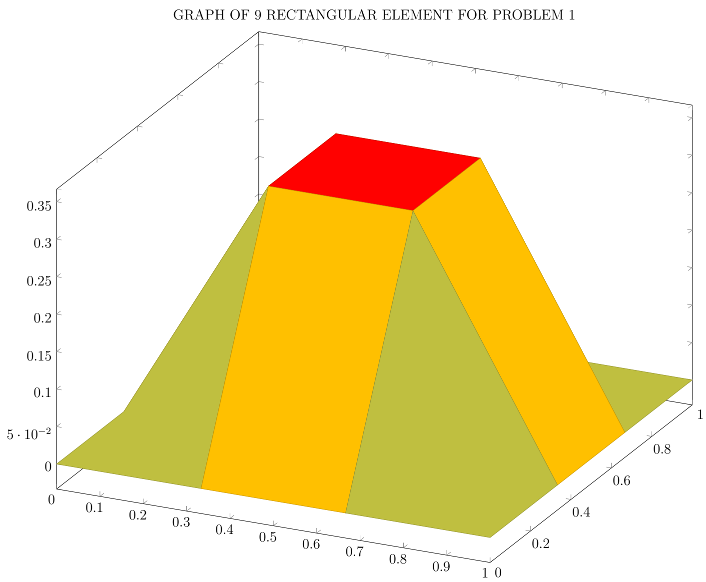
4 Rectangular Elements and 4 Triangular Elements to Problem 1 plotted side to side

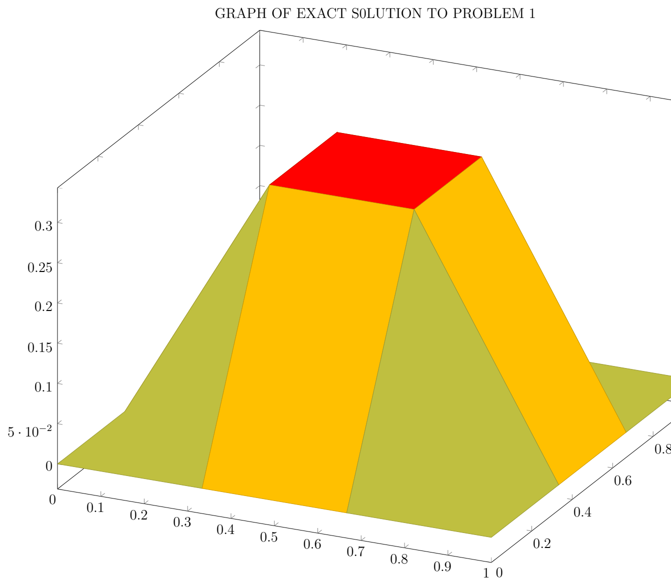




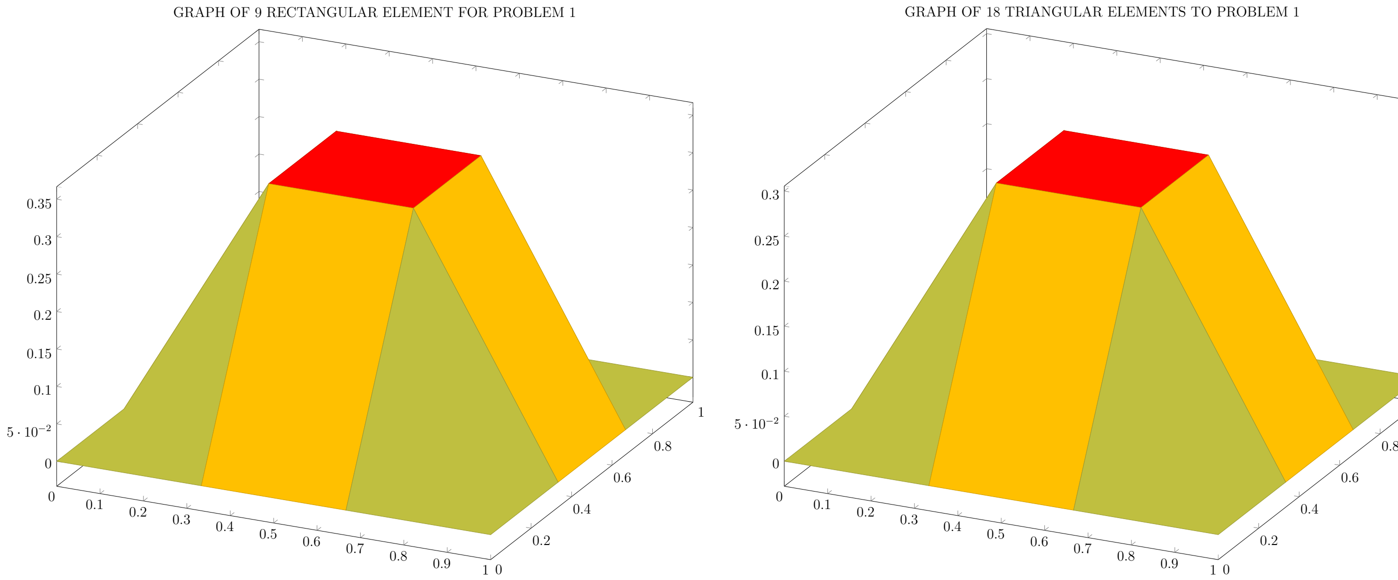
PLOT 2: (SOLUTION TO PROBLEM ONE WITH BOUNDARY CONDITIONS)

FEM Solution with 9 Rectangular Elements and Exact Solution to Problem 1 plotted side to side.

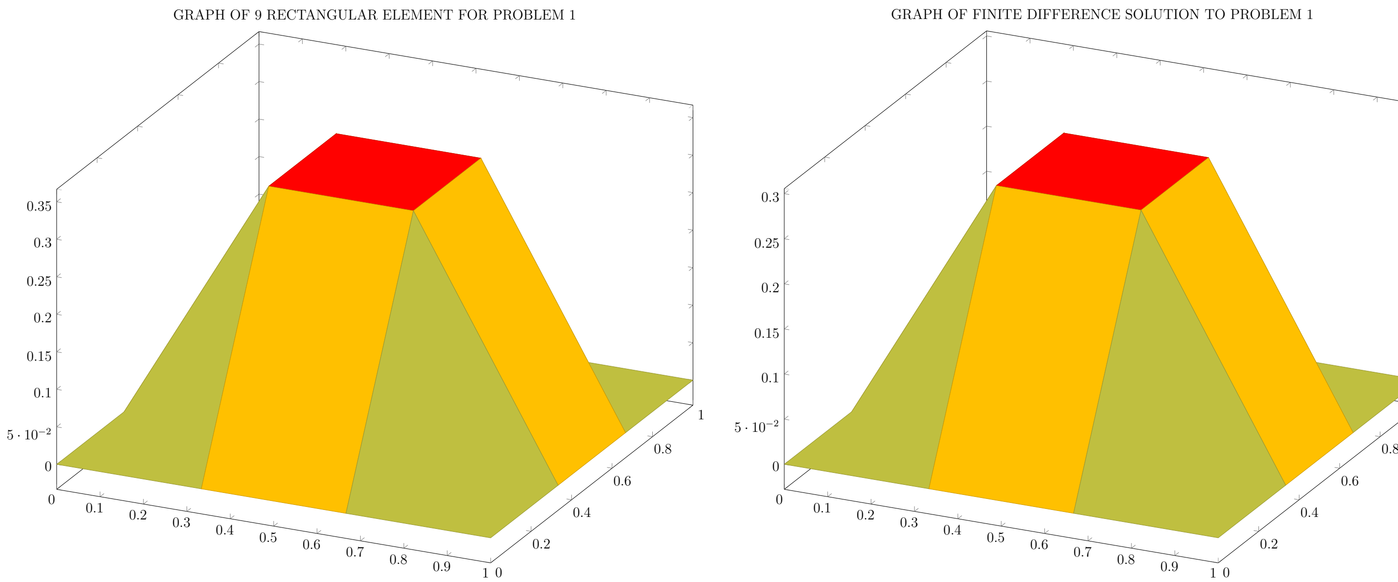




FEM Solution to Problem 1 with 9 Rectangular Elements and 18 Triangular Elements plotted side to side.



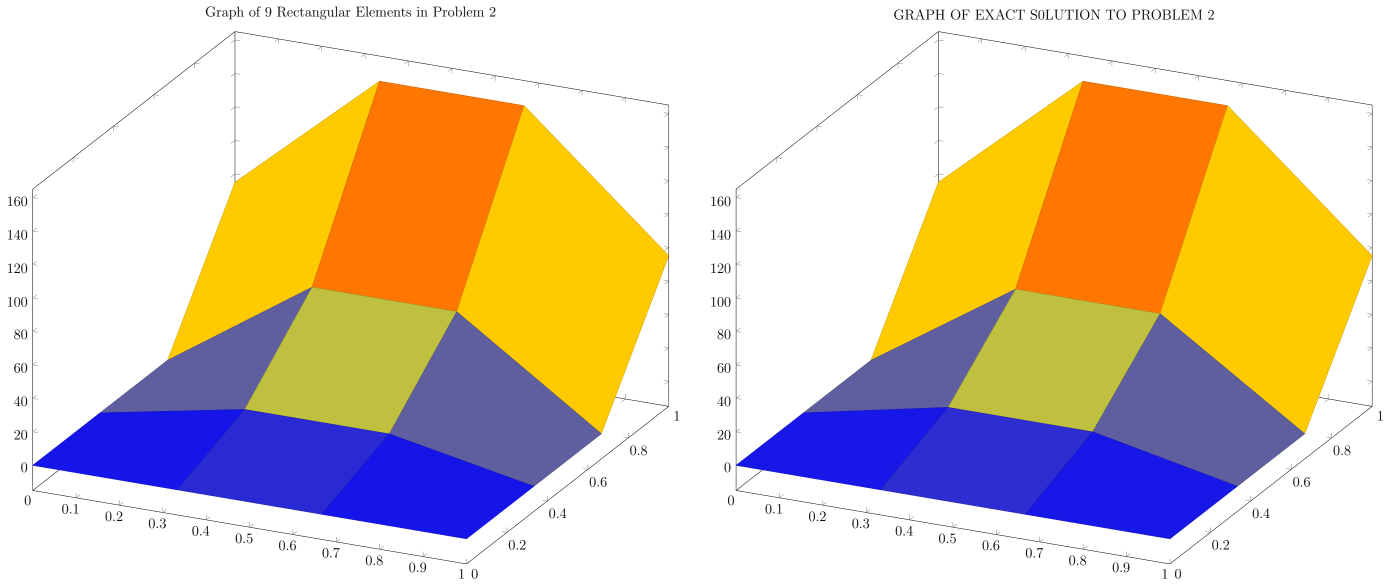
FEM with 9 Rectangular Elements and FD Solution to Problem 1 plotted side to side.



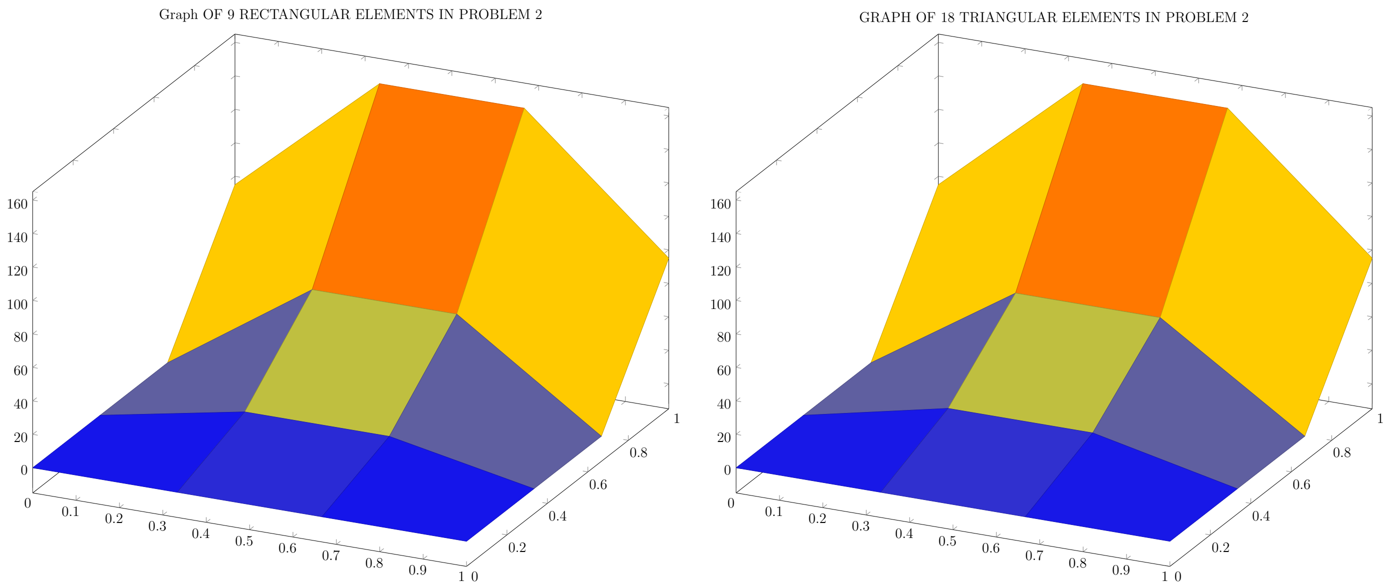
6.1 Appendix B

PLOTS 2 show the graphical solution to problem two for different number of elements compared for FEM, FDM,and Analytical method. PLOT 2: (SOLUTION TO PROBLEM TWO WITH BOUNDARY CONDITIONS)

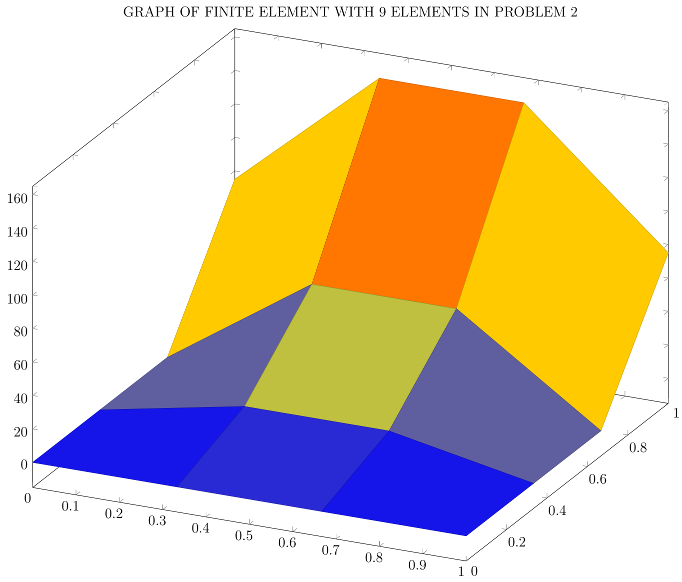
FEM with 9 Rectangular Elements and Exact Solution to Problem 2 plotted side to side

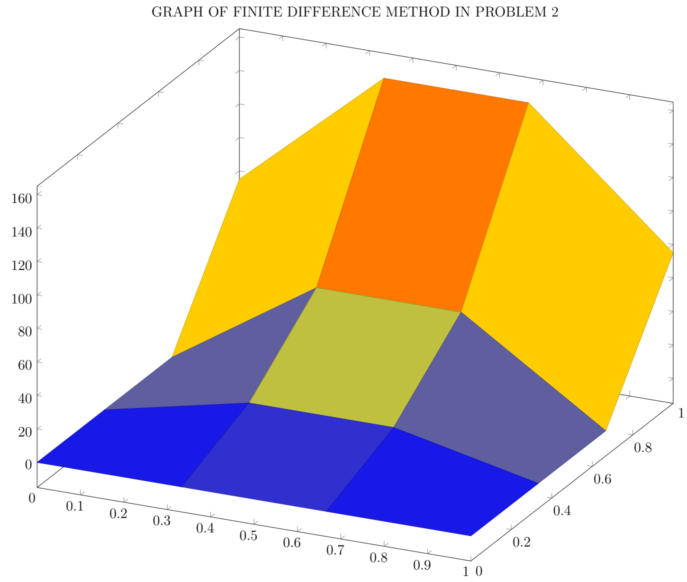


Finite Element Solution to Problem 2 with 9 Rectangular Elements and 18 Triangular Elements plotted side to side



FEM with 9 Rectangular Elements and FD Solution to Problem 2 plotted side to side


GRAPH OF FINITE ELEMENT WITH 9 ELEMENTS IN PROBLEM 2
160
140
120
100
80
60
40
20
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1


GRAPH OF FINITE DIFFERENCE METHOD IN PROBLEM 2
160
140
120
100
80
60
40
20
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1