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An Odd Order Numerical Integrator for Analyzing Environmental Models

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Abstract

In this work, we consider a class of formulae for the numerical solution of Initial Value Problem (IVP) that is applicable in the study and analysis of Environmental pollution Models. These classes of integrators are assumed to be an approximate reections of environmental problems.

1. Introduction

In [3], we considered the class of formulae for the numerical solution of the form: $y' = f(x, y)$; $y(x_0) = a$. This paper is to present an explicit method to compute a trig-polynomial interpolation, more precisely the coefficients of the trigonometric polynomial which is of the form:

$$F(x) = P_m(x) + b\cos(Nx + A), \ m > 0 \tag{1}$$

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where

$$P_m(x) = \sum_{j=0}^{m} a_j x^j \tag{2}$$

such that $\theta_x = Nx + A$ and $F(x) = \sum_{j=0}^{m} a_j x^j + b\cos(Nx + A)$, $m > 0$ whenever $a_j,\ b,\ A$ and N are real, m is a positive integer.

2. Derivation of order in some Numerical Integrator

In this section, we consider the derivation of order one, three and five Numerical Integrators.

2.1 Derivation of order one Numerical Integrator

Consider equation (1) and let $y(x) = F(x)$ in (2), if $x = x_n$ then

$$y(x_n) - P_m(x_n) - b\cos(Nx + A) = 0 \tag{3}$$

hence

$$y(x_{n+1}) - P_m(x_{n+1}) - b\cos(Nx_{n+1} + A) = 0$$

$$y^{(s)}(x_n) - P_m^{(s)}(x_n) - \frac{b}{\delta x^s}\cos^s(Nx + A) \qquad : x = x_n$$

for $s = 1,\ 2,\ \cdots,\ m+1$

We now define the linear function $\theta(x)$ as $Nx + A$

this implies that $\theta(x_n) = \theta(x) = Nx + A$ and $\theta_{n+1} = Nx_n + A + nh = \theta_n + Nh$.

when $m = 1$, the polynomial $P_m(x)$ becomes linear as:

$$F(x_n) = a_0 + a_1 x_n + b\cos\theta_n \tag{4}$$

from equation (3), we obtain the first derivative with respect to θ as:

$$F'(x_n) = a_1 + b\left[N(-\sin\theta_n)\right] = a_1 - bN\sin\theta_n = f(x, y)$$

$$F''(x_n) = -bN^2\cos\theta_n = f'(x, y)$$

hence

$$a_1 = f(x, y) + bN\sin\theta_n$$

The undetermined coefficients are obtained as follow:

$$b = \frac{-f'(x, y)}{N^2\cos\theta_n} = -f'(x, y)\, N^{-2}\sec\theta_n$$

$$a_1 = f(x, y) - \frac{-f'(x, y)}{N\cos\theta_n}\sin\theta_n = f(x, y) - f'(x, y)\, N^{-1}\tan\theta_n$$

such that

$$y_{n+1} - y_n = a_1(x_{n+1} - x_n) + b(\cos\theta_{n+1} - \cos\theta_n)$$

From (4), the desired integrator is derived as:

$$y_{n+1} = y_n + hf(x,y) - N^{-1}\text{h}f'(x,y)\tan\theta_n - N^{-2}f'(x,y)(\cos\theta_{n+1} - \cos\theta_n)\sec\theta_n$$

2.2 Derivation of order three Numerical Integrator

The order three numerical integrator for the case $m = 3$ is obtain by

$$P_3(x) = a_0 + a_1x_n + a_2x^2 + a_3x^3$$

It follows that the Trigo-polynomial interpolant is given as:

$$F(x_n) = a_0 + a_1x_n + a_2x_n^2 + a_3x_n^3 + b\cos(Nx + A)$$

This can be written as:

(5) $$F(x_n) = a_0 + a_1x_n + a_2x_n^2 + a_3x_n^3 + b\cos\theta_n$$

The first four derivatives of (5) yields

(6) $$F'(x_n) = a_1 + 2a_2x_n + 3a_3x_n^2 - bN\sin\theta_n = f(x,y)$$

(7) $$F''(x_n) = 2a_2 + 6a_3x_n - bN^2\cos\theta_n = f'(x,y)$$

(8) $$F'''(x_n) = 6a_3 + bN^3\sin\theta_n = f''(x,y)$$

(9) $$F^{iv}(x_n) = bN^4\cos\theta_n = f'''(x,y)$$

From equation (9), one of the four undetermined coefficients is obtained as:

$$b = \frac{f'''(x,y)}{N^4\cos\theta_n} = f'''(x,y)N^{-4}\sec\theta_n$$

From equation (8)

$$6a_3 = f''(x,y) - bN^3\sin\theta_n$$

such that

$$a_3 = \frac{1}{6}\left[f''(x,y) - f'''(x,y)N^{-1}\tan\theta_n\right]$$

From equation (7), we obtain

$$2a_2 = f'(x,y) - 6a_3x_n + bN^2\cos\theta_n$$

such that

$$a_2 = \frac{1}{2}\left[f'(x,y) - x_nf''(x,y) + x_nf'''(x,y)N^{-1}\tan\theta_n + f'''(x,y)N^{-2}\right]$$

From equation (6), we obtain:

$$a_1 = f(x,y) - x_n\left[f'(x,y) - f^v(x,y)N^{-4}\right] + \frac{x_n^2}{2}\left[f''(x,y) + f^v(x,y)N^{-3}\tan\theta_n\right]$$
$$-\frac{x_n^3}{6}[f'''(x,y) + f^v(x,y)N^{-2}] + \frac{x_n^4}{24}\left[f^{iv}(x,y) - f^v(x,y)N^{-1}\tan\theta_n\right] - f^v(x,y)N^5\tan\theta_n$$

with the stipulated condition (3), the desired integrator is derived as:

$$y_{n+1} = y_n + hf(x,y) - x_n h\left[f^{'}(x,y) + f^{'''}(x,y)N^{-2}\right] + \frac{x_n^2}{2}h\left[f^{''}(x,y) - f^{'''}(x,y)N^{-1}\tan\theta_n\right]$$
$$-hf^{'''}(x,y)N^{-1}\tan\theta_n + f^{'''}(x,y)N^{-4}\sec\theta_n\ (\cos\theta_{n+1} - \cos\theta_n)$$

2.3 Derivation of order five Numerical Integrator

The derivation of order five Numerical Integrator when $m = 5$
(10)

$$F(x_n) = a_0 + a_1x_n + a_2x_n^2 + a_3x_n^3 + a_4x_n^4 + a_5x_n^5 + b\cos(Nx + A) = a_0 + a_1x_n + a_2x_n^2$$
$$+ a_3x_n^3 + a_3x_n^3 + a_4x_n^4 + a_5x_n^5 + b\cos\theta_n$$

The undetermined coefficients are determined by differentiating equation (11) as follows:

(11) $$F^{'}(x_n) = a_1 + 2a_2x_n + 3a_3x_n^2 + 4a_4x_n^4 + 5a_5x_n^5 - bN\sin\theta_n = f(x,y)$$

(12) $$F^{''}(x_n) = 2a_2 + 6a_3x_n + 12a_4x_n^4 + 20a_5x_n^5 - bN^2\cos\theta_n = f^{'}(x,y)$$

(13) $$F^{'''}(x_n) = 6a_3 + 24a_4x_n^4 + 60a_5x_n^5 + bN^3\sin\theta_n = f^{''}(x,y)$$

(14) $$F^{iv}(x_n) = 24a_4 + 120a_5x_n^5 + bN^4\cos\theta_n = f^{'''}(x,y)$$

(15) $$F^{v}(x_n) = 120a_5 - bN^5\sin\theta_n = f^{iv}(x,y)$$

(16) $$F^{vi}(x_n) = -bN^6\cos\theta_n = f^{v}(x,y)$$

From equation (16), we determine

$$b = -\frac{f^{v}(x,y)}{N^6\cos\theta_n} = -f^{v}(x,y)N^{-6}\sec\theta_n$$

From equation (15)

$$120a_5 = f^{iv}(x,y) + bN^5\sin\theta_n$$
$$a_5 = \frac{1}{120}\left[f^{iv}(x,y) - f^{v}(x,y)N^{-1}\tan\theta_n\right]$$

From equation (14)

$$24a_4 = f^{'''}(x,y) - 120a_5x_n^5 - bN^4\cos\theta_n$$
$$a_4 = \frac{1}{24}\left[f^{'''}(x,y) - x_n\left(f^{iv}(x,y) - f^{v}(x,y)N^{-1}\tan\theta_n\right) + f^{v}(x,y)N^{-2}\right]$$

From equation (13)

$$6a_3 = f^{''}(x,y) - 24a_4x_n^4 - 60a_5x_n^5 - bN^3\sin\theta_n$$
$$a_3 = \frac{1}{6}\left[f^{''}(x,y) - x_n\left(f^{'''}(x,y) + f^{v}(x,y)N^{-2}\right) + \frac{x_n^2}{2}\left(f^{iv}(x,y) - f^{v}(x,y)N^{-1}\tan\theta_n\right)\right]$$

$$+\frac{1}{6}f^{v}\left(x,y\right)N^{-3}\text{tan}\theta_{n}$$

From equation (12)

$$2a_{2}=f^{'}\left(x,y\right)-6a_{3}x_{n}-12a_{4}x_{n}^{4}-20a_{5}x_{n}^{5}+bN^{2}\text{cos}\theta_{n}$$

$$a_{2}=\frac{1}{2}\left[f^{'}-x_{n}\left(f^{''}+f^{v}N^{-3}\text{tan}\theta_{n}\right)+\frac{x_{n}^{2}}{2}\left(f^{'''}+f^{v}N^{-2}\right)-\frac{x_{n}^{3}}{6}\left(f^{iv}-f^{v}N^{-1}\text{tan}\theta_{n}\right)-f^{v}N^{-4}\right]$$

From equation (11)

$$a_{1}=f(x,y)-x_{n}\left[f'(x,y)-f^{v}(x,y)N^{-4}\right]+\frac{x_{n}^{2}}{2}\left[f''(x,y)+f^{v}(x,y)N^{-3}\tan\theta_{n}\right]$$

$$-\frac{x_{n}^{3}}{6}\left[f^{'''}(x,y)+f^{v}(x,y)N^{-2}\right]+\frac{x_{n}^{4}}{24}\left[f^{(}4)(x,y)-f^{v}(x,y)N^{-1}\text{tan}\theta_{n}\right]-f^{v}\left(x,y\right)N^{5}\text{tan}\theta_{n}$$

At this, the desired Integrator is given as;

$$y_{n+1}=y_{n}+hf\left(x,y\right)-x_{n}h\left[f^{'(x,y)-f^{v}(x,y)N^{-4}}\right]+\frac{x_{n}^{2}}{2}h\left[f^{''}\left(x,y\right)+f^{v}\left(x,y\right)N^{-3}\text{tan}\theta_{n}\right]$$

$$-\frac{x_{n}^{3}}{6}h\left[f^{'''}\left(x,y\right)+f^{v}\left(x,y\right)N^{-2}\right]+\frac{x_{n}^{4}}{24}h\left[f^{iv}\left(x,y\right)-f^{v}\left(x,y\right)N^{-1}\text{tan}\theta_{n}\right]$$

$$-hf^{v}\left(x,y\right)N^{-5}\text{tan}\theta_{n}-f^{v}\left(x,y\right)N^{-6}\text{sec}\theta_{n}\left(\text{cos}\theta_{n+1}-\text{cos}\theta_{n}\right)$$

3. Implementation of the Integrators

We shall apply these integrators for the numerical solutions of some Initial Value Problem which are models on environmental pollution and climatic changes.

3.1 Problem 1.

$y' = -y^2\ ; \quad 0 \leq x \leq 1$

The theoretical (exact) solution is $y\,(x) = \ 1/(1+x)$

Table of result for Problem 1

The table of integrators performs well at m=1 and the truncation errors are approximately equal to zero.

Xn	Y(x)	Yn	-An	Nn	ERROR
0.0000000	1.000000000	1.000000000	-1.0000000	-1.0000000	0.00000000
500.00000	0.001996008	0.001996088	-0.9803480	-1.0000023	0.00000008
1000.0000	0.000999001	0.000999021	-0.9800431	-1.0000001	0.00000002
1500.0000	0.000666223	0.000666231	-0.9803212	-1.0000012	0.00000001
2000.0000	0.000499750	0.000499755	-0.9799110	-0.9999999	0.00000001
2500.0000	0.000399840	0.000399843	-0.9802156	-1.0000002	0.00000000
3000.0000	0.000333222	0.000333224	-0.9797055	-0.9999993	0.00000000
3500.0000	0.000285633	0.000285634	-0.9804835	-1.0000004	0.00000000
4000.0000	0.000249938	0.000249939	-0.9790376	-0.9999995	0.00000000
4500.0000	0.000222173	0.000222174	-0.9790530	-0.9999993	0.00000000
5000.0000	0.000199960	0.000199961	-0.9794670	-1.0000000	0.00000000
5500.0000	0.000181785	0.000181786	-0.9793017	-1.0000001	0.00000000
6000.0000	0.000166639	0.000166639	-0.9819345	-1.0000015	0.00000000
6500.0000	0.000153822	0.000153823	-0.9784266	-0.9999998	0.00000000
7000.0000	0.000142837	0.000142837	-0.9785526	-0.9999996	0.00000000
7500.0000	0.000133316	0.000133316	-0.9808069	-1.0000011	0.00000000
8000.0000	0.000124984	0.000124985	-0.9824676	-1.0000017	0.00000000
8500.0000	0.000117633	0.000117634	-0.9780953	-1.0000001	0.00000000
9000.0000	0.000111099	0.000111099	-0.9798390	-1.0000006	0.00000000
9500.0000	0.000105252	0.000105252	-0.9774979	-0.9999998	0.00000000
10000.0000	0.000099990	0.000099990	-0.9811407	-1.0000012	0.00000000

3.2 Problem 2.

Consider $y' = -\sqrt{(1-y^2\ )},\ y\ (0)\ = 1.0$

The theoretical solution is given as:

$$y\,(x) = Tan^{-1}\left(x + \frac{\pi}{4}\right); \text{h} = 0.1$$

Table of result for Problem 2

The performance of the integrator on the Initial Value Problem.

Xn	Y(x)	Yn	-An	Nn	ERROR
0.0000	1.000000000	1.000000000	1.0000000	3.0000000	0.000000000
0.1000	0.995004177	0.995064855	1.2107641	3.1107640	0.000060678
0.2000	0.980066597	0.980229199	1.4529072	3.2529070	0.000162601
0.3000	0.955336511	0.955612421	1.7457384	3.4457383	0.000275910
0.4000	0.921060979	0.921452105	2.1289327	3.7289326	0.000391126
0.5000	0.877582550	0.878085434	2.6975126	4.1975126	0.000502884
0.6000	0.825335562	0.825943172	3.7501729	5.1501727	0.000607610
0.7000	0.764842093	0.765544534	6.9666533	8.2666531	0.000702441
0.8000	0.696706593	0.697491825	-35.716167	-34.516166	0.000785232
0.9000	0.621609867	0.622464299	-2.9859352	-1.8859353	0.000854433
1.0000	0.540302217	0.541211247	-0.8059914	0.1940085	0.000909030
1.1000	0.453596085	0.454544306	0.0578970	0.9578969	0.000948220

3.3 Problem 3.

$y' = cos^2\, y,\, y(0.1) = 0.099668652$

The theoretical solution is given as $y\,(x) = Tan^{-1}\left(x + \frac{\pi}{4}\right); y\,(0.1) = 0.099668652, \text{h} = 0.3$

Table of result for Problem 3

Xn	Y(x)	Yn	-An	Nn	ERROR
0.100000	0.099668652	0.099668652	0.009427764	4.826868057	0.000000000
0.300000	0.291456819	0.291452616	0.237386107	4.057555676	0.000004202
0.600000	0.540419519	0.540265858	0.788471103	4.439555168	0.000153661
0.900000	0.732815146	0.732703030	1.087498903	4.822109699	0.000112116
1.200000	0.876058102	0.875980437	1.264924049	4.945718288	0.000077665
1.500000	0.982793808	0.982690990	1.223112702	5.310365677	0.000102818
1.800000	1.063697934	1.063622117	-0.28772291	7.430562973	0.000075817
2.100000	1.126377106	1.126319647	-0.23918172	6.596050739	0.000057459
2.400000	1.176005244	1.175960302	1.394127011	5.055659771	0.000044942
2.700000	1.216090679	1.216054797	2.045941114	4.751595497	0.000035882
3.099999	1.258754134	1.258726716	2.609899282	4.676197052	0.000027418
3.399999	1.284744740	1.284721971	2.970306158	4.680755138	0.000022769
3.699999	1.306832552	1.306813240	3.311149597	4.700228214	0.000019312
3.999998	1.325817585	1.325801015	3.642016888	4.724144459	0.000016570
4.299998	1.342299581	1.342285275	3.966991901	4.748342514	0.000014305
4.599998	1.356735587	1.356723189	4.288060188	4.771152496	0.000012398
4.899998	1.369479179	1.369468212	4.606306076	4.791974545	0.000010967
5.199997	1.380807996	1.380798221	4.922381878	4.810687065	0.000009775
5.499997	1.390942693	1.390934110	5.236718178	4.827377796	0.000008583
5.799997	1.400061011	1.400053263	5.549615383	4.842218876	0.000007749
6.099997	1.408307076	1.408300161	5.861298561	4.855407238	0.000006914

Conclusion:

The Integrators under consideration have been shown to perform favorably, especially when the order of the polynomial is greater than one and at smaller mesh sizes. The oscillatory parameters are also computed to ensure an accurate numerical approximation. At some points, the values of
$x_n, \quad A_n \quad and\ N_n$ are closely related in the computation.

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