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Strengthened of Some Results on Fuzzy Soft Multiset Theory

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Abstract

In this paper, we present the definition of fuzzy soft multiset and its basic operations such as complement, union and intersection as introduced in 2012. We define the operations such as restricted intersection, restricted union, AND-product, OR-product, restricted difference and restricted symmetric difference. Basic algebraic properties of these operations are presented with relevant examples. De Morgan's inclusions and laws are established in fuzzy soft multiset context.

1. Introduction

Most of the complicated and challenging problems faced in engineering, economics, environmental science, medical and social sciences have various levels of uncertainties and imprecision embedded in them. The solutions of such problems involve the use of mathematical principles based on uncertainties and imprecision. In an attempt to solve these problems, different theories were developed, such as probability theory [1], fuzzy set theory [2], theory of interval mathematics [3], theory of rough sets [4], and vague sets [5], which were considered as mathematical

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tools for dealing with uncertainties. But all these theories have their limitations in dealing with the uncertainties. The major problem associated with these theories is the inadequacies of the parameterization tools. To surmount these limitations, Molodtsov [6] introduced the concept of soft set theory as a new general mathematical tool for dealing with uncertainties and imprecision that is free from the difficulties that have troubled the traditional mathematical approaches. Molodtsov pointed out the application of soft set in several directions, such as game theory, operation research, perron integration among others.
This theory has proven useful in many different fields such as decision making [7], data analysis [8], forecasting and so on.
Research on soft sets has been progressing, since its introduction by Molodtsov in 1999 up to the present and several results have been achieved both in theory and practice. Maji *et al.* [9] defined different algebraic operations in soft set theory and published a detailed theoretical study on soft sets. Ali *et al.* [10] further presented and investigated some new algebraic operations for soft sets. Sezgin and Atagun [11] proved that certain De Morgan's law holds in soft set theory with respect to different operations on soft sets and discuss the basic properties of operations on soft sets such as intersection, extended intersection, restricted union and restricted difference. Maji *et al.* [12] extended crisp soft set to fuzzy soft set.
Alkhazaleh *et al.* in 2012 introduced fuzzy soft multiset and define some basic terms including union and intersection operations with examples. They finally applied it to decision making using Row-Maji algorithm using scores. In this paper, we define some operations such as restricted union, restricted intersection, AND-product, OR-product, restricted difference and restricted *symmetric* difference with relevant examples and illustrations. Basic properties of the operations were presented. We state and proved various De Morgan's inclusions and laws.

2. Preliminaries

2.1 Soft Set

We first recall some basic notions in soft set theory. Let U be an initial universe set, E be a set of parameters or attributes with respect to U, $P(U)$ be the power set of U and A⊆E.
Definition 2.1.1 [6]: . A pair $(F,\ A)$ is called a **soft set** over U, where F is a mapping given by $F : A \ \rightarrow P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U. For $x \in A$, $F(x)$ may be considered as the set of x-elements or as the set of x-approximate elements of the soft set $(F,\ A)$.
The soft set $(F,\ A)$ can be represented as a set of ordered pairs as follows:

$$(F,\ A) = \{(x,\ F(x)) : x \in A,\ F(x) \in P(U)\}.$$

Example 2.1.1
Let $U = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ consisting of six students and $A = \{a_1,\ a_2,\ a_3\}$ be the set of parameters under consideration, where each parameter e_i, $i= 1, 2, 3$ stands for, brilliant, average, healthy, respectively. In this case to define a soft set means to point out brilliant students, average students and healthy students. Such that $F(a_1) = \{ S_1, S_2, S_5\}$, $F(a_2) =\{ S_3, S_4, S_6\}$ and $F(a_3) = \{S_1, S_4, S_5, S_6 \}$. Then the soft set (F, A) over U is given by
$(F, A) = \{(a_1, \{ S_1, S_2, S_5\}), (a_2,\{ S_3, S_4, S_6\}),(a_3, \{ S_1, S_4, S_5, S_6\})\}$.
Definition 2.1.2 [9]**:** Let $(F,\ A)$ and (G, B) be two soft sets over U. Then

(1) $(F,\ A)$ is said to be a **soft subset** of (G, B), denoted by

$(F,\ A) \tilde{\subseteq} (\mathrm{G},\ \mathrm{B})$, if $A \subseteq \mathrm{B}$ and $F(x) \subseteq \mathrm{G}(x)$, $\forall x \in \mathrm{A}$

(1) $(F,\ A)$ and $(G,\ B)$ are said to be **soft equal**, denoted by

$(F,\ A) = (\mathrm{G},\ \mathrm{B})$, if $(F,\ A) \tilde{\subseteq} (\mathrm{G},\ \mathrm{B})$ and $(\mathrm{G},\ \mathrm{B}) \tilde{\subseteq} (F,\ A)$
Definition 2.1.3. [9]: Let $(F,\ A)$ be a soft set over U. Then , the support of $(F,\ A)$ written **supp**$(F,\ A)$ is the set defined as **supp**$(F,\ A) = \{x \in A :\ F(x) \neq \emptyset\}$.
(i) $(F,\ A)$ is called a **non-null** soft set if supp $(F,\ A) \neq \emptyset$.
(ii) $(F,\ A)$ is called a **relative null** soft set denoted by $\emptyset_A\ if\ F(x) = \emptyset,\ \forall x \in A$
(iii) $(F,\ A)$ is called a **relative whole** soft set, denoted by $U_A\ if\ F(x) = U,\ \forall x \in A$.
Definition 2.1.4. [8]: Let $(F,\ A)$ be a soft set over U. If $F(x) \neq \emptyset$ for all $x \in A$, then $(F,\ A)$ is called a non-empty soft set.
Definition 2.1.5[9]: Let $(F,\ A)$ and $(G,\ B)$ be two soft sets over U. Then the **union** of $(F,\ A)$ and $(G,\ B)$, denoted by $(F, A) \tilde{\cup} (G,\ B)$ is a soft set defined as $(F, A) \tilde{\cup} (G,\ B) = (H,\ C),\ where\ C = A \cup B\ and\ \forall x \in C$,

$$H(x) = \begin{cases} F(x), & if\ x \in A - B \\ G(x), & if x \in B - A \\ F(x) \cup\ G(x), & if\ x \in A \cap B \end{cases}.$$

Definition 2.1.6 [10]**:** Let $(F,\ A)$ and $(G,\ B)$ be two soft sets over U. Then the **restricted union** of $(F,\ A)$ and $(G,\ B)$, denoted by $(F,\ A)\ \tilde{\cup}_R(G,\ B)$ is a soft set defined as;
$(F,\ A)\ \tilde{\cup}_R(G,\ B) = (H,\ C)$, where $C = A \cap B\ \neq \emptyset$ and $\forall\ x \in C$
$H(x) = F(x) \cup G(x)$.
Definition 2.1.7 [10]**:** Let $(F,\ A)$ and $(G,\ B)$ be two soft sets over U. Then the **extended intersection** of $(F,\ A)$ and $(G,\ B)$, denoted by $(F,\ A)\ \tilde{\cap}_E\ (G,\ B)$, is a soft set defined as $(F,\ A) \tilde{\cap}_E\ (G,\ B) = (H,\ C)$ where $C = A \cup B$ and $\forall\ x \in C$,

$$H(x) = \begin{cases} F(x), & if\ x \in A - B \\ G(x), & if x \in B - A \\ F(x) \cap\ G(x), & if\ x \in A \cap B \end{cases}$$

Definition 2.1.8 [10]**:** Let (F, A) and $(G,\ B)$ be two soft sets over U. Then the **restricted intersection** of (F, A) and $(G,\ B)$ denoted by $(F, A) \Cap (G,\ B)$, is a soft set defined as $(F, A) \Cap (G,\ B) = (H,\ C)$, where $C = A \cap B$ and $\forall\ x \in C$, $H(x) = F(x) \cap G\ (x)$.

Definition 2.1.9 [10]: Let $(F,\ A)$ and $(G,\ B)$ be two soft sets over U. Then the **AND-product** or **AND-intersection** of $(F,\ A)$ and $(G,\ B)$ denoted by $(F,\ A)\widetilde{\bigwedge}\ (G,\ B)$ is a soft set defined as

$$(F,\ A)\widetilde{\bigwedge}\ (G,\ B) = (H,\ C)\,,\ where\ C = A \times B\ and\ \forall\ (x,\ y) \in A \times B$$

$H(x,\ y) = F(x) \cap\ G(y)$.

Definition 2.1.10 [10]**:** Let $(F,\ A)$ and $(G,\ B)$ be two soft sets over U. Then the **OR-product** or **OR-union** of $(F,\ A)$ and $(G,\ B)$, denoted by $(F,\ A)\widetilde{\bigvee}$ $(G,\ B)$ is a soft set defined as

$$(F,\ A)\widetilde{\bigvee}(G,\ B) = (H, C)\,,\ where\ C = A \times B\ and\ \forall\ (x, y) \in A \times B,$$

$H(x,\ y) = F(x) \cup\ G(y)$.

2.2 Fuzzy Soft Set

Let U be an initial universe set and E be a set of parameters (which are fuzzy words or sentences involving fuzzy words). Let $P(U)$ denotes the set of all fuzzy subsets of U, and $A \subseteq E$.

Definition 2.2.1 [12]: A pair $(F,\ A)$ is called a fuzzy soft set over U, where F is a mapping given by $F : A\ \longrightarrow P(U)$. F is called fuzzy approximation function of the fuzzy set $(F,\ A)$ and the values $F(x)$ are fuzzy subsets of U, $\forall\ x \in A$. Therefore, a fuzzy soft set $(F,\ A)$ over U can be represented by the set of ordered pair $(F,\ A) = \{(x,\ F\,(x)) : x \in A,\ F(x) \in P(U)\}$.

Example 2.2.1: Suppose that $U = \{h_1, h_2, h_3, h_4, h_5\}$ be a universe set and $E = \{x_1, x_2, x_3, x_4\}$ be a set of parameters. $A = \{x_1, x_2, x_3\} \subseteq E$, $F\,(x_1) = \left\{\frac{h_2}{0.8}, \frac{h_4}{0.6}\right\}$,
$F\,(x_2) = U$ and $F\,(x_3) = \left\{\frac{h_1}{0.3}, \frac{h_4}{0.4}, \frac{h_5}{0.9}\right\}$, then the fuzzy soft set $(F,\ A)$ is written as,

$$(F,\ A) = \left\{\left(x_1, \left\{\frac{h_2}{0.8}, \frac{h_4}{0.6}\right\}\right), (x_2,\ U)\,, \left(x_3, \left\{\frac{h_1}{0.3}, \frac{h_4}{0.4}, \frac{h_5}{0.9}\right\}\right)\right\}.$$

2.3 Soft Multiset:

Let $\{U_i : i \in I\}$ be a collection of universes such that $\bigcap_{i\in I} U_i = \emptyset$ and let $\{E_{U_i} : i \in I\}$ be a collection of sets of parameters. Let $U = \biguplus_{\boldsymbol{i\in I}} P\,(U_i)$, where $P\,(U_i)$ denotes the power sets of $U_i^{\prime}s$, $E = \biguplus_{\boldsymbol{i\in I}} E_{U_i}$ and $A \subseteq E$.

Definition 2.3.1 [13]: A pair (F, A) is called a soft multiset over U, where F is a mapping given by $F : A \longrightarrow U$.
In other words, a soft multiset over U is a parameterized family of subsets of U. For $a \in A, F(a)$ may be considered as the set of $a-$approximate elements of the soft multiset (F, A). Based on the definition, any change in the order of the universes will produce a different soft multiset.

Example 2.3.1:
Suppose that there are three universes U_1, U_2 and U_3. Let us consider a soft multiset (F, A) which describes the "attractiveness of houses", "cars" and "hotels" that Mr. X is considering for accommodation purchase, transportation purchase, and venue to hold a wedding celebration respectively.
Let $U_1 = \{h_1, h_2, h_3, h_4, h_5, h_6\}, U_2 = \{c_1, c_2, c_3, c_4, c_5\}$ and $U_3 = \{v_1, v_2, v_3, v_4\}$.
Let $E_U = \{E_{U_1}, E_{U_2}, E_{U_3}\}$ be a collection of sets of decision parameters related to the above universes, where

$$E_{U_1} = \left\{ \begin{array}{c} e_{U_1}, 1 = expensive,\ e_{U_1}, 2 = cheap,\ e_{U_1}, 3 = beautiful, \\ e_{U_1}, 4 = wooden,\ e_{U_1}, 5 = in\ green\ surroundings \end{array} \right\},$$

$$E_{U_2} = \left\{ \begin{array}{c} e_{U_2}, 1 = expensive,\ e_{U_2}, 2 = cheap,\ e_{U_2}, 3 = Model\ 2000\ and\ above, \\ e_{U_2}, 4 = Black,\ e_{U_2}, 5 = Made\ in\ Japan,\ e_{U_2}, 6 = Made\ in\ Malaysia \end{array} \right\},$$

$$E_{U_3} = \left\{ \begin{array}{c} e_{U_3}, 1 = expensive,\ e_{U_3}, 2 = cheap,\ e_{U_3}, 3 = majestic, \\ e_{U_3}, 4 = in\ Kuala\ Lumpur,\ e_{U_3}, 5 = in\ Kajang \end{array} \right\}.$$

Let $U = \biguplus_{i=1}^{3} P(U_i)$, $E = \biguplus_{i=1}^{3} E_{U_i}$ *and* $A \subseteq E$, such that

$$\begin{aligned} A = \{ & a_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1),\ a_2 = (e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1), \\ & a_3 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1),\ a_4 = (e_{U_1}, 5, e_{U_2}, 4, e_{U_3}, 2), \\ & a_5 = (e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 3),\ a_6 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 2), \\ & a_7 = (e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 1),\ a_8 = (e_{U_1}, 1, e_{U_2}, 3, e_{U_3}, 2)\}. \end{aligned}$$

Suppose that

$$\begin{aligned} F(a_1) &= (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\}), \\ F(a_2) &= (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2\}), \\ F(a_3) &= (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset), \\ F(a_4) &= (\{h_1, h_4, h_6\}, \emptyset, \{v_1, v_4\}), \\ F(a_5) &= (\{h_1, h_4\}, \{c_1, c_3\}, \{v_1\}), \\ F(a_6) &= (\{h_1, h_4, h_5\}, \{c_1, c_3\}, U_3), \\ F(a_7) &= (\{h_1, h_4\}, \emptyset, \{v_3\}), \\ F(a_8) &= (\{h_2, h_3, h_6\}, \{c_1, c_3\}, \{v_1, v_4\}). \end{aligned}$$

Then we can view the soft multiset (F, A) as consisting of the following collection of approximations:

$$(F, A) = \{(a_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})),$$

$$(a_2,\ (\{h_2,h_3,h_6\},\{c_1,c_3,c_4,c_5\},\{v_2\})),$$
$$(a_3,(\{h_1,h_4,h_5\},\{c_1,c_3\},\emptyset)),\ (a_4,\ (\{h_1,h_4,h_6\},\emptyset,\{v_1,v_4\})),$$
$$(a_5,\ (\{h_1,h_4\},\{c_1,c_3\},\{v_1\})),(a_6,(\{h_1,h_4,h_5\},\{c_1,c_3\},U_3)),$$
$$(a_7,(\{h_1,h_4\},\emptyset,\{v_3\})),\ (a_8,\ (\{h_2,h_3,h_6\},\{c_1,c_3\},\{v_1,v_4\}))\}.$$

3. Fuzzy Soft Multiset:

Let $\{U_i : i \in I\}$ be a collection of universes such that $\bigcap_{i\in I} U_i = \emptyset$ and let $\{E_{U_i} : i \in I\}$ be a collection of set of parameters. Let $U = \biguplus_{\boldsymbol{i\in I}} FS(U_{\boldsymbol{i}})$, where $FS(U_i)$ denotes the set of all fuzzy subsets U_i, $E = \biguplus_{\boldsymbol{i\in I}} E_{U_i}$ and $A \subseteq E$.

Definition 3.1 [14]: A pair (F, A) is called a fuzzy soft multiset over U, where F is a mapping given by $F : A \longrightarrow U$.
In other words, a fuzzy soft multiset over U is a parameterized family of fuzzy subsets of U. For $a \in A, F(a)$ may be considered as the set of $a-$approximate elements of the fuzzy soft multiset (F, A). Based on the definition, any change in the order of the universes will produce a different fuzzy soft multiset.
Example 3.1:
Suppose that there are three universes U_1, U_2 and U_3. Let us consider a fuzzy soft multiset (F, A) which describes the "attractiveness of houses", "cars" and "hotels" that Mr. X with a budget is considering for accommodation purchase, transportation purchase, and venue to hold a wedding celebration respectively.
Let $U_1 = \{h_1, h_2, h_3, h_4, h_5\}, U_2 = \{c_1, c_2, c_3, c_4\}$ and $U_3 = \{v_1, v_2, v_3\}$.
Let $E_U = \{E_{U_1}, E_{U_2}, E_{U_3}\}$ be a collection of sets of decision parameters related to the above universes, where

$$E_{U_1} = \left\{ \begin{array}{c} e_{U_1},1 = expensive,\ e_{U_1},2 = cheap, \\ e_{U_1},3 = wooden,\ e_{U_1},4 = in\ green\ surroundings \end{array} \right\},$$

$$E_{U_2} = \{e_{U_2},1 = expensive, e_{U_2},2 = cheap, e_{U_2},3 = Sporty\},$$

$$E_{U_3} = \left\{ \begin{array}{c} e_{U_3},1 = expensive,\ e_{U_3},2 = cheap,\ e_{U_3},3 = in\ Kuala\ Lumpur, \\ e_{U_3},4 = Majestic \end{array} \right\}.$$

Let $U = \biguplus_{i=1}^{3} FS(U_{\boldsymbol{i}})$, $E = \biguplus_{i=1}^{3} E_{U_i}$ *and* $A \subseteq E$, such that

$$A = \{a_1 = (e_{U_1},1, e_{U_2},1, e_{U_3},1),\ a_2 = (e_{U_1},1, e_{U_2},2, e_{U_3},1),$$
$$a_3 = (e_{U_1},2, e_{U_2},3, e_{U_3},1),\ a_4 = (e_{U_1},5, e_{U_2},4, e_{U_3},2),$$
$$a_5 = (e_{U_1},4, e_{U_2},3, e_{U_3},3),\ a_6 = (e_{U_1},2, e_{U_2},3, e_{U_3},2),$$

Suppose that

$$F(a_1) = \left(\left\{\frac{h_1}{0.2}, \frac{h_2}{0.4}, \frac{h_3}{0.8}, \frac{h_4}{0.5}, \frac{h_5}{0}\right\}, \left\{\frac{c_1}{0.8}, \frac{c_2}{0.5}, \frac{c_3}{0.4}, \frac{c_4}{0..6}\right\}, \left\{\frac{v_1}{0.8}, \frac{v_2}{0.7}, \frac{v_3}{0.7}\right\}\right),$$

$$F(a_2)= \left(\left\{\frac{h_1}{0.2},\ \frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.4},\ \frac{c_2}{0.5},\frac{c_3}{0.8},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.4},\ \frac{v_2}{0.4},\frac{v_3}{0.3}\right\}\right),$$
$$F(a_3)= \left(\left\{\frac{h_1}{0.7},\ \frac{h_2}{0.7},\frac{h_3}{0.1},\frac{h_4}{0.8},\frac{h_5}{0.7}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.4},\frac{v_3}{0.2}\right\}\right),$$
$$F(a_4)= \left(\left\{\frac{h_1}{0.9},\ \frac{h_2}{0.5},\frac{h_3}{0.5},\frac{h_4}{0.2},\frac{h_5}{0.7}\right\},\left\{\frac{c_1}{0},\ \frac{c_2}{0.2},\frac{c_3}{0.7},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7},\frac{v_3}{0.9}\right\}\right),$$
$$F(a_5)= \left(\left\{\frac{h_1}{0.9},\ \frac{h_2}{0.5},\frac{h_3}{0.5},\frac{h_4}{0.2},\frac{h_5}{0.7}\right\},\left\{\frac{c_1}{0.7},\ \frac{c_2}{0.8},\frac{c_3}{0.5},\frac{c_4}{0.4}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.5},\frac{v_3}{0.7}\right\}\right),$$
$$F(a_6)= \left(\left\{\frac{h_1}{0.7},\ \frac{h_2}{0.7},\frac{h_3}{0.1},\frac{h_4}{0.8},\frac{h_5}{0.7}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7},\frac{v_3}{0.6}\right\}\right).$$

Then we can view the fuzzy soft multiset $(F,\ A)$ as consisting of the following collection of approximations:

$$(F,\ A)=\left\{\begin{array}{l}
\left(a_1\left(\left\{\frac{h_1}{0.2},\ \frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7},\frac{v_3}{0.7}\right\}\right)\right),\\
\left(a_2\left(\left\{\frac{h_1}{0.2},\ \frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.4},\ \frac{c_2}{0.5},\frac{c_3}{0.8},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.4},\ \frac{v_2}{0.4},\frac{v_3}{0.3}\right\}\right)\right),\\
\left(a_3\left(\left\{\frac{h_1}{0.7},\ \frac{h_2}{0.7},\frac{h_3}{0.1},\frac{h_4}{0.8},\frac{h_5}{0.7}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.4},\frac{v_3}{0.2}\right\}\right)\right),\\
\left(a_4\left(\left\{\frac{h_1}{0.9},\ \frac{h_2}{0.5},\frac{h_3}{0.5},\frac{h_4}{0.2},\frac{h_5}{0.7}\right\},\left\{\frac{c_1}{0},\ \frac{c_2}{0.2},\frac{c_3}{0.7},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7},\frac{v_3}{0.9}\right\}\right)\right),\\
\left(a_5\left(\left\{\frac{h_1}{0.9},\ \frac{h_2}{0.5},\frac{h_3}{0.5},\frac{h_4}{0.2},\frac{h_5}{0.7}\right\},\left\{\frac{c_1}{0.7},\ \frac{c_2}{0.8},\frac{c_3}{0.5},\frac{c_4}{0.4}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.5},\frac{v_3}{0.7}\right\}\right)\right),\\
\left(a_6\left(\left\{\frac{h_1}{0.7},\ \frac{h_2}{0.7},\frac{h_3}{0.1},\frac{h_4}{0.8},\frac{h_5}{0.7}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7},\frac{v_3}{0.6}\right\}\right)\right)
\end{array}\right\}.$$

Each approximation has two parts: a predicate name and an approximate value set.

We can logically explain the above example as follows: we know that
$a_1=(e_{U_1},1,e_{U_2},1,e_{U_3},1)$, where $e_{U_1},1=$ *expensive house*, $e_{U_2},1=$ *expensive car* and
$e_{U_3},1=$ *expensive hotel*. Then

$$F(a_1)= \left(\left\{\frac{h_1}{0.2},\ \frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7},\frac{v_3}{0.7}\right\}\right).$$

We can see that, the membership value for house h_1 is 0.2, so this house is not expensive for Mr. X; also we can see that the membership value for house h_3 is 0.8, this means that the house h_3 is expensive and since the membership value for house h_5 is 0, then this house is absolutely not expensive. Now, since the first set is concerning expensive houses, then we can explain the second set as follows: the membership value for car c_1 is 0.8, so this car is expensive (however, this car may not be expensive if the first set is concerning cheap houses), also we can see that the membership value for car c_3 is 0.4, this means that, this car is not very expensive for him and since the membership value for car c_4 is 0.6, then this car is quite expensive. Now, since the first set is concerning expensive houses and the

second set is concerning expensive cars, then we can also explain the third set as follows: since the membership value for v_1 is 0.8, so this hotel is expensive (but this hotel may not be expensive if the first set is concerning cheap houses and the second set is concerning cheap cars), also we can see that, the membership value for venue v_2 and v_3 is 0.7, this means that these venues are almost expensive. Therefore, depending on the previous explanation we can say the following.
If the $\left\{\frac{h_1}{0.2}, \frac{h_2}{0.4}, \frac{h_3}{0.8}, \frac{h_4}{0.5}, \frac{h_5}{0}\right\}$ is the fuzzy set of expensive houses, then the fuzzy set of relatively expensive cars is $\left\{\frac{c_1}{0.8}, \frac{c_2}{0.5}, \frac{c_3}{0.4}, \frac{c_4}{0.6}\right\}$ and if $\left\{\frac{h_1}{0.2}, \frac{h_2}{0.4}, \frac{h_3}{0.8}, \frac{h_4}{0.5}, \frac{h_5}{0}\right\}$ is the fuzzy set of expensive houses and $\left\{\frac{c_1}{0.8}, \frac{c_2}{0.5}, \frac{c_3}{0.4}, \frac{c_4}{0.6}\right\}$ is the fuzzy set of relatively expensive cars, then the fuzzy set of relatively expensive hotels is $\left\{\frac{v_1}{0.8}, \frac{v_2}{0.7}, \frac{v_3}{0.7}\right\}$.
it is obvious that, the relation in fuzzy soft multiset is conditional relation.
Definition 3.2: For any fuzzy soft multiset $(F,\ A)$, a pair $(e_{U_{ij}},\ F_{e_{U_{ij}}})$ is called a
$U_i - fuzzy$ soft multiset part $e_{U_{ij}} \in a_k$ and $F_{e_{U_{ij}}} \subseteq F(A)$ is a fuzzy approximate value set where $a_k \in A,\ k = \{1,\ 2,\ \ldots, n\},\ i \in \{1,\ 2,\ \ldots, m\}$ and $j \in \{1,\ 2,\ \ldots, r\}$.

Example 3.2: Consider example 3.1. then

$$(e_{U_{ij}},\ F_{eu_{ij}}) = \left\{ \begin{array}{c} \left(e_{U_1}, 1, \left\{\frac{h_1}{0.2}, \frac{h_2}{0.4}, \frac{h_3}{0.8}, \frac{h_4}{0.5}, \frac{h_5}{0}\right\}\right), \left(e_{U_1}, 1, \left\{\frac{h_1}{0.2}, \frac{h_2}{0.4}, \frac{h_3}{0.8}, \frac{h_4}{0.5}, \frac{h_5}{0}\right\}\right), \\ \left(e_{U_1}, 2, \left\{\frac{h_1}{0.7}, \frac{h_2}{0.7}, \frac{h_3}{0.1}, \frac{h_4}{0.8}, \frac{h_5}{0.7}\right\}\right), \left(e_{U_1}, 4, \left\{\frac{h_1}{0.9}, \frac{h_2}{0.5}, \frac{h_3}{0.5}, \frac{h_4}{0.2}, \frac{h_5}{0.7}\right\}\right), \\ \left(e_{U_1}, 4, \left\{\frac{h_1}{0.9}, \frac{h_2}{0.5}, \frac{h_3}{0.5}, \frac{h_4}{0.2}, \frac{h_5}{0.7}\right\}\right), \left(e_{U_1}, 2, \left\{\frac{h_1}{0.7}, \frac{h_2}{0.7}, \frac{h_3}{0.1}, \frac{h_4}{0.8}, \frac{h_5}{0.7}\right\}\right) \end{array} \right\},$$

is a U_1-fuzzy soft multiset part of $(F,\ A)$.
Definition 3.3: For any two fuzzy soft multiset $(F,\ A)$ and $(G,\ B)$ over U, $(F,\ A)$ is called a fuzzy soft multisubset of $(G,\ B)$ if

(1) $A \subseteq B$ and
(2) $\forall\ e_{U_{ij}} \in a_k$, $(e_{U_{ij}},\ F_{e_{U_{ij}}})$ is a fuzzy subset of $(e_{U_{ij}},\ G_{e_{U_{ij}}})$, where $a_k \in A$,

$k = \{1,\ 2,\ \ldots, n\}$, $i \in \{1,\ 2,\ \ldots, m\}$ and $j \in \{1,\ 2,\ \ldots, r\}$.
The relationship is denoted by $(F,\ A)\,\widetilde{\subseteq}(G,\ B)$. In this case $(G,\ B)$ is called a fuzzy soft multisuperset of $(F,\ A)$.
Definition 3.4: Two fuzzy soft multisets $(F,\ A)$ and $(G,\ B)$ over U, are said to be equal if $(F,\ A)$ is a fuzzy soft multisubset of $(G,\ B)$ and $(G,\ B)$ is a fuzzy soft multisubset of $(F,\ A)$.
Example 3.3: Consider example 3.1. Let

$$A = \left\{ \begin{array}{c} a_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1),\ a_2 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1), \\ a_3 = (e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 3), a_4 = (e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 1) \end{array} \right\},$$

$$B=\left\{\begin{array}{c} b_1=(e_{U_1},1,e_{U_2},1,e_{U_3},1)\,,\ b_2=(e_{U_1},1,e_{U_2},2,e_{U_3},1)\,,\\ b_3=(e_{U_1},2,e_{U_2},3,e_{U_3},1)\,,b_4=(e_{U_1},5,e_{U_2},4,e_{U_3},2)\\ b_5=(e_{U_1},4,e_{U_2},3,e_{U_3},3)\,,b_6=(e_{U_1},3,e_{U_2},1,e_{U_3},1)\end{array}\right\},$$

Clearly, $A\subseteq B$. Let $(F,\ A)$ and $(G,\ B)$ be two fuzzy soft multisets over the same U, such that

$$(F,\ A)=\left\{\begin{array}{c} \left(a_1,\left(\left\{\frac{h_1}{0.2},\ \frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7},\frac{v_3}{0.7}\right\}\right)\right),\\ \left(a_2,\left(\left\{\frac{h_1}{0.7},\ \frac{h_2}{0.7},\frac{h_3}{0.1},\frac{h_4}{0.8},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.4},\frac{v_3}{0.2}\right\}\right)\right),\\ \left(a_3,\left(\left\{\frac{h_1}{1},\ \frac{h_2}{0.8},\frac{h_3}{0.7},\frac{h_4}{0},\frac{h_5}{0.7}\right\},\left\{\frac{c_1}{0.7},\ \frac{c_2}{0.8},\frac{c_3}{0.5},\frac{c_4}{0.4}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.5},\frac{v_3}{0.7}\right\}\right)\right),\\ \left(a_4,\left(\left\{\frac{h_1}{0.8},\ \frac{h_2}{0.6},\frac{h_3}{0.1},\frac{h_4}{0.5},\frac{h_5}{1}\right\},\left\{\frac{c_1}{0.5},\ \frac{c_2}{0.3},\frac{c_3}{0.1},\frac{c_4}{0.2}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.3},\frac{v_3}{0.4}\right\}\right)\right)\end{array}\right\},$$

$$(G,\ B)=\left\{\begin{array}{c} \left(b_1,\left(\left\{\frac{h_1}{0.3},\ \frac{h_2}{0.4},\frac{h_3}{0.9},\frac{h_4}{0.7},\frac{h_5}{0.2}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.6},\frac{c_4}{0.7}\right\},\left\{\frac{v_1}{0.9},\ \frac{v_2}{0.7},\frac{v_3}{0.9}\right\}\right)\right),\\ \left(b_2,\left(\left\{\frac{h_1}{0.4},\ \frac{h_2}{0.6},\frac{h_3}{0.7},\frac{h_4}{0.7},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{1},\ \frac{c_2}{0.9},\frac{c_3}{0.9},\frac{c_4}{0.8}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.5},\frac{v_3}{0.4}\right\}\right)\right),\\ \left(b_3,\left(\left\{\frac{h_1}{0.9},\ \frac{h_2}{0.9},\frac{h_3}{1},\frac{h_4}{0.8},\frac{h_5}{0.6}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.8},\frac{c_3}{0.5},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.7},\ \frac{v_2}{0.6},\frac{v_3}{0.5}\right\}\right)\right),\\ \left(b_4,\left(\left\{\frac{h_1}{0.9},\ \frac{h_2}{0.6},\frac{h_3}{0.7},\frac{h_4}{0.5},\frac{h_5}{0.5}\right\},\left\{\frac{c_1}{0.3},\ \frac{c_2}{0.5},\frac{c_3}{0.7},\frac{c_4}{0.7}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.8},\frac{v_3}{0.1}\right\}\right)\right),\\ \left(b_5,\left(\left\{\frac{h_1}{1},\ \frac{h_2}{0.9},\frac{h_3}{0.8},\frac{h_4}{0.4},\frac{h_5}{0.7}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.9},\frac{c_3}{0.5},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.6},\frac{v_3}{0.8}\right\}\right)\right),\\ \left(b_6,\left(\left\{\frac{h_1}{0.8},\ \frac{h_2}{0.7},\frac{h_3}{1},\frac{h_4}{0.9},\frac{h_5}{0.6}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.7},\frac{c_3}{0.6},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.9},\frac{v_3}{0.9}\right\}\right)\right)\end{array}\right\}.$$

Therefore $(F,A)\,\widetilde{\subseteq}(G,B)$.

Definition 3.5: The complement of a fuzzy soft multiset (F,A) is denoted by $(F,\ A)^c$ and is defined by $(F,\ A)^c=(F^c,\ A)$, where $F^c:A\ \to U$ is a mapping given by $F^c\,(a)=\ c\,(F\,(a))\,,\ \forall\ a\in A$ where c is any fuzzy complement.

Example 3.4: Consider example 3.1. By using the basic fuzzy complement, which is
$c\,(x)=1-x$, we have

$$(F,\ A)^c=\left\{\begin{array}{c} \left(a_1\left(\left\{\frac{h_1}{0.8},\ \frac{h_2}{0.6},\frac{h_3}{0.2},\frac{h_4}{0.5},\frac{h_5}{1}\right\},\left\{\frac{c_1}{0.2},\ \frac{c_2}{0.5},\frac{c_3}{0.6},\frac{c_4}{0.4}\right\},\left\{\frac{v_1}{0.2},\ \frac{v_2}{0.3},\frac{v_3}{0.3}\right\}\right)\right),\\ \left(a_2\left(\left\{\frac{h_1}{0.8},\ \frac{h_2}{0.6},\frac{h_3}{0.2},\frac{h_4}{0.5},\frac{h_5}{1}\right\},\left\{\frac{c_1}{0.6},\ \frac{c_2}{0.5},\frac{c_3}{0.2},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.6},\ \frac{v_2}{0.6},\frac{v_3}{0.7}\right\}\right)\right),\\ \left(a_3\left(\left\{\frac{h_1}{0.3},\ \frac{h_2}{0.3},\frac{h_3}{0.9},\frac{h_4}{0.2},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{0.2},\ \frac{c_2}{0.4},\frac{c_3}{0.7},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.6},\frac{v_3}{0.8}\right\}\right)\right),\\ \left(a_4\left(\left\{\frac{h_1}{0.1},\ \frac{h_2}{0.5},\frac{h_3}{0.5},\frac{h_4}{0.8},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{1},\ \frac{c_2}{0.8},\frac{c_3}{0.3},\frac{c_4}{0.4}\right\},\left\{\frac{v_1}{0.2},\ \frac{v_2}{0.3},\frac{v_3}{0.1}\right\}\right)\right),\\ \left(a_5\left(\left\{\frac{h_1}{0.1},\ \frac{h_2}{0.5},\frac{h_3}{0.5},\frac{h_4}{0.8},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{0.3},\ \frac{c_2}{0.2},\frac{c_3}{0.5},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.5},\frac{v_3}{0.3}\right\}\right)\right),\\ \left(a_6\left(\left\{\frac{h_1}{0.3},\ \frac{h_2}{0.3},\frac{h_3}{0.9},\frac{h_4}{0.2},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{0.2},\ \frac{c_2}{0.4},\frac{c_3}{0.7},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.2},\ \frac{v_2}{0.3},\frac{v_3}{0.4}\right\}\right)\right)\end{array}\right\}.$$

Definition 3.6: A fuzzy soft multiset $(F,\ A)$ over U is called a **Seminull fuzzy soft multiset,** denoted by $(F,\ A)_{\approx\emptyset}$, if at least one of a fuzzy soft multiset parts of $(F,\ A)=\emptyset$.

Example 3.5: Consider example 3.1. Let us consider a fuzzy soft multiset $(F,\ A)$ which describes "stone houses," "cars" and "hotels" with

$$A=\{a_1=(e_{U_1},3,e_{U_2},1,e_{U_3},1)\,,\ a_2=(e_{U_1},3,e_{U_2},3,e_{U_3},1)\,,a_3=(e_{U_1},4,e_{U_2},3,e_{U_3},3)\}.$$

Then a seminull fuzzy soft multiset $(F,\ A)_{\approx\emptyset_1}$ is given as

$$(F,\ A)_{\approx\emptyset_1}=\left\{\begin{array}{c}\left(a_1,\left(\left\{\frac{h_1}{0},\frac{h_2}{0},\frac{h_3}{0},\frac{h_4}{0},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.8},\frac{v_2}{0.7},\frac{v_3}{0.7}\right\}\right)\right),\\ \left(a_2,\left(\left\{\frac{h_1}{0},\frac{h_2}{0},\frac{h_3}{0},\frac{h_4}{0},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.4},\ \frac{c_2}{0.5},\frac{c_3}{0.8},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.4},\frac{v_2}{0.4},\frac{v_3}{0.3}\right\}\right)\right),\\ \left(a_3,\left(\left\{\frac{h_1}{0},\frac{h_2}{0},\frac{h_3}{0},\frac{h_4}{0},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\frac{v_2}{0.4},\frac{v_3}{0.2}\right\}\right)\right)\end{array}\right\}.$$

Definition 3.7: A fuzzy soft multiset $(F,\ A)$ over U is a **null fuzzy soft multiset** written $(F,\ A)_{\emptyset}$, if all the fuzzy soft multiset parts of $(F,\ A)=\emptyset$.

Example 3.6: Consider example 3.1. Let us consider a fuzzy soft multiset $(F,\ A)$ which describes "stone houses," "cheap cars" and "hotels in Kuala Lumpur" with

$$A=\{a_1=(e_{U_1},3,e_{U_2},2,\ e_{U_3},3)\,,\ a_2=(e_{U_1},3,e_{U_2},1,e_{U_3},3)\}.$$

$$(F,\ A)_{\emptyset}=\left\{\begin{array}{c}\left(a_1,\left(\left\{\frac{h_1}{0},\frac{h_2}{0},\frac{h_3}{0},\frac{h_4}{0},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0},\ \frac{c_2}{0},\frac{c_3}{0},\frac{c_4}{0.}\right\},\left\{\frac{v_1}{0},\frac{v_2}{0},\frac{v_3}{0}\right\}\right)\right),\\ \left(a_2,\left(\left\{\frac{h_1}{0},\frac{h_2}{0},\frac{h_3}{0},\frac{h_4}{0},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0},\ \frac{c_2}{0},\frac{c_3}{0},\frac{c_4}{0}\right\},\left\{\frac{v_1}{0},\frac{v_2}{0},\frac{v_3}{0}\right\}\right)\right)\end{array}\right\}.$$

Definition 3.8: A fuzzy soft multiset $(F,\ A)$ over U is called a **semi-absolute fuzzy soft multiset**, denoted by $(F,\ A)_{\approx U_i}$, if $(e_{U_i,}j,\ \ Fe_{U_i},j)=\ U_i$ for at least one i, $a_k\in A$, $k=\{1,2,\ldots,n\}\,,\ i\in\{1,2,\ldots,m\}$ and $j\in\{1,2,\ldots,r\}$.

Example 3.7: Consider example 3.1. Let us consider a fuzzy soft multiset $(F,\ A)$ which
Describes "wooden houses," "cars" and "hotels"

$$A=\{a_1=(e_{U_1},3,e_{U_2},1,e_{U_3},1)\,,\ a_2=(e_{U_1},3,e_{U_2},3,e_{U_3},1)\,,a_3=(e_{U_1},3,e_{U_2},3,e_{U_3},3)\}.$$

Then a semi-absolute fuzzy soft multiset $(F,\ A)_{\approx U_1}$ is given as

$$(F,\ A)_{\approx U_1}=\left\{\begin{array}{c}\left(a_1,\left(\left\{\frac{h_1}{1},\frac{h_2}{1},\frac{h_3}{1},\frac{h_4}{1},\frac{h_5}{1}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.8},\frac{v_2}{0.7},\frac{v_3}{0.7}\right\}\right)\right),\\ \left(a_2,\left(\left\{\frac{h_1}{1},\frac{h_2}{1},\frac{h_3}{1},\frac{h_4}{1},\frac{h_5}{1}\right\},\left\{\frac{c_1}{0.4},\ \frac{c_2}{0.5},\frac{c_3}{0.8},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.4},\frac{v_2}{0.4},\frac{v_3}{0.3}\right\}\right)\right),\\ \left(a_3,\left(\left\{\frac{h_1}{1},\frac{h_2}{1},\frac{h_3}{1},\frac{h_4}{1},\frac{h_5}{1}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\frac{v_2}{0.4},\frac{v_3}{0.2}\right\}\right)\right)\end{array}\right\}.$$

Definition 3.9: A fuzzy soft multiset $(F,\ A)$ over U is called an **absolute fuzzy soft multiset** denoted by $(F,\ A)_U$, if $(e_{U_i},j,\ \ Fe_{U_i},j)=\ U_i,\ \forall\ i$.

Example 3.8: Consider example 3.1. Let us consider a fuzzy soft multiset $(F,\ A)$ which describes "wooden houses," "expensive classic cars" and "hotels in

Kuala Lumpur" with

$$A=\{a_1=(e_{U_1},3,e_{U_2},1,e_{U_3},3)\,,\ a_2=(e_{U_1},3,e_{U_2},3,e_{U_3},3)\}.$$

Then an absolute fuzzy soft multiset $(\widetilde{\Gamma},\ A)_U$ is given as

$$(F,\ A)_U=\left\{\begin{array}{c}\left(a_1,\left(\left\{\frac{h_1}{1},\frac{h_2}{1},\frac{h_3}{1},\frac{h_4}{1}\right\},\left\{\frac{c_1}{1},\ \frac{c_3}{1},\frac{c_4}{1}\right\},\left\{\frac{v_1}{1},\frac{v_2}{1}\right\}\right)\right),\\ \left(a_2,\left(\left\{\frac{h_1}{1},\frac{h_2}{1},\frac{h_3}{1},\frac{h_4}{1}\right\},\left\{\frac{c_1}{1},\ \frac{c_3}{1},\frac{c_4}{1}\right\},\left\{\frac{v_1}{1},\frac{v_2}{1}\right\}\right)\right)\end{array}\right\}.$$

Definition 3.10: Let $(F,\ A)$ and $(G,\ B)$ be two fuzzy soft multisets over U. Then the union of two fuzzy soft multiset $(F,\ A)$ and $(G,\ B)$ denoted by $(F,\ A)\widetilde{\cup}(G,\ B)$ is defined as
$(F,\ A)\widetilde{\cup}(G,\ B)=(H,\ C)$, where $C=A\cup B$ and for all $e\in C$,

$$H(e)=\left\{\begin{array}{ll}F(e)\,, & if\ e\in A/B\\ G(e)\,, & if\ e\in B/A\\ F(e)\cup G(e)\,, & if\ e\in A\cap B\end{array}\right..$$

where $F(e)\cup G(e)=\max(F(e)\,,G(e))$.

Example 3.9: Let

$$A=\left\{\begin{array}{c}a_1=(e_{U_1},1,e_{U_2},1,e_{U_3},1)\,,\ a_2=(e_{U_1},2,e_{U_2},3,e_{U_3},1)\,,\\ a_3=(e_{U_1},4,e_{U_2},3,e_{U_3},3)\,,a_4=(e_{U_1},3,e_{U_2},1,e_{U_3},1)\end{array}\right\},$$

$$B=\left\{\begin{array}{c}b_1=(e_{U_1},1,e_{U_2},1,e_{U_3},1)\,,\ b_2=(e_{U_1},1,e_{U_2},2,e_{U_3},1)\,,\\ b_3=(e_{U_1},2,e_{U_2},3,e_{U_3},1)\,,b_4=(e_{U_1},5,e_{U_2},4,e_{U_3},2)\\ b_5=(e_{U_1},4,e_{U_2},3,e_{U_3},3)\,,b_6=(e_{U_1},3,e_{U_2},1,e_{U_3},1)\end{array}\right\},$$

Suppose $(F,\ A)$ and $(G,\ B)$ are two fuzzy soft multisets over U, such that

$$(F,\ A)=\left\{\begin{array}{c}\left(a_1,\left(\left\{\frac{h_1}{0.2},\ \frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7},\frac{v_3}{0.7}\right\}\right)\right),\\ \left(a_2,\left(\left\{\frac{h_1}{0.7},\ \frac{h_2}{0.7},\frac{h_3}{0.1},\frac{h_4}{0.8},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.4},\frac{v_3}{0.2}\right\}\right)\right),\\ \left(a_3,\left(\left\{\frac{h_1}{1},\ \frac{h_2}{0.8},\frac{h_3}{0.7},\frac{h_4}{0},\frac{h_5}{0.7}\right\},\left\{\frac{c_1}{0.7},\ \frac{c_2}{0.8},\frac{c_3}{0.5},\frac{c_4}{0.4}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.5},\frac{v_3}{0.7}\right\}\right)\right),\\ \left(a_4,\left(\left\{\frac{h_1}{0.8},\ \frac{h_2}{0.6},\frac{h_3}{0.1},\frac{h_4}{0.5},\frac{h_5}{1}\right\},\left\{\frac{c_1}{0.5},\ \frac{c_2}{0.3},\frac{c_3}{0.1},\frac{c_4}{0.2}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.3},\frac{v_3}{0.4}\right\}\right)\right)\end{array}\right\},$$

$$(G,\ B)=\left\{\begin{array}{c}\left(b_1,\left(\left\{\frac{h_1}{0.3},\ \frac{h_2}{0.4},\frac{h_3}{0.9},\frac{h_4}{0.7},\frac{h_5}{0.2}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.6},\frac{c_4}{0.7}\right\},\left\{\frac{v_1}{0.9},\ \frac{v_2}{0.7},\frac{v_3}{0.9}\right\}\right)\right),\\ \left(b_2,\left(\left\{\frac{h_1}{0.4},\ \frac{h_2}{0.6},\frac{h_3}{0.7},\frac{h_4}{0.7},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{1},\ \frac{c_2}{0.9},\frac{c_3}{0.9},\frac{c_4}{0.8}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.5},\frac{v_3}{0.4}\right\}\right)\right),\\ \left(b_3,\left(\left\{\frac{h_1}{0.9},\ \frac{h_2}{0.9},\frac{h_3}{1},\frac{h_4}{0.8},\frac{h_5}{0.6}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.8},\frac{c_3}{0.5},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.7},\ \frac{v_2}{0.6},\frac{v_3}{0.5}\right\}\right)\right),\\ \left(b_4,\left(\left\{\frac{h_1}{0.9},\ \frac{h_2}{0.6},\frac{h_3}{0.7},\frac{h_4}{0.5},\frac{h_5}{0.5}\right\},\left\{\frac{c_1}{0.3},\ \frac{c_2}{0.5},\frac{c_3}{0.7},\frac{c_4}{0.7}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.8},\frac{v_3}{0.1}\right\}\right)\right),\\ \left(b_5,\left(\left\{\frac{h_1}{1},\ \frac{h_2}{0.9},\frac{h_3}{0.8},\frac{h_4}{0.4},\frac{h_5}{0.7}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.9},\frac{c_3}{0.5},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.6},\frac{v_3}{0.8}\right\}\right)\right),\\ \left(b_6,\left(\left\{\frac{h_1}{0.8},\ \frac{h_2}{0.7},\frac{h_3}{1},\frac{h_4}{0.9},\frac{h_5}{0.6}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.7},\frac{c_3}{0.6},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.9},\frac{v_3}{0.9}\right\}\right)\right)\end{array}\right\}.$$

Therefore

$$(H,\ C) = \left\{ \begin{array}{c} \left(c_1, \left(\left\{\frac{h_1}{0.3}, \frac{h_2}{0.4}, \frac{h_3}{0.9}, \frac{h_4}{0.7}, \frac{h_5}{0.2}\right\}, \left\{\frac{c_1}{0.8}, \frac{c_2}{0.6}, \frac{c_3}{0.6}, \frac{c_4}{0.7}\right\}, \left\{\frac{v_1}{0.9}, \frac{v_2}{0.7}, \frac{v_3}{0.9}\right\}\right)\right), \\ \left(c_2, \left(\left\{\frac{h_1}{0.9}, \frac{h_2}{0.9}, \frac{h_3}{1}, \frac{h_4}{0.8}, \frac{h_5}{0.6}\right\}, \left\{\frac{c_1}{0.8}, \frac{c_2}{0.8}, \frac{c_3}{0.5}, \frac{c_4}{0.5}\right\}, \left\{\frac{v_1}{0.7}, \frac{v_2}{0.6}, \frac{v_3}{0.5}\right\}\right)\right), \\ \left(c_3, \left(\left\{\frac{h_1}{1}, \frac{h_2}{0.8}, \frac{h_3}{0.7}, \frac{h_4}{0}, \frac{h_5}{0.7}\right\}, \left\{\frac{c_1}{0.7}, \frac{c_2}{0.8}, \frac{c_3}{0.5}, \frac{c_4}{0.4}\right\}, \left\{\frac{v_1}{0.5}, \frac{v_2}{0.5}, \frac{v_3}{0.7}\right\}\right)\right), \\ \left(c_4, \left(\left\{\frac{h_1}{0.8}, \frac{h_2}{0.6}, \frac{h_3}{0.1}, \frac{h_4}{0.5}, \frac{h_5}{1}\right\}, \left\{\frac{c_1}{0.5}, \frac{c_2}{0.3}, \frac{c_3}{0.1}, \frac{c_4}{0.2}\right\}, \left\{\frac{v_1}{0.5}, \frac{v_2}{0.3}, \frac{v_3}{0.4}\right\}\right)\right), \\ \left(c_5, \left(\left\{\frac{h_1}{0.4}, \frac{h_2}{0.6}, \frac{h_3}{0.7}, \frac{h_4}{0.7}, \frac{h_5}{0.3}\right\}, \left\{\frac{c_1}{1}, \frac{c_2}{0.9}, \frac{c_3}{0.9}, \frac{c_4}{0.8}\right\}, \left\{\frac{v_1}{0.8}, \frac{v_2}{0.5}, \frac{v_3}{0.4}\right\}\right)\right), \\ \left(c_6, \left(\left\{\frac{h_1}{0.9}, \frac{h_2}{0.6}, \frac{h_3}{0.7}, \frac{h_4}{0.5}, \frac{h_5}{0.5}\right\}, \left\{\frac{c_1}{0.3}, \frac{c_2}{0.5}, \frac{c_3}{0.7}, \frac{c_4}{0.7}\right\}, \left\{\frac{v_1}{0.8}, \frac{v_2}{0.8}, \frac{v_3}{0.1}\right\}\right)\right), \\ \left(c_7, \left(\left\{\frac{h_1}{1}, \frac{h_2}{0.9}, \frac{h_3}{0.8}, \frac{h_4}{0.4}, \frac{h_5}{0.7}\right\}, \left\{\frac{c_1}{0.8}, \frac{c_2}{0.9}, \frac{c_3}{0.5}, \frac{c_4}{0.5}\right\}, \left\{\frac{v_1}{0.5}, \frac{v_2}{0.6}, \frac{v_3}{0.8}\right\}\right)\right), \\ \left(c_8, \left(\left\{\frac{h_1}{0.8}, \frac{h_2}{0.7}, \frac{h_3}{1}, \frac{h_4}{0.9}, \frac{h_5}{0.6}\right\}, \left\{\frac{c_1}{0.8}, \frac{c_2}{0.7}, \frac{c_3}{0.6}, \frac{c_4}{0.5}\right\}, \left\{\frac{v_1}{0.8}, \frac{v_2}{0.9}, \frac{v_3}{0.9}\right\}\right)\right) \end{array} \right\}.$$

where

$$C = \{a_1 = b_1 = c_1,\ a_2 = b_3 = c_2, a_3 = c_3,\ a_4 = c_4,\ b_2 = c_5, b_4 = c_6, b_5 = c_7, b_6 = c_8,\}$$

Definition 3.11: The **extended intersection** of two fuzzy soft multisets $(F,\ A)$ and $(G,\ B)$ over U, denoted by $(F,\ A)\widetilde{\cap}_E(G,\ B)$, is the fuzzy soft multiset $(H,\ D)$, where $D = A \cup B$ and for all $e \in D$,

$$\tilde{H}(e) = \begin{cases} F(e), & if\ e \in A/B \\ G(e), & if\ e \in B/A \\ F(e) \cap G(e), & if\ e \in A \cap B \end{cases}.$$

where $F(e) \cap G(e) = \min(F(e), G(e))$

Example 3.10: Let consider example 3.9 with

$$A = \left\{ \begin{array}{c} a_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1),\ a_2 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1), \\ a_3 = (e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 3), a_4 = (e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 1) \end{array} \right\},$$

$$B = \left\{ \begin{array}{c} b_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1),\ b_2 = (e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1), \\ b_3 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1), b_4 = (e_{U_1}, 5, e_{U_2}, 4, e_{U_3}, 2) \\ b_5 = (e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 3), b_6 = (e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 1) \end{array} \right\}.$$

hence

$$(H,\ D)=\left\{\begin{array}{l}\left(d_1,\left(\left\{\frac{h_1}{0.2},\ \frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7},\frac{v_3}{0.7}\right\}\right)\right),\\ \left(d_2,\left(\left\{\frac{h_1}{0.7},\ \frac{h_2}{0.7},\frac{h_3}{0.1},\frac{h_4}{0.8},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.4},\frac{v_3}{0.2}\right\}\right)\right),\\ \left(d_3,\left(\left\{\frac{h_1}{1},\ \frac{h_2}{0.8},\frac{h_3}{0.7},\frac{h_4}{0},\frac{h_5}{0.7}\right\},\left\{\frac{c_1}{0.7},\ \frac{c_2}{0.8},\frac{c_3}{0.5},\frac{c_4}{0.4}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.5},\frac{v_3}{0.7}\right\}\right)\right),\\ \left(d_4,\left(\left\{\frac{h_1}{0.8},\ \frac{h_2}{0.6},\frac{h_3}{0.1},\frac{h_4}{0.5},\frac{h_5}{1}\right\},\left\{\frac{c_1}{0.5},\ \frac{c_2}{0.3},\frac{c_3}{0.1},\frac{c_4}{0.2}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.3},\frac{v_3}{0.4}\right\}\right)\right),\\ \left(d_5,\left(\left\{\frac{h_1}{0.4},\ \frac{h_2}{0.6},\frac{h_3}{0.7},\frac{h_4}{0.7},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{1},\ \frac{c_2}{0.9},\frac{c_3}{0.9},\frac{c_4}{0.8}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.5},\frac{v_3}{0.4}\right\}\right)\right),\\ \left(d_6,\left(\left\{\frac{h_1}{0.9},\ \frac{h_2}{0.6},\frac{h_3}{0.7},\frac{h_4}{0.5},\frac{h_5}{0.5}\right\},\left\{\frac{c_1}{0.3},\ \frac{c_2}{0.5},\frac{c_3}{0.7},\frac{c_4}{0.7}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.8},\frac{v_3}{0.1}\right\}\right)\right),\\ \left(d_7,\left(\left\{\frac{h_1}{1},\ \frac{h_2}{0.9},\frac{h_3}{0.8},\frac{h_4}{0.4},\frac{h_5}{0.7}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.9},\frac{c_3}{0.5},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.6},\frac{v_3}{0.8}\right\}\right)\right),\\ \left(d_8,\left(\left\{\frac{h_1}{0.8},\ \frac{h_2}{0.7},\frac{h_3}{1},\frac{h_4}{0.9},\frac{h_5}{0.6}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.7},\frac{c_3}{0.6},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.9},\frac{?_{,3}}{0.9}\right\}\right)\right)\end{array}\right\}.$$

where

$$C=\{a_1=b_1=\ d_1,\ a_2=b_3=\ d_2,a_3=\ d_3,\ a_4=\ d_4,\ b_2=\ d_5,b_4=\ d_6,b_5=\ d_7,b_6=\ d_8,\}$$

Definition 3.12: Let $(F,\ A)$ and $(G,\ B)$ be two fuzzy soft multisets over U. Then the **restricted union** of $(F,\ A)$ and $(G,\ B)$ denoted by $(F,\ A)\widetilde{\cup}_R(G,\ B)$ is defined as:
$(F,\ A)\widetilde{\cup}_R(G,\ B)=(H,\ C)$, where $C=A\cap B\neq\emptyset$ and for all $e\in C$,

$$H\left(e\right)=\ F(e)\cup G(e).$$

Where $F\left(e\right)\cup G\left(e\right)=\max(F\left(e\right),G\left(e\right))$
Example 3.11: Consider example 3.9. Let

$$A=\left\{\begin{array}{c}a_1=\left(e_{U_1},1,e_{U_2},1,e_{U_3},1\right),\ a_2=\left(e_{U_1},2,e_{U_2},3,e_{U_3},1\right),\\ a_3=\left(e_{U_1},4,e_{U_2},3,e_{U_3},3\right),a_4=\left(e_{U_1},3,e_{U_2},1,e_{U_3},1\right)\end{array}\right\},$$

$$B=\left\{\begin{array}{c}b_1=\left(e_{U_1},1,e_{U_2},1,e_{U_3},1\right),\ b_2=\left(e_{U_1},1,e_{U_2},2,e_{U_3},1\right),\\ b_3=\left(e_{U_1},2,e_{U_2},3,e_{U_3},1\right),b_4=\left(e_{U_1},5,e_{U_2},4,e_{U_3},2\right)\\ b_5=\left(e_{U_1},4,e_{U_2},3,e_{U_3},3\right),b_6=\left(e_{U_1},3,e_{U_2},1,e_{U_3},1\right)\end{array}\right\},$$

Suppose $(F,\ A)$ and $(G,\ B)$ are two fuzzy soft multisets over U, such that

$$(F,\ A)=\left\{\begin{array}{l}\left(a_1,\left(\left\{\frac{h_1}{0.2},\ \frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7},\frac{v_3}{0.7}\right\}\right)\right),\\ \left(a_2,\left(\left\{\frac{h_1}{0.7},\ \frac{h_2}{0.7},\frac{h_3}{0.1},\frac{h_4}{0.8},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.4},\frac{v_3}{0.2}\right\}\right)\right),\\ \left(a_3,\left(\left\{\frac{h_1}{1},\ \frac{h_2}{0.8},\frac{h_3}{0.7},\frac{h_4}{0},\frac{h_5}{0.7}\right\},\left\{\frac{c_1}{0.7},\ \frac{c_2}{0.8},\frac{c_3}{0.5},\frac{c_4}{0.4}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.5},\frac{v_3}{0.7}\right\}\right)\right),\\ \left(a_4,\left(\left\{\frac{h_1}{0.8},\ \frac{h_2}{0.6},\frac{h_3}{0.1},\frac{h_4}{0.5},\frac{h_5}{1}\right\},\left\{\frac{c_1}{0.5},\ \frac{c_2}{0.3},\frac{c_3}{0.1},\frac{c_4}{0.2}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.3},\frac{v_3}{0.4}\right\}\right)\right)\end{array}\right\},$$

$$(G,\ B)=\left\{\begin{array}{l}\left(b_1,\left(\left\{\frac{h_1}{0.3},\ \frac{h_2}{0.4},\frac{h_3}{0.9},\frac{h_4}{0.7},\frac{h_5}{0.2}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.6},\frac{c_4}{0.7}\right\},\left\{\frac{v_1}{0.9},\ \frac{v_2}{0.7},\frac{v_3}{0.9}\right\}\right)\right),\\ \left(b_2,\left(\left\{\frac{h_1}{0.4},\ \frac{h_2}{0.6},\frac{h_3}{0.7},\frac{h_4}{0.7},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{1},\ \frac{c_2}{0.9},\frac{c_3}{0.9},\frac{c_4}{0.8}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.5},\frac{v_3}{0.4}\right\}\right)\right),\\ \left(b_3,\left(\left\{\frac{h_1}{0.9},\ \frac{h_2}{0.9},\frac{h_3}{1},\frac{h_4}{0.8},\frac{h_5}{0.6}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.8},\frac{c_3}{0.5},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.7},\ \frac{v_2}{0.6},\frac{v_3}{0.5}\right\}\right)\right),\\ \left(b_4,\left(\left\{\frac{h_1}{0.9},\ \frac{h_2}{0.6},\frac{h_3}{0.7},\frac{h_4}{0.5},\frac{h_5}{0.5}\right\},\left\{\frac{c_1}{0.3},\ \frac{c_2}{0.5},\frac{c_3}{0.7},\frac{c_4}{0.7}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.8},\frac{v_3}{0.1}\right\}\right)\right),\\ \left(b_5,\left(\left\{\frac{h_1}{1},\ \frac{h_2}{0.9},\frac{h_3}{0.8},\frac{h_4}{0.4},\frac{h_5}{0.7}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.9},\frac{c_3}{0.5},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.6},\frac{v_3}{0.8}\right\}\right)\right),\\ \left(b_6,\left(\left\{\frac{h_1}{0.8},\ \frac{h_2}{0.7},\frac{h_3}{1},\frac{h_4}{0.9},\frac{h_5}{0.6}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.7},\frac{c_3}{0.6},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.9},\frac{v_3}{0.9}\right\}\right)\right)\end{array}\right\}.$$

$$(F,\ A)\,\widetilde{\cup}_R\,(G,\ B)=\left\{\begin{array}{l}\left(c_1,\left(\left\{\frac{h_1}{0.3},\ \frac{h_2}{0.4},\frac{h_3}{0.9},\frac{h_4}{0.7},\frac{h_5}{0.2}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.6},\frac{c_4}{0.7}\right\},\left\{\frac{v_1}{0.9},\ \frac{v_2}{0.7},\frac{v_3}{0.9}\right\}\right)\right),\\ \left(c_2,\left(\left\{\frac{h_1}{0.8},\ \frac{h_2}{0.9},\frac{h_3}{1},\frac{h_4}{0.8},\frac{h_5}{0.6}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.8},\frac{c_3}{0.5},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.7},\ \frac{v_2}{0.6},\frac{v_3}{0.5}\right\}\right)\right)\end{array}\right\}.$$

$$C=A\cap B=\ \{a_1=\ b_1=\ c_1,\ a_2=\ b_3=\ c_2\}.$$

Theorem 3.1: If $(F,A),(G,B)$ *and* (H,C) are three fuzzy soft multisets over U, then

$$\begin{array}{rl}(1) & (F,A)\,\widetilde{\cup}_R\,(F,A)=(F,A),\\ (2) & (F,A)\,\widetilde{\cup}_R\,\big((G,B)\,\widetilde{\cup}_R\,(H,C)\big)=\ \big((F,A)\,\widetilde{\cup}_R\,(G,B)\big)\,\widetilde{\cup}_R\,(H,C),\\ (3) & (F,A)\,\widetilde{\cup}_R(G,A)_{\approx\Phi_i}=(R,A),\\ (4) & (F,A)\,\widetilde{\cup}_R(G,A)_{\Phi}=(F,A),\\ (5) & (F,A)\,\widetilde{\cup}_R(G,B)_{\approx\Phi_i}=(R,D),\text{ where } D=A\cap B,\\ (6) & (F,A)\,\widetilde{\cup}_R(G,B)_{\Phi}=\left\{\begin{array}{ll}(F,A) & if\ A=B,\\ (R,D) & otherwise\end{array}\right.,\quad where\ D=A\cap B,\\ (7) & (F,A)\,\widetilde{\cup}_R(G,A)_{\approx U_i}=(R,A)_{\approx A_i},\\ (8) & (F,A)\,\widetilde{\cup}_R(G,A)_{\mathrm{U}}=(G,A)_{\mathrm{A}},\\ (9) & (F,A)\,\widetilde{\cup}_R(G,B)_{\approx U_i}=\left\{\begin{array}{ll}(R,D)_{\approx A_i} & if\ A=B,\\ (R,D) & otherwise\end{array}\right.,\quad where\ D=A\cap B,\\ (10) & (F,A)\,\widetilde{\cup}_R(G,B)_{U}=\left\{\begin{array}{ll}(G,B)_{A} & if\ A\subseteq B,\\ (R,D) & otherwise\end{array}\right.,\quad where\ D=A\cap B.\end{array}$$

Proof:

(1) $(F,\ A)\,\widetilde{\cup}_R\,(F,\ A)=(F,A)$, since $A=A\cap A\neq\emptyset,\ \forall e\in A$, $F(e)=\ F(e)\cup F(e)=F(e)$. Hence, the result.

(2) We first consider left hand side:

$(G,\ B)\,\widetilde{\cup}_R\,(H,\ C) = (M, D)$, where $D = B \cap C \neq \emptyset,\ \forall\ e \in D$,

$$M\,(e) = G\,(e) \cup H(e).$$

Now, $(F,\ A)\,\widetilde{\cup}_R\,\big((G,\ B)\,\widetilde{\cup}_R\,(H,\ C)\big) = (F,\ A)\,\widetilde{\cup}_R\,(M, D) = (J, Z)$,
where $Z = A \cap D = A \cap B \cap C \neq \emptyset,\ \forall\ e \in Z$,

$J\,(e) = F\,(e) \cup M\,(e) = F\,(e) \cup G\,(e) \cup H\,(e)$ .... [1]

Let us consider right hand side:
$(F, A)\,\widetilde{\cup}_R\,(G,\ B) = (N, K)$, where $K = A \cap B \neq \emptyset,\ \forall\ e \in K$,

$$N\,(e) = F\,(e) \cup G\,(e)\,.$$

Also, $\big((F, A)\,\widetilde{\cup}_R\,(G,\ B)\big)\,\widetilde{\cup}_R\,(H,\ C) = (N, K)\,\widetilde{\cup}_R\,(H,\ C) = (W,\ L)$, where
$L = K \cap C = A \cap B \cap C \neq \emptyset$, then $\forall\ e \in L$,

$W\,(e) = N\,(e) \cup H\,(e) = F\,(e) \cup G\,(e) \cup H\,(e)$ .... [2].

Since [1] and [2] are the same map, hence the result.

(3) $(F,\ A)\,\widetilde{\cup}_R\,(G,\ A)_{\approx\emptyset_i} = (R,\ A)_{\approx\emptyset_i}$, where $A = A \cap A \neq \emptyset,\ \forall\ e \in A$,
$R\,(e) = F\,(e) \cup G(e)$, hence the result.

(4) $(F, A)\,\widetilde{\cup}_R(G, A)_\Phi = (F, A)\,,\ where\ A = A \cap A \neq \emptyset,\ \forall\ e \in A$
$F\,(e) = F\,(e) \cup G\,(e) = F(e)$, since every $G\,(e) = \emptyset$.
Therefore, the result has been established.
Following similar argument (5)-(10) can be proved.

Definition 3.13: Let $(F,\ A)$ and $(G,\ B)$ be two fuzzy soft multisets over U. Then the **restricted intersection** of $(F,\ A)$ and $(G,\ B)$ denoted by $(F,\ A)\Cap(G,\ B)$ is defined as
$(F,\ A)\Cap(G,\ B) = (H,\ C)$, where $C = A \cap B \neq \emptyset$ and for all $e \in C$,

$$V\,(e) = F(e) \cap G(e).$$

Where $F\,(e) \cap G\,(e) = \min(F\,(e)\,, G\,(e))$

Example 3.12: Consider example 3.9.

$$(F,\ A)\Cap(G,\ B) = \left\{\begin{array}{l} \left(c_1, \left(\left\{\frac{h_1}{0.3}, \frac{h_2}{0.4}, \frac{h_3}{0.9}, \frac{h_4}{0.7}, \frac{h_5}{0.2}\right\}, \left\{\frac{c_1}{0.8}, \frac{c_2}{0.6}, \frac{c_3}{0.6}, \frac{c_4}{0.7}\right\}, \left\{\frac{v_1}{0.9}, \frac{v_2}{0.7}, \frac{v_3}{0.9}\right\}\right)\right), \\ \left(c_2, \left(\left\{\frac{h_1}{0.8}, \frac{h_2}{0.9}, \frac{h_3}{1}, \frac{h_4}{0.8}, \frac{h_5}{0.6}\right\}, \left\{\frac{c_1}{0.8}, \frac{c_2}{0.8}, \frac{c_3}{0.5}, \frac{c_4}{0.5}\right\}, \left\{\frac{v_1}{0.7}, \frac{v_2}{0.6}, \frac{v_3}{0.5}\right\}\right)\right) \end{array}\right\}.$$

Where $C = A \cap B = \{a_1 = b_1 = c_1,\ a_2 = b_3 = c_2\}$.

Theorem 3.2: If $(F,\ A)$, $(G,\ B)$ and $(H,\ C)$ are three fuzzy soft multisets over U, then

(1) $(F,\ A)\Cap(F,\ A) = (F,\ A)$,
(2) $(F,\ A)\Cap((G, B)\Cap(H,\ C)) = ((F, A)\Cap(G,\ B))\Cap(H,\ C)$,
(3) $(F,\ A)\Cap(G,\ A)_U = (F,\ A)$,
(4) $(F,\ A)\Cap(G,\ A)_{\approx\emptyset_i} = (R,\ A)_{\approx\emptyset_i}$, where R is defined by definition 3.13.
(5) $(F,\ A)\Cap(G,\ A)_\emptyset = (R,\ A)_\emptyset$, where R is as defined by definition 3.13.

(6) $(F,\ A) \Cap (G,\ B)_{\approx\emptyset_i} = \begin{cases} (R,\ A)_{\approx\emptyset_i}, & if \quad A \subseteq B \\ (R,\ D), & otherwise, \end{cases}$ $where\ D = A \cap B$,

(7) $(F,\ A) \Cap (G,\ B)_{\approx U_i} =\ (R,\ D)$ where $D = A \cap B$ and R is as defined by definition 3.13.

(8) $(F,\ A) \Cap (G,\ A)_U =\ (F,\ A)$.

(9) $(F,\ A) \Cap (G,\ B)_{\emptyset} = \begin{cases} (G,\ B)_{\emptyset}, & if\ A = B, \\ (R,\ D), & \text{otherwise}, \end{cases}$ $where\ D = A \cap B$,

(10) $(F,\ A) \Cap (G,\ B)_U = \begin{cases} (F,\ A), & if\ A = B, \\ (R,\ D), & otherwise, \end{cases}$ $where\ D = A \cap B$, R is as defined by definition 3.13.

Proof:

(1) $(F,\ A) \Cap (F,\ A) = (F, A)$, since $A = A \cap A \neq \emptyset,\ \forall e \in A$, $F(e) =\ F(e) \cap F(e) = F(e)$. Hence, the result.

(2) We first consider left hand side:
$(G,\ B) \Cap (H,\ C) = (M, D)$, where $D = B \cap C \neq \emptyset,\ \forall\ e \in D$,

$$M(e) = G(e) \cap H(e).$$

Now, $(F,\ A) \Cap ((G,\ B) \Cap (H,\ C)) = (F,\ A) \Cap (M, D) = (N, L)$,
where $L = A \cap D = A \cap B \cap C \neq \emptyset,\ \forall\ e \in L$,

$$N(e) = F(e) \cup M(e) =\ F(e) \cap G(e) \cap H(e) \ldots\ldots [3]$$

Let us consider right hand side:
$(F, A) \Cap (G,\ B) = (W, Q)$, where $Q = A \cap B \neq \emptyset,\ \forall\ e \in Q$,

$$W(e) =\ F(e) \cap G(e).$$

Also, $((F, A) \Cap (G,\ B)) \Cap (H,\ C) = (W, Q) \Cap (H,\ C) = (K,\ V)$, where
$V = Q \cap C = A \cap B \cap C \neq \emptyset$, then $\forall\ e \in V$,

$$K(e) =\ W(e) \cap H(e) = F(e) \cap G(e) \cap H(e) \ldots\ldots [4]$$

Since [3] and [4] are the same map, hence the result.

(3) $(F,\ A) \Cap (G,\ A)_{\approx\emptyset_i} =\ (R,\ A)_{\approx\emptyset_i}$, where $A = A \cap A \neq \emptyset,\ \forall\ e \in A$, $R(e) = F(e) \cup G(e)$, hence the result.

Following similar argument (3), (5)-(10) can be proved.

Proposition 3.1:
If $(F, A), (G, B)\ and\ (H, C)$ are three fuzzy soft multisets over U, then

(1) $(F, A)\, \widetilde{\cup} \big((G, B)\, \widetilde{\cap}_E\, (H, C)\big) =\ \big((F, A)\, \widetilde{\cup}\, (G, B)\big)\, \widetilde{\cap}_E \big((F, A)\, \widetilde{\cup}\, (H, C)\big)$,

(2) $(F, A)\, \widetilde{\cap}_E \big((G, B)\, \widetilde{\cup}\, (H, C)\big) = \big((F, A)\, \widetilde{\cap}_E\, (G, B)\big)\, \widetilde{\cup} \big((F, A)\, \widetilde{\cap}_E\, (H, C)\big)$.

Proof: The proofs are straightforward.

Proposition 3.2:
If $(F, A), (G, B)\ and\ (H, C)$ are three fuzzy soft multisets over U, then

(1) $(F, A)\, \widetilde{\cup}_R \left((G, B) \Cap (H, C)\right) =\ \big((F, A)\, \widetilde{\cup}_R\, (G, B)\big) \Cap \big((F, A)\, \widetilde{\cup}_R\, (H, C)\big)$,

(2) $(F,A)\Cap\left((G,B)\widetilde{\cup}_R(H,C)\right)=((F,A)\Cap(G,B))\widetilde{\cup}_R((F,A)\Cap(H,C))$.

Proof: The proofs are straightforward.

4. De Morgan's Inclusions and Laws

We shall prove the following De Morgan's inclusions and Laws.

Theorem 4.1. If (F,A) *and* (G,B) are two fuzzy soft multisets over U, then

(1) $\left((F,A)\widetilde{\cup}(G,B)\right)^C\widetilde{\subseteq}(F,A)^C\widetilde{\cup}(G,B)^C$,

(2) $(F,\ A)^C\Cap(G,B)^C\widetilde{\subseteq}\left((F,A)\Cap\left(\tilde{G},\ B\right)\right)^C$.

Proof

(1) Let $(F,A)\widetilde{\cup}(G,B)=(H,\ A\cup B)$. Therefore,

$$\left((F,A)\widetilde{\cup}(G,B)\right)^C=(H,\ A\cup B)^C,$$

$$=\left(\tilde{H}^C,\ \neg(A\cup B)\right).$$

Take $\neg\boldsymbol{\alpha}\in\neg(A\cup B)$,

$$H^C(\neg\boldsymbol{\alpha})=(H(\alpha))^C,$$

$$=\begin{cases}(F(\alpha))^C, & if\ \neg\boldsymbol{\alpha}\in\neg A/\neg B\\ (G(\alpha))^C, & if\ \neg\boldsymbol{\alpha}\in\neg B/\neg A\\ (F(\alpha)\cup G(\alpha))^C, & if\neg\boldsymbol{\alpha}\in\neg A\cap\neg B\end{cases}.$$

$$=\begin{cases}F^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha}\in\neg A/\neg B\\ G^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha}\in\neg B/\neg A\\ F^C(\neg\boldsymbol{\alpha})\cap\ G^C(\neg\boldsymbol{\alpha}), & if\neg\boldsymbol{\alpha}\in\neg A\cap\neg B\end{cases}$$

Consider

$$(F,A)^C\widetilde{\cup}(G,B)^C=(F^C,\neg A)\widetilde{\cup}(G^C,\neg B),$$

$=(J,\ \neg A\cup\neg B)$, (say), where

$$J(\neg\boldsymbol{\alpha})=\begin{cases}F^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha}\in\neg A/\neg B\\ G^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha}\in\neg B/\neg A\\ F^C(\neg\boldsymbol{\alpha})\cup\ G^C(\neg\boldsymbol{\alpha}), & if\neg\boldsymbol{\alpha}\in\neg A\cap\neg B\end{cases}.$$

Obviously, $H^C(\neg\boldsymbol{\alpha})\ \subseteq\ J(\neg\boldsymbol{\alpha})$, hence, (1) holds.

(2) Consider $(F,\ A)^C\Cap(G,\ B)^C=\left(F^C,\ \neg A\right)\Cap\left(G^C,\ \neg B\right)$
$=(K,\ \neg A\cap\neg B)$, (say), where

$$K(\neg\boldsymbol{\alpha})=F^C(\neg\boldsymbol{\alpha})\cap G^C(\neg\boldsymbol{\alpha}),\forall\neg\boldsymbol{\alpha}\in\neg A\cap\neg B.$$

On the other hand,
$((F,\ A)\Cap(G,\ B))^C=(M,\ A\cap B)^C$, (say)

$$=(M^C,\neg(A\cap B)).$$

Now for $\neg\boldsymbol{\alpha} \in \neg(A \cap B)$,

$$M^C(\neg\boldsymbol{\alpha}) = (M(\boldsymbol{\alpha}))^C,$$
$$= (F(\boldsymbol{\alpha}) \cap G(\boldsymbol{\alpha}))^C,$$
$$= F^C(\neg\boldsymbol{\alpha}) \cup G^C(\neg\boldsymbol{\alpha}).$$

Clearly, $K(\neg\boldsymbol{\alpha}) = F^C(\neg\boldsymbol{\alpha}) \cap G^C(\neg\boldsymbol{\alpha}) \subseteq F^C(\neg\boldsymbol{\alpha}) \cup G^C(\neg\boldsymbol{\alpha}) = M^C(\neg\boldsymbol{\alpha})$.

Theorem 4.2: If (F, A) *and* (G, B) are two fuzzy soft multisets over U. Then the following De Morgan's inclusions hold.

(1) $(F, A)^C \Cap (G, B)^C \widetilde{\subseteq} \left((F, A) \widetilde{\cup} (G, B)\right)^C$.

(2) $((F, A) \Cap (G, B))^C \widetilde{\subseteq} (F, A)^C \widetilde{\cup} (G, B)^C$,

Proof (1) Consider $(F, A)^C \Cap (G, B)^C = \left(F^C, \neg A\right) \Cap \left(G^C, \neg B\right)$
$= (H^C, \neg A \cap \neg B)$, (say), where

$$H^C(\neg\boldsymbol{\alpha}) = F^C(\neg\boldsymbol{\alpha}) \cap G^C(\neg\boldsymbol{\alpha}), \forall \neg\boldsymbol{\alpha} \in \neg A \cap \neg B.$$

Again, let $(F, A) \widetilde{\cup} (G, B) = (V, A \cup B)$
$\left((F, A) \widetilde{\cup} (G, B)\right)^C = (V, A \cup B)^C$, (say)

$$= (V^C, \neg(A \cup B)).$$

For $\neg\boldsymbol{\alpha} \in \neg(A \cup B)$, we have

$$(V^C(\neg\boldsymbol{\alpha})) = (V(\alpha))^C = \begin{cases} (F(\alpha))^C, & if\ \neg\boldsymbol{\alpha} \in \neg A / \neg B \\ (G(\alpha))^C, & if\ \neg\boldsymbol{\alpha} \in \neg B / \neg A \\ (F(\alpha) \cup G(\alpha))^C, & if \neg\boldsymbol{\alpha} \in \neg A \cap \neg B \end{cases},$$

$$= \begin{cases} F^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A / \neg B \\ G^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg B / \neg A \\ F^C(\neg\boldsymbol{\alpha}) \cap G^C(\neg\boldsymbol{\alpha}), & if \neg\boldsymbol{\alpha} \in \neg A \cap \neg B \end{cases}.$$

Obviously, $H^C(\neg\boldsymbol{\alpha}) \subseteq (V^C(\neg\boldsymbol{\alpha})$.
(2) Suppose that $(F, A) \Cap (G, B) = (D, A \cap B)$, where
$D(\boldsymbol{\alpha}) = F(\alpha) \cap G(\alpha)$, for all $\alpha \in A \cap B$.
Therefore, $((F, A) \Cap (G, B))^C = (D, A \cap B)^C$

$$= \left(D^C, \neg(A \cap B)\right).$$

Let us take $\neg\boldsymbol{\alpha} \in \neg(A \cap B)$, then

$$D^C(\neg\boldsymbol{\alpha}) = (D(\boldsymbol{\alpha}))^C = (F(\alpha) \cap G(\alpha))^C$$
$$= (F(\boldsymbol{\alpha}))^C \cup (G(\boldsymbol{\alpha}))^C$$
$$D^C(\neg\boldsymbol{\alpha}) = F^C(\neg\boldsymbol{\alpha}) \cup G^C(\neg\boldsymbol{\alpha}).$$

Now consider, $(F, A)^C \widetilde{\cup} (G, B)^C = (F^C, \neg A) \widetilde{\cup} (G^C, \neg B)$
$= (T, \neg A \cup \neg B)$, (say)

For $\neg\boldsymbol{\alpha} \in \neg A \cup \neg B$, we have

$$T(\neg\boldsymbol{\alpha}) = \begin{cases} F^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ G^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ F^C(\neg\boldsymbol{\alpha}) \cup\ G^C(\neg\boldsymbol{\alpha}), & if \neg\boldsymbol{\alpha} \in \neg A \cap \neg B \end{cases}.$$

Clearly, $D^C(\neg\boldsymbol{\alpha}) \subset T(\neg\boldsymbol{\alpha})$.

Theorem 4.3: Let $(F,\ A)$ *and* $(G,\ B)$ be two fuzzy soft multisets over U. Then the following
De Morgan's law holds.

(1) $\left((F,\ A)\,\widetilde{\cup}_R\,(G,\ B)\right)^C = (F,\ A)^C \Cap (G,\ B)^C$.

(2) $((F,\ A) \Cap (G,\ B))^C = (F,\ A)^C \widetilde{\cup}_R (G,\ B)^C$.

Proof:

(1) Let $(F,\ A)\,\widetilde{\cup}_R\,(G,\ B) = (H,\ C)$, where $C = A \cap B \neq \emptyset$. For all $\boldsymbol{\alpha} \in (A \cap B)$, we have

$$H(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha}) \cup G(\boldsymbol{\alpha}) .$$

Now, $\left((F,\ A)\,\widetilde{\cup}_R\,(G,\ B)\right)^C = (H,\ (A \cap B))^C = (H^C, \neg(A \cap B))$.
For all $\neg\boldsymbol{\alpha} \in \neg(A \cap B)$, we have

$$H^C(\neg\boldsymbol{\alpha}) = (H(\boldsymbol{\alpha}))^C = (F(\boldsymbol{\alpha}) \cup G(\boldsymbol{\alpha}))^C$$

$$= F^C(\neg\boldsymbol{\alpha}) \cap G^C(\neg\boldsymbol{\alpha})$$

Also, $(F,\ A)^C \Cap (G,\ B)^C = \left(F^C, \neg A\right) \Cap \left(G^C, \neg B\right) = (\mathrm{K}, \neg(A \cap B)\ )$.
For all $\neg\boldsymbol{\alpha} \in \neg(A \cap B)$, we obtain

$$\mathrm{K}(\neg\boldsymbol{\alpha}) = F^C(\neg\boldsymbol{\alpha}) \cap G^C(\neg\boldsymbol{\alpha}).$$

Since, $H^C(\neg\boldsymbol{\alpha}) = \mathrm{K}(\neg\boldsymbol{\alpha})$, therefore the result has been established.

(2) $(F,\ A) \Cap (G,\ B) = (H,\ A \cap B)$, where $A \cap B \neq \emptyset$. For all $\boldsymbol{\alpha} \in (A \cap B)$, we have $H(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha}) \cap G(\boldsymbol{\alpha})$.
Now, $((F,\ A) \Cap (G,\ B))^C = (H,\ (A \cap B))^C = (H^C, \neg(A \cap B))$.
For all $\neg\boldsymbol{\alpha} \in \neg(A \cap B)$, we have

$$H^C(\neg\boldsymbol{\alpha}) = (H(\boldsymbol{\alpha}))^C = (F(\boldsymbol{\alpha}) \cap G(\boldsymbol{\alpha}))^C,$$

$$= F^C(\neg\boldsymbol{\alpha}) \cup G^C(\neg\boldsymbol{\alpha}).$$

Also, $(F,\ A)^C \widetilde{\cup}_R\ (G,\ B)^C = \left(F^C, \neg A\right) \widetilde{\cup}_R \left(G^C, \neg B\right) = (\mathrm{J}, \neg A \cap \neg B\ )$.
For all $\neg\boldsymbol{\alpha} \in \neg A \cap \neg B$, we obtain

$$\mathrm{J}(\neg\boldsymbol{\alpha}) = F^C(\neg\boldsymbol{\alpha}) \cup G^C(\neg\boldsymbol{\alpha}).$$

Since, $H^C(\neg\boldsymbol{\alpha}) = \mathrm{J}(\neg\boldsymbol{\alpha})$, hence, (2) has been established.

Definition 4.1: The **AND-product** on two fuzzy soft multisets $(F,\ A)$ and $(G,\ B)$ which is denoted by $(F,\ A)\,\widetilde{\bigwedge}\,(G,\ B)$ is defined as $(F,\ A)\,\widetilde{\bigwedge}\,(G,\ B) = (M,\ \ A \times B)$, where $M(a,b) = F(a) \cap G(b)$, for all $(a,\ b) \in A \times B$.

Example 4.1: Consider Example 3.9. Let

$$A=\{a_1=(e_{U_1},1,e_{U_2},1,e_{U_3},1)\,,a_2=(e_{U_1},2,e_{U_2},3,e_{U_3},1)\}\,,$$

$$B=\ \{b_1=(e_{U_1},1,e_{U_2},1,e_{U_3},1)\,,b_2=(e_{U_1},1,e_{U_2},2,e_{U_3},1)\,,b_3=(e_{U_1},2,e_{U_2},3,e_{U_3},1)\}\,.$$

Suppose $(F,\ A)$ and $(G,\ B)$ are two fuzzy soft multisets over U, such that

$$(F,\ A)=\left\{\begin{array}{c}\left(a_1,\left(\left\{\frac{h_1}{0.2},\ \frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7},\frac{v_3}{0.7}\right\}\right)\right),\\ \left(a_2,\left(\left\{\frac{h_1}{0.7},\ \frac{h_2}{0.7},\frac{h_3}{0.1},\frac{h_4}{0.8},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.4},\frac{v_3}{0.2}\right\}\right)\right)\end{array}\right\},$$

$$(G,\ B)=\left\{\begin{array}{c}\left(b_1,\left(\left\{\frac{h_1}{0.3},\ \frac{h_2}{0.4},\frac{h_3}{0.9},\frac{h_4}{0.7},\frac{h_5}{0.2}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.6},\frac{c_4}{0.7}\right\},\left\{\frac{v_1}{0.9},\ \frac{v_2}{0.7},\frac{v_3}{0.9}\right\}\right)\right),\\ \left(b_2,\left(\left\{\frac{h_1}{0.4},\ \frac{h_2}{0.6},\frac{h_3}{0.7},\frac{h_4}{0.7},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{1},\ \frac{c_2}{0.9},\frac{c_3}{0.9},\frac{c_4}{0.8}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.5},\frac{v_3}{0.4}\right\}\right)\right),\\ \left(b_3,\left(\left\{\frac{h_1}{0.9},\ \frac{h_2}{0.9},\frac{h_3}{1},\frac{h_4}{0.8},\frac{h_5}{0.6}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.8},\frac{c_3}{0.5},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.7},\ \frac{v_2}{0.6},\frac{v_3}{0.5}\right\}\right)\right)\end{array}\right\}.$$

$$A\times B=\ \{(a_1,\ b_1)\,,(a_1,\ b_2)\,,(a_1,\ b_3)\,,(a_2,\ b_1)\,,(a_2,\ b_2)\,,(a_2,\ b_3)\}\,.$$

$$M\,(a_1,\ b_1)=F\,(a_1)\cap G\,(\ b_1)\,,$$

$$=\left(\left\{\frac{h_1}{0.2},\ \frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7},\frac{v_3}{0.7}\right\}\right).$$

$$M\,(a_1,\ b_2)=F\,(a_1)\cap G\,(\ b_2)\,,$$

$$=\left(\left\{\frac{h_1}{0.2},\ \frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.5},\frac{v_3}{0.4}\right\}\right).$$

$$M\,(a_1,\ b_3)=F\,(a_1)\cap G\,(\ b_3)\,,$$

$$=\left(\left\{\frac{h_1}{0.2},\ \frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.7},\ \frac{v_2}{0.6},\frac{v_3}{0.5}\right\}\right).$$

$$M\,(a_2,\ b_1)=F\,(a_2)\cap G\,(\ b_2)\,,$$

$$=\left(\left\{\frac{h_1}{0.3},\ \frac{h_2}{0.4},\frac{h_3}{0.9},\frac{h_4}{0.7},\frac{h_5}{0.2}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.4},\frac{v_3}{0.2}\right\}\right).$$

$$M\,(a_2,\ b_2)=F\,(a_2)\cap G\,(\ b_2)\,,$$

$$= \left(\left\{\frac{h_1}{0.4},\ \frac{h_2}{0.6},\frac{h_3}{0.8},\frac{h_4}{0.7},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.4},\frac{v_3}{0.2}\right\}\right).$$

$$M\left(a_2,\ b_3\right) = F\left(a_2\right) \cap G\left(\ b_3\right),$$

$$= \left(\left\{\frac{h_1}{0.7},\ \frac{h_2}{0.7},\frac{h_3}{1},\frac{h_4}{0.8},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.4},\frac{v_3}{0.2}\right\}\right).$$

Therefore

$$(M,\ A\times B) = \left\{\begin{array}{l} \left(\left(a_1,\ b_1\right),\left(\left\{\frac{h_1}{0.2},\ \frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7},\frac{v_3}{0.7}\right\}\right)\right),\\ \left(\left(a_1,\ b_2\right),\left(\left\{\frac{h_1}{0.2},\ \frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.5},\frac{v_3}{0.4}\right\}\right)\right),\\ \left(\left(a_1,\ b_3\right),\left(\left\{\frac{h_1}{0.2},\ \frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.7},\ \frac{v_2}{0.6},\frac{v_3}{0.5}\right\}\right)\right),\\ \left(\left(a_2,\ b_1\right),\left(\left\{\frac{h_1}{0.3},\ \frac{h_2}{0.4},\frac{h_3}{0.9},\frac{h_4}{0.7},\frac{h_5}{0.2}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.4},\frac{v_3}{0.2}\right\}\right)\right),\\ \left(\left(a_2,\ b_2\right),\left(\left\{\frac{h_1}{0.4},\ \frac{h_2}{0.6},\frac{h_3}{0.8},\frac{h_4}{0.7},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.4},\frac{v_3}{0.2}\right\}\right)\right),\\ \left(\left(a_2,\ b_3\right),\left(\left\{\frac{h_1}{0.7},\ \frac{h_2}{0.7},\frac{h_3}{1},\frac{h_4}{0.8},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\ \frac{v_2}{0.4},\frac{v_3}{0.2}\right\}\right)\right) \end{array}\right\}.$$

Definition 4.2: The **OR-product** on two fuzzy soft multisets $(F,\ A)$ and $(G,\ B)$ which is denoted by $(F,\ A)\widetilde{\bigvee}(G,\ B)$ is defined as $(F,\ A)\widetilde{\bigvee}(G,\ B) = (N,\ \ A\times B)$, where
$N\left(a,b\right) = F(a) \cup G(b)$, for all $(a,\ b) \in A\times B$.

Example 4.2: Consider example 3.9. Let

$$A = \left\{a_1 = \left(e_{U_1},1,e_{U_2},1,e_{U_3},1\right),a_2 = \left(e_{U_1},2,e_{U_2},3,e_{U_3},1\right)\right\},$$

$$B = \ \left\{b_1 = \left(e_{U_1},1,e_{U_2},1,e_{U_3},1\right),b_2 = \left(e_{U_1},1,e_{U_2},2,e_{U_3},1\right),b_3 = \left(e_{U_1},2,e_{U_2},3,e_{U_3},1\right)\right\}.$$

$$A\times B = \ \left\{\left(a_1,\ b_1\right),\left(a_1,\ b_2\right),\left(a_1,\ b_3\right),\left(a_2,\ b_1\right),\left(a_2,\ b_2\right),\left(a_2,\ b_3\right)\right\}.$$

$$N\left(a_1,\ b_1\right) = F\left(a_1\right) \cup G\left(\ b_1\right),$$

$$= \left(\left\{\frac{h_1}{0.3},\ \frac{h_2}{0.4},\frac{h_3}{0.9},\frac{h_4}{0.7},\frac{h_5}{0.2}\right\},\left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6},\frac{c_3}{0.6},\frac{c_4}{0.7}\right\},\left\{\frac{v_1}{0.9},\ \frac{v_2}{0.7},\frac{v_3}{0.9}\right\}\right).$$

$$N\left(a_1,\ b_2\right) = F\left(a_1\right) \cup G\left(\ b_2\right),$$

$$= \left(\left\{\frac{h_1}{0.4},\ \frac{h_2}{0.6},\frac{h_3}{0.8},\frac{h_4}{0.7},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{1},\ \frac{c_2}{0.9},\frac{c_3}{0.9},\frac{c_4}{0.8}\right\},\left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7},\frac{v_3}{0.7}\right\}\right).$$

$$N(a_1,\ b_3) = F(a_1) \cup G(\ b_3),$$

$$= \left(\left\{\frac{h_1}{0.8},\ \frac{h_2}{0.9}, \frac{h_3}{1}, \frac{h_4}{0.8}, \frac{h_5}{0.6}\right\}, \left\{\frac{c_1}{0.8},\ \frac{c_2}{0.8}, \frac{c_3}{0.5}, \frac{c_4}{0.6}\right\}, \left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7}, \frac{v_3}{0.7}\right\}\right).$$

$$N(a_2,\ b_1) = F(a_2) \cup G(\ b_2),$$

$$= \left(\left\{\frac{h_1}{0.7},\ \frac{h_2}{0.7}, \frac{h_3}{1}, \frac{h_4}{0.8}, \frac{h_5}{0.3}\right\}, \left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6}, \frac{c_3}{0.6}, \frac{c_4}{0.7}\right\}, \left\{\frac{v_1}{0.9},\ \frac{v_2}{0.7}, \frac{v_3}{0.9}\right\}\right).$$

$$N(a_2,\ b_2) = F(a_2) \cup G(\ b_2),$$

$$= \left(\left\{\frac{h_1}{0.7},\ \frac{h_2}{0.7}, \frac{h_3}{1}, \frac{h_4}{0.8}, \frac{h_5}{0.3}\right\}, \left\{\frac{c_1}{1},\ \frac{c_2}{0.9}, \frac{c_3}{0.9}, \frac{c_4}{0.8}\right\}, \left\{\frac{v_1}{0.8},\ \frac{v_2}{0.5}, \frac{v_3}{0.4}\right\}\right).$$

$$N(a_2,\ b_3) = F(a_2) \cup G(\ b_3),$$

$$= \left(\left\{\frac{h_1}{0.8},\ \frac{h_2}{0.9}, \frac{h_3}{1}, \frac{h_4}{0.8}, \frac{h_5}{0.6}\right\}, \left\{\frac{c_1}{0.8},\ \frac{c_2}{0.8}, \frac{c_3}{0.5}, \frac{c_4}{0.8}\right\}, \left\{\frac{v_1}{0.7},\ \frac{v_2}{0.6}, \frac{v_3}{0.5}\right\}\right).$$

Hence

$$(N,\ A \times B) = \left\{\begin{array}{l} \left((a_1,\ b_1), \left(\left\{\frac{h_1}{0.3},\ \frac{h_2}{0.4}, \frac{h_3}{0.9}, \frac{h_4}{0.7}, \frac{h_5}{0.2}\right\}, \left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6}, \frac{c_3}{0.6}, \frac{c_4}{0.7}\right\}, \left\{\frac{v_1}{0.9},\ \frac{v_2}{0.7}, \frac{v_3}{0.9}\right\}\right)\right), \\ \left((a_1,\ b_2), \left(\left\{\frac{h_1}{0.4},\ \frac{h_2}{0.6}, \frac{h_3}{0.8}, \frac{h_4}{0.7}, \frac{h_5}{0.3}\right\}, \left\{\frac{c_1}{1},\ \frac{c_2}{0.9}, \frac{c_3}{0.9}, \frac{c_4}{0.8}\right\}, \left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7}, \frac{v_3}{0.7}\right\}\right)\right), \\ \left((a_1,\ b_3), \left(\left\{\frac{h_1}{0.8},\ \frac{h_2}{0.9}, \frac{h_3}{1}, \frac{h_4}{0.8}, \frac{h_5}{0.6}\right\}, \left\{\frac{c_1}{0.8},\ \frac{c_2}{0.8}, \frac{c_3}{0.5}, \frac{c_4}{0.6}\right\}, \left\{\frac{v_1}{0.8},\ \frac{v_2}{0.7}, \frac{v_3}{0.7}\right\}\right)\right), \\ \left((a_2,\ b_1), \left(\left\{\frac{h_1}{0.7},\ \frac{h_2}{0.7}, \frac{h_3}{1}, \frac{h_4}{0.8}, \frac{h_5}{0.3}\right\}, \left\{\frac{c_1}{0.8},\ \frac{c_2}{0.6}, \frac{c_3}{0.6}, \frac{c_4}{0.7}\right\}, \left\{\frac{v_1}{0.9},\ \frac{v_2}{0.7}, \frac{v_3}{0.9}\right\}\right)\right), \\ \left((a_2,\ b_2), \left(\left\{\frac{h_1}{0.7},\ \frac{h_2}{0.7}, \frac{h_3}{1}, \frac{h_4}{0.8}, \frac{h_5}{0.3}\right\}, \left\{\frac{c_1}{1},\ \frac{c_2}{0.9}, \frac{c_3}{0.9}, \frac{c_4}{0.8}\right\}, \left\{\frac{v_1}{0.8},\ \frac{v_2}{0.5}, \frac{v_3}{0.4}\right\}\right)\right), \\ \left((a_2,\ b_3), \left(\left\{\frac{h_1}{0.8},\ \frac{h_2}{0.9}, \frac{h_3}{1}, \frac{h_4}{0.8}, \frac{h_5}{0.6}\right\}, \left\{\frac{c_1}{0.8},\ \frac{c_2}{0.8}, \frac{c_3}{0.5}, \frac{c_4}{0.8}\right\}, \left\{\frac{v_1}{0.7},\ \frac{v_2}{0.6}, \frac{v_3}{0.5}\right\}\right)\right) \end{array}\right\}.$$

Proposition 4.1: Let $(F,\ A), (G,\ B)$ *and* $(H,\ C)$ be three fuzzy soft multiset over U. Then

(1) $(F,\ A)\widetilde{\bigwedge}\left((G,\ B)\widetilde{\bigwedge}(H,\ C)\right) = \left((F,\ A)\widetilde{\bigwedge}(G,\ B)\right)\widetilde{\bigwedge}(H,\ C)$.

(2) $(F,\ A)\widetilde{\bigvee}\left((G,\ B)\widetilde{\bigvee}(H,\ C)\right) = \left((F,\ A)\widetilde{\bigvee}(G,\ B)\right)\widetilde{\bigvee}(H,\ C)$.

(3) $(F,\ A)\widetilde{\bigwedge}(F,\ A) = (F,\ A)$.

Proof (1) By using definition 4.1.
$(F,\ A)\widetilde{\bigwedge}\left((G,\ B)\widetilde{\bigwedge}(H,\ C)\right) = (F,\ A)\widetilde{\bigwedge}(G,\ B\times C) = (N, A\times B\times C)$, where for all $(b,\ c)\in B\times C$, $M(b,c) = G(b)\cap H(c)$ and for all $(a,\ b,\ c)\in A\times B\times C$,
$N(a,\ b,\ c) = F(a)\cap M(b,c) = F(a)\cap(G(b)\cap H(c)) = (F(a)\cap G(b))\cap H(c) = Q(a,b)\cap H(c)$ with $Q(a,b) = F(a)\cap G(b)$.
$(Q,\ A\times B)\widetilde{\bigwedge}(H,\ C) = \left((G,\ B)\widetilde{\bigwedge}(H,\ C)\right)\widetilde{\bigwedge}(H,\ C)$. Hence (1) has been proved.
(2) Similar to proof of (1), (2) can be proved.
The proof of (3) is straight forward, hence omitted.

Definition 4.3: The **restricted difference** of two fuzzy soft multisets $(F,\ A)$ and $(G,\ B)$ over U denoted by $(F,\ A)\sim_R(G,\ B)$ and is defined as $(F,\ A)\sim_R(G,\ B) = (H,\ C)$, where $C = A\cap B \neq \emptyset$ and for all $\boldsymbol{\alpha}\in C$, $\mathrm{H}(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha}) - G(\boldsymbol{\alpha})$. The difference of sets $F(\boldsymbol{\alpha})$ and $G(\boldsymbol{\alpha})$ is denoted by $F(\boldsymbol{\alpha}) - G(\boldsymbol{\alpha})$ and is defined as $F(\boldsymbol{\alpha}) - G(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha})\cap G^{\mathrm{C}}(\boldsymbol{\alpha})$.

Example 4.3: Consider example 3.9. Let

$$A = \{a_1 = (e_{U_1},1,e_{U_2},1,e_{U_3},1), a_2 = (e_{U_1},2,e_{U_2},3,e_{U_3},1)\},$$

,

$$B = \{b_1 = (e_{U_1},1,e_{U_2},1,e_{U_3},1), b_2 = (e_{U_1},1,e_{U_2},2,e_{U_3},1), b_3 = (e_{U_1},2,e_{U_2},3,e_{U_3},1)\}.$$

Suppose $(F,\ A)$ and $(G,\ B)$ are two fuzzy soft multisets over U, such that

$$(F,\ A) = \left\{\begin{array}{c} \left(a_1,\left(\left\{\frac{h_1}{0.2},\frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.6}\right\},\left\{\frac{v_1}{0.8},\frac{v_2}{0.7},\frac{v_3}{0.7}\right\}\right)\right), \\ \left(a_2,\left(\left\{\frac{h_1}{0.7},\frac{h_2}{0.7},\frac{h_3}{0.1},\frac{h_4}{0.8},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{0.8},\frac{c_2}{0.6},\frac{c_3}{0.3},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.5},\frac{v_2}{0.4},\frac{v_3}{0.2}\right\}\right)\right) \end{array}\right\},$$

$$(G,\ B) = \left\{\begin{array}{c} \left(b_1,\left(\left\{\frac{h_1}{0.3},\frac{h_2}{0.4},\frac{h_3}{0.9},\frac{h_4}{0.7},\frac{h_5}{0.2}\right\},\left\{\frac{c_1}{0.8},\frac{c_2}{0.6},\frac{c_3}{0.6},\frac{c_4}{0.7}\right\},\left\{\frac{v_1}{0.9},\frac{v_2}{0.7},\frac{v_3}{0.9}\right\}\right)\right), \\ \left(b_2,\left(\left\{\frac{h_1}{0.4},\frac{h_2}{0.6},\frac{h_3}{0.7},\frac{h_4}{0.7},\frac{h_5}{0.3}\right\},\left\{\frac{c_1}{1},\frac{c_2}{0.9},\frac{c_3}{0.9},\frac{c_4}{0.8}\right\},\left\{\frac{v_1}{0.8},\frac{v_2}{0.5},\frac{v_3}{0.4}\right\}\right)\right), \\ \left(b_3,\left(\left\{\frac{h_1}{0.9},\frac{h_2}{0.9},\frac{h_3}{1},\frac{h_4}{0.8},\frac{h_5}{0.6}\right\},\left\{\frac{c_1}{0.8},\frac{c_2}{0.8},\frac{c_3}{0.5},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.7},\frac{v_2}{0.6},\frac{v_3}{0.5}\right\}\right)\right) \end{array}\right\}.$$

where $C = A\cap B = \{a_1 = b_1 = c_1,\ a_2 = b_3 = c_2\}$.

$$F(c_1) = \left\{\left(c_1,\left(\left\{\frac{h_1}{0.2},\frac{h_2}{0.4},\frac{h_3}{0.8},\frac{h_4}{0.5},\frac{h_5}{0}\right\},\left\{\frac{c_1}{0.8},\frac{c_2}{0.5},\frac{c_3}{0.4},\frac{c_4}{0.5}\right\},\left\{\frac{v_1}{0.8},\frac{v_2}{0.7},\frac{v_3}{0.7}\right\}\right)\right)\right\},$$

$$F(c_2) = \left\{ \left(c_2, \left(\left\{ \frac{h_1}{0.7}, \frac{h_2}{0.7}, \frac{h_3}{1}, \frac{h_4}{0.8}, \frac{h_5}{0.3} \right\}, \left\{ \frac{c_1}{0.8}, \frac{c_2}{0.6}, \frac{c_3}{0.3}, \frac{c_4}{0.5} \right\}, \left\{ \frac{v_1}{0.5}, \frac{v_2}{0.4}, \frac{v_3}{0.2} \right\} \right) \right) \right\},$$

$$G^c(c_1) = \left\{ \left(\neg c_1, \left(\left\{ \frac{h_1}{0.8}, \frac{h_2}{0.6}, \frac{h_3}{0.2}, \frac{h_4}{0.5}, \frac{h_5}{1} \right\}, \left\{ \frac{c_1}{0.2}, \frac{c_2}{0.5}, \frac{c_3}{0.6}, \frac{c_4}{0.5} \right\}, \left\{ \frac{v_1}{0.2}, \frac{v_2}{0.3}, \frac{v_3}{0.3} \right\} \right) \right) \right\},$$

$$G^c(c_2) = \left\{ \left(\neg c_2, \left(\left\{ \frac{h_1}{0.3}, \frac{h_2}{0.3}, \frac{h_3}{0}, \frac{h_4}{0.2}, \frac{h_5}{0.7} \right\}, \left\{ \frac{c_1}{0.2}, \frac{c_2}{0.4}, \frac{c_3}{0.7}, \frac{c_4}{0.5} \right\}, \left\{ \frac{v_1}{0.5}, \frac{v_2}{0.6}, \frac{v_3}{0.8} \right\} \right) \right) \right\}.$$

$$H(c_1) = F(c_1) \cap G^c(c_1),$$

$$= \left(\left\{ \frac{h_1}{0.2}, \frac{h_2}{0.4}, \frac{h_3}{0.2}, \frac{h_4}{0.5}, \frac{h_5}{0} \right\}, \left\{ \frac{c_1}{0.2}, \frac{c_2}{0.5}, \frac{c_3}{0.4}, \frac{c_4}{0.5} \right\}, \left\{ \frac{v_1}{0.2}, \frac{v_2}{0.3}, \frac{v_3}{0.3} \right\} \right).$$

$$H(c_2) = F(c_2) \cap G^c(c_2),$$

$$= \left(\left\{ \frac{h_1}{0.3}, \frac{h_2}{0.3}, \frac{h_3}{0}, \frac{h_4}{0.2}, \frac{h_5}{0.3} \right\}, \left\{ \frac{c_1}{0.2}, \frac{c_2}{0.4}, \frac{c_3}{0.3}, \frac{c_4}{0.5} \right\}, \left\{ \frac{v_1}{0.5}, \frac{v_2}{0.4}, \frac{v_3}{0.2} \right\} \right).$$

Therefore

$$(H, C) = \left\{ \begin{array}{c} \left(c_1, \left(\left\{ \frac{h_1}{0.2}, \frac{h_2}{0.4}, \frac{h_3}{0.2}, \frac{h_4}{0.5}, \frac{h_5}{0} \right\}, \left\{ \frac{c_1}{0.2}, \frac{c_2}{0.5}, \frac{c_3}{0.4}, \frac{c_4}{0.5} \right\}, \left\{ \frac{v_1}{0.2}, \frac{v_2}{0.3}, \frac{v_3}{0.3} \right\} \right) \right), \\ \left(c_2, \left(\left\{ \frac{h_1}{0.3}, \frac{h_2}{0.3}, \frac{h_3}{0}, \frac{h_4}{0.2}, \frac{h_5}{0.3} \right\}, \left\{ \frac{c_1}{0.2}, \frac{c_2}{0.4}, \frac{c_3}{0.3}, \frac{c_4}{0.5} \right\}, \left\{ \frac{v_1}{0.5}, \frac{v_2}{0.4}, \frac{v_3}{0.2} \right\} \right) \right) \end{array} \right\}.$$

Definition 4.4: The **restricted symmetric difference** of two soft multisets (F, A) and (G, B) over U denoted by $(F, A)\,\Delta_R(G, B)$, such that $A \cap B \neq \emptyset$ and is defined as

$$(F, A)\,\Delta_R\,(G, B) = ((F, A) \sim_R (G, B)) \cup ((G, B) \sim_R (F, A)),$$

$$= ((F, A) \cap (G, B)^C) \cup ((G, B) \cap (F, A)^C).$$

Conclusion

In this paper, we have defined restricted union, restricted intersection, AND-Product, OR-Product, restricted difference and restricted symmetric difference with relevant and illustrative examples. Basic properties of these operations were investigated. De Morgan's inclusions and laws were stated and proved.

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