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SIGNED FULL $1-1$ SYMMETRIC GROUP AND SIGNED FULL TRANSFORMATION SEMIGROUPS

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Abstract

The cardinalities, idempotents and chain decomposition of Signed full $1-1$ symmetric group and Signed full tramsformation are obtained.

1. Introduction

Let SS_n and ST_n be Signed full $1-1$ symmetric group and Signed full tramsformation respectively. Let α be a transformation from set $X_n \longrightarrow Z^*$. The set X_n is the set of natural numbers and $X \subseteq N$ such that $X_n = \{1, 2, 3, \ldots .n\}$ and $Z^* = \{\pm 1 \pm 2, \pm 3, \ldots, \pm n\}$. Mogbonju (2015) defined the signed transformation semigroups the set of all mapping from $X_n \longrightarrow Z^*$. The signed (partial) transformation semigroup is defined in the form $\alpha : dom(\alpha) \subseteq X_n \longrightarrow lm(\alpha) \subset Z^*$ is said to be full or total if $dom(\alpha) = X_n \subset Z^*$. $Dom(\alpha)$ stand for the domain of α while $lm(\alpha)$ represents image of α. Richard (2008) defined symmetric groups as the set of all bijection mappings. Richaard (2008) defined permutation matrices with a permutation α of X_n and associated $n \times n$ permutation matrix $\prod(\alpha)$. James and Kerber(1981) definedd signed permutation from $X_n \longrightarrow X_n$. Thus, just as the study of symmetric groups, alternating groups has made a significant contribution to group theory, so has the study of semigroups; see for example:Bashar(2010), Fernandes et. al.(2011), Howie(2002), Laradji and Umar (1992), Adeniji and

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Makanjuola (2008), Adeshola(2013). It is well known that T_n have order n^n. An element e is an idempotent if e^2. The set of all idempotent elements is S is denote by $E(S)$. Garba(1990), Tainter(1968), Vergner(1964) showed that the number of idempotent in full and partial transformation semigroups is $|E(T_n)| = \sum_{r=1}^{n}\binom{n}{r}^{n-r}$ and $|E(PT_n)| = \sum_{r=1}^{n+1}\binom{n}{n-r}r^{n+1-r}$

Adeniji(2012), Mogbonju(2015) define chain decomposition, η_β of elements of a semigroup as the set of fragments of each β such that $i \in dom(\beta), j \in Im(\beta)$ such that $i\beta = j \Longrightarrow i_k \longrightarrow j_k$ where $k = 1, 2, 3, \ldots, n$.

Adeniji(2012) studied chain decomposition of elements of identity difference full transformation semigroup. In his work, he obtained the total number of chains; L_n in the chain decomposition of element of $IDT_n = n^2$ He also showed that the total number of chains in the chain decomposition of IDT_n and total number of chains in the chain decomposition of all idempotent element of IDT_n are equal.

2.1 Methodology

Richard(2008) initiated the study of signed symmetry group. In this paper, the elements in each of the semigroups were listed using matrix notations, the approach will facilitate the study of the structures of each of the semigroups better.

2.1.1 Matrix notation of SS_n and ST_n :

The following notations will be used in representing elements in signed symmetric groups and transformation Semigroups.

Let $\alpha \in SS_n$; we can indicate $\alpha(j) = i$ by placing a 1 in the (i, j) - entry of an $n \times n$ matrix.

Example 2.1.

If $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \in S_4$

which means $\alpha(1) = 2$, $\alpha(2) = 1$, $\alpha(3) = 4$, $\alpha(4) = 3$ can be written in matrix notation as

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Similar notation can be used for elements in SS_n and ST_n

Example 2.2

If $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -3 & 1 & -2 & -4 \end{pmatrix} \in SS_4$

One can represent these elements in matrix form by placing ± 1 in the (i,j) - entry to indicate $j \to \pm i$.
So,

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Example 2.3

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -2 \end{pmatrix} \in ST_4$

then the matrix notation is given as;

$$\alpha = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2.2 Elements in SS_n.

The set of elements SS_2 is as follows;

$$|SS_2| = \left\{ \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\} = 8$$

$$|lm(\alpha^+)| = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\} = 2$$

$$|lm(\alpha^-)| = \left\{ \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} \right\} = 2$$

$$|lm(\alpha^*)| = \left\{ \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \right\} = 4$$

Table 2.1 : Elements in SS_n

n	$\|lm(\alpha^+)\|$	$\|lm(\alpha^-)\|$	$\|lm(\alpha^*)\|$	$SS_n = 2n! + (2^n - 2)n!$
1	1	1		2
2	2	2	4	8
3	6	6	36	48
4	24	24	336	384
5	120	120	3300	3840
6	720	720	44640	46080

$|lm(\alpha^+)|$ = number of the elements with positive integers only in the image of α.
$|lm(\alpha^-)|$ = number of the elements with negative integers only in the image of

α.
$|lm(\alpha^*)|$ = number of elements both positive and negative integers in the image of α.

Table 2.2 : Values of elements in ST_n

$n/Im(\alpha)$	$\lvert lm(\alpha^+)\rvert$	$\lvert lm(\alpha^-)\rvert$	$\lvert lm(\alpha^*)\rvert$	$ST_n = 2n^n + n^n(2^n - 2)$
1	1	1		2
2	4	4	8	16
3	27	27	162	216
4	256	256	3584	4096
5	3125	3125	93750	100000

2.3 Combinatorics in SS_n and ST_n :
Adeniji(2012) studied chain decomposition of elements of Identity difference full transformation, IDT_n, Identity difference partial transformation, IDP_n and identity difference partial one - one semigroups, IDI_n. In his work, he obtained the total number of chains; L_n in the chain decomposition of all elements of IDT_n is n^2 and L_n of IDI_n is $n^2 + 2$.
Chain decompositions η_β of elements of a semigroup is defined as the set of fragments of each $\beta \in ST_n$, for each $i \in dom(\beta)$, $j \in lm(\beta)$, such that $i\beta = j \Longrightarrow \beta_k : i_k \longrightarrow j_k$ where $k = 1, 2, 3, \ldots, n$.

For example
Let $\beta_1 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ j_1 & j_2 & j_3 & j_4 & j_5 \end{pmatrix}$

Then, the chain decomposition of
$\beta_1 = \left\{ \binom{i_1}{j_1}, \binom{i_2}{j_2}, \binom{i_3}{j_3}, \binom{i_4}{j_4}, \binom{i_5}{j_5} \right\}$

Let $S = SS_n$. The combinatorial relations between numbers associated with semigroup SS_n and ST_n are defined as:
$|S|$ = the cardinality of S.
$|H_n|$ = the total number of chains in the chain decomposition of all elements of S.
$|E(S)|$ = the total number of idempotent in S.
$|E_n|$ = the total number of chains in the chain decomposition of all the idempotent elements of S.
$|N(S)|$ = total number of nilpotent elements in S.

The following is an example illustrate chain decomposition in SS_3:
$|S| = SS_3 = 48$ i.e

$$|H_3| = \Big\{ \binom{1}{-3}, \binom{1}{-2}, \binom{1}{-1}, \binom{2}{-3}, \binom{2}{-2}, \binom{2}{-1}, \binom{3}{-3}, \binom{3}{-2}, \binom{3}{-1}, \binom{1}{1},$$

$$\binom{1}{2}, \binom{1}{3}, \binom{2}{1}, \binom{2}{2}, \binom{2}{3}, \binom{3}{1}, \binom{3}{2}, \binom{3}{3} \Big\} = 18$$

$$|E(S)| = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\} = 1$$

$$|E_3| = \left\{ \binom{1}{1}, \binom{2}{2}, \binom{3}{3} \right\} = 3$$

3.0 Results

Theorem 3.1 : Let $S = SS_n$ and if $\alpha \in SS_n$, $n \geq 1$ then $|S| = 2n! + (2^n - 2)n!$.
Proof : Let $X_n = \{1, 2, 3, \ldots, n\}$ and $lm(\alpha) \subset Z^*$. It follows from a counting argument. If $|lm(\alpha)^+|$ then $|\alpha(S)| = n!$ and $|\alpha(S)| = n!$ if $|lm(\alpha)^-|$, then $|lm(\alpha)^+|$ $= |lm(\alpha)^-| = n!$. Since $lm(\alpha)$ is either i or $-i$ where $i = 1, 2, 3, \ldots, n$ and by the binomial theorem for a positive integer n where
$\sum_{k=0}^{n} \binom{n}{k} = 2^n$, we have $|\alpha(S)| = (2^n - 2)n!$, when $|lm(\alpha^*)|$. Multiplying and summing which is equivalent to $2^n n!$. Hence the result.

Theorem 3.2 : Let $S = ST_n$, then $|S| = 2n^n + n^n(2^n - 2)$
Proof : The k images of $\alpha : X_n \longrightarrow Z^*$ be chosen in $\binom{n}{k}$ ways and each image has pre - image in $1-1$ fashions. Since $\alpha \in ST_n$ and $\alpha : dom(\alpha) \subseteq X_n \longrightarrow lm(\alpha) \subset Z^*$ and the choices of $k's$ are independently, we have $\alpha(S) = n^n$ when $|lm(\alpha)^+|$ which is equal to $|lm(\alpha)^-|$. Since the semigroup is bijection and k is the number of elements in $lm(\alpha)$ then the remaining $n-k$ elements of $lm(\alpha)$ fall into category of $lm(\alpha) \in Z^*$. These elements occurs in n^n groups with $n^n(2^n+2)$ elements in each group which is equivalent to $(2n)^n$. Hence the prove.

Table 3.1 : Relation between values in SS_n

n	$\lvert S\rvert$	$\lvert H_n\rvert$	$\lvert E(S)\rvert$	$\lvert E_n\rvert$
1	2	2	1	1
2	8	8	1	2
3	48	18	1	3
4	384	32	1	4
5	3840	50	1	5
6	46080	72	1	6

The following results are from Table 3.1 above. $|S|$ are known from theorem 2.1.

Proposition 3.1: The total number of chain $|H_n|$ in the chain decompositions of all elements of SS_n is $2n^2$.
Proof : Let $X_n = \{1, 2, 3, \ldots, n\}$ such that
$\alpha : dom(\alpha) \subseteq X_n \longrightarrow Z^*$, each $i \longrightarrow j$ and $lm(\alpha) \subset Z^*$. Each i maps all j the $lm(\alpha)$ is either i or $-i$ which occurs in twice n - times. Hence the results.

Proposition 3.2: The total number of chains in the chain decompositions of all idempotent elements, $|E_n|$ of SS_n is n.
Proof : Let $\alpha : Dom(\alpha) \subseteq X_n \longrightarrow lm(\alpha) \subset Z^*$, then the chains decomposition in SS_n is defined $\alpha : i \longrightarrow j$ where $i \in dom(\alpha)$ and $j \in lm(\alpha)$. That is each i maps all j which occurs in n ways.

Table 3.2 : Values of Relations in ST_n

n	$\|S\|$	$\|H_n\|$	$\|E(S)\|$	$\|E_n\|$
1	2	2	1	1
2	16	8	5	6
3	216	18	25	15
4	4096	32	125	28
5	100000	50	625	45

The following results emerge from table 3.2. $|S|$ of ST_n are known from theorem 2.2.
Proposition 3.3 : The total number of chain $|H_n|$ in the decompositions of all elements of ST_n is $2n^2$.
Proof : The proof of this is as in Theorem 3.1

Proposition 3.4 : The total number of chains in the chain decompositions of all idempotent elements, E_n of ST_n is $2n^2 - n$.
Proof : Let $\alpha \in ST_n$, $\alpha : X_n \longrightarrow Z^*$ and $\alpha(i) = j$. For each $i \in dom(\alpha)$ and $lm(\alpha)$ is either i or $-i$ for $i = 1, 2, 3 \ldots, n$, each i maps all j which occurs in n times $2n - 1$ ways. Hence $|H_n| = 2n^2 - n$.

4.0 Summary and Conclusion

4.0 Summary of Results :

The following table contains the summary of various results obtained.

Table 4.1 Summary of Results :

S	$\|S\|$	$\|E(S)\|$	$\|E_n\|$	$\|H_n\|$
SS_n	$2n! + (2^n - 2)n!$	1	n	$2n^2$
ST_n	$2n^n + n^n(2^n - 2)$	5^{n-1}	$2n^2 - n$	$2n^2$

4.2 Sequences generated from results :

In the course of this research study, the following results with sequences were obtained for all n. Let $\alpha : X_n \longrightarrow Z^*$ where $X_n = \{1, 2, 3, \ldots, n\}$ and $Z^* = \{\pm 1 \pm 2, \pm 3, \ldots, \pm n\}$.

Let $S = SS_n$ (i) $|S|= 2n!+(2^n-2)n!$ which generates the sequence **2,8,48,346,3840,...**
(ii) $|E(S)| = 1$
(iii) $|E_n| = n$ which generates the sequence **1,2,3,4,5,...**
(iv) $|H_n| = 2n$ which generates the sequence **2,8,18,32,50,...**

Let $S = ST_n$
(v) $|S| = 2n^n+n^n(2^n-2)$ which generates the sequence **2,16,216,4096,100000,...**
(vi) $|E(S)|= 5^{n-1}$ which generates the sequence **1,5,25,125,625,...**
(vii) $|E_n| = 2n^2 - n$ which generates the sequence **1,6,16,25,45,...**
(viii) $|H_n| = 2n^2$ which generates the sequence **2,8,18,32,50,...**

4.3 Conclusion

The sequences generated have application in coding theory, combinatorial algebra computational theory

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