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Fixed Point Results on Certain Contractive Conditions in Complete G-Metric Space

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Abstract

In this work, results on the existence and uniqueness of a fixed point in a complete G-metric space are obtained. Ciric, λ-generalized and Geraghty contractions formulated in G-metric settings are involved by applying Picard iterative procedure. Our results extends and also improve some results in the literature.

1. Introduction

Fixed point theory has been a major aspect of functional analysis and it is used in profounding solution to linear or nonlinear differential and integral equations that based its solutions on initial guess. After the publication of Banach contraction condition in [1], fixed point theory has became the major focus of many researchers, in proving the existence and uniqueness of solution of the fixed point. Also, there have been many works on the generalization of Banach contraction by weakening the contraction.

In 1962, Rakotch established a new contraction condition by considering the contractive constant as a monotone decreasing function. Moreover, existence and

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uniqueness of solution of fixed point has also been to the direction of ascertaining the convergence and stability of the fixed point which is made possible by the iterative procedure introduced by Picard. After this iterative procedure, there have been some generalizations of the scheme by Mann [8], Ishikawa [6], Kranolseskij [7] and Noor [11] to mention but a few.
In 1906, Frechet [3] gave the famous definition of distance by introducing a function d that assigns a nonnegative real number $d(x, y)$ (the distance between x and y) to every pair of (x, y) of elements or points of a nonempty set X satisfying certain axioms and (X, d) is called a metric space.
In the early sixties, the notion of D-metric space was introduced by Gahler [4] which he claimed to be a generalization of ordinary metric space.
Furthermore, Mustafa and Sims [10] proved a new generalization of metric space, which they called G-metric space after proving that most of the properties of D-metric space were incorrect. For further work, see [[9],[14],[15] and [16]].
In this article, some contractive conditions coupled with the generalized metric space (G-metric) were fused to show the existence and uniqueness of solution of the fixed point in G-metric space.

2. Preliminary

This section is concerned with some useful definitions, Lemmas and propositions.
Definition 2.1[10]: Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$, be a function satisfying:
(G1): $G(x, y, z) = 0$ if $x = y = z$
(G2): $0 < G(x, x, y)$; $\forall\ x, y \in X$, with $x \neq y$,
(G3): $G(x, x, y) \leq G(x, y, z)$, $\forall\ x, y, z \in X$ with $z \neq y$,
(G4): $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$
(G5): $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, $\forall\ x, y, z, a \in X$,
then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is a G-metric space.
Clearly these properties are satisfied when $G(x, y, z)$ is the perimeter of the triangle with vertices at x, y and z in $\mathbb{R}^2$, taking a in the interior of the triangle shows that $(G5)$ is the best possible.
Remark: If (X, d) is a usual metric space, then E_s and E_m define the relationship between (X, d) and G-metrics on X, however, for this to be true, it is necessary that d satisfy the triangle inequality.
Proposition 2.1 [10]: Let (X, G) be a G-metric space, then for any x, y, z and $a \in X$ it follows that:

(1) if $G(x, y, z) = 0$ then $x = y = z$,
(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(3) $G(x, y, z) \leq 2G(y, x, x)$,

(4) $G(x,y,z) \leq G(x,a,z) + G(a,y,z)$,
(5) $G(x,y,z) \leq \frac{2}{3}\left(G(x,y,a) + G(x,a,z) + G(a,y,z)\right)$,
(6) $G(x,y,z) \leq (G(x,a,a) + G(y,a,a) + G(z,a,a))$,
(7) $|G(x,y,z) - G(x,y,a)| \leq \max\{G(a,z,z), G(z,a,a)\}$,
(8) $|G(x,y,z) - G(x,y,a)| \leq G(x,a,z)$,
(9) $|G(x,y,z) - G(y,z,z)| \leq \max\{G(x,z,z), G(z,x,x)\}$,
(10) $|G(x,y,z) - G(y,x,x)| \leq \max\{G(y,x,x), G(x,y,y)\}$

Definition 2.2[10]: Let (X,G) be a G-metric space. The sequence $\{x_n\} \subseteq X$ is $G-convergent$ to x if it converges to x in G-metric topology, $\tau(G)$.

Proposition 2.2[10]: Let (X,G) be G-metric space, then for a sequence $\{x_n\} \subseteq X$ and point $x \in X$, the following are equivalent.

(1) (x_n) is G-convergent to x.
(2) $d_G(x_n,x) \to 0$, as $n \to \infty$ (that is, (x_n) converges to x relative to the metric d_G).
(3) $G(x_n,x_n,x) \to 0$, as $n \to \infty$.
(4) $G(x_n,x,x) \to 0$, as $n \to \infty$.
(5) $G(x_m,x_n,x) \to 0$, as $m,n \to \infty$.

Definition 2.3: Let (X,G) be a G-metric space, then a sequence $(x_n) \subseteq X$ is said to be $G-Cauchy$ if for every $\epsilon > 0$, there exists $N \in$ such that $G(x_n,x_m,x_l) < \epsilon$ for all $n,m,l \geq N$.

The next proposition follows directly from the above definitions.

Corollary 2.1: Every G-convergent sequence in a G-metric space is G-Cauchy.

Corollary 2.2: If a G-Cauchy sequence in a G-metric space (X,G) contains a G-convergent subsequence, then the sequence itself is G-convergent.

Definition 2.4: A G-metric space (X,G) is said to be G-complete if every G-Cauchy sequence in (X,G) is G-convergent in (X,G).

Definition 2.5[10]: Let (X,G) be a G-metric space. A mapping $T : X \to X$ is said to be a Geraghty contraction, if there exists $\beta \in \varsigma$ such that for any $x,y,z \in X$,

$$G(Tx,Ty,Tz) \leq \beta\left[G(x,y,z)\right]G(x,y,z)$$

where the class ς denotes those function $\beta : [0,\infty) \to [0,\infty)$ satisfying the following condition $\beta(t_n) \to 1 \Rightarrow t_n \to 0$.

Definition 2.6[10]: A mapping $T : X \to X$ is said to be a λ-generalized contraction if and only if for every $x,y,z \in X$ there exists non-negative numbers $q(x,y,z)$, $r(x,y,z)$, $s(x,y,z)$ and $t(x,y)$ such that

$$\sup_{(x,y,z)\in X}\left[q(x,y,z) + r(x,y,z) + s(x,y,z) + 2t(x,y,z)\right] = \lambda < 1.$$

and

$$G(Tx,Ty,Tz) \leq q(x,y,z)G(x,y,z) + r(x,y,z)G(x,Tx,Tx) + s(x,y,z)G(y,Ty,Ty)$$
$$+ t(x,y,z)[G(x,Ty,Ty) + G(y,Tx,Tx)]$$

Definition 2.7[10]: The mapping $T : X \to X$ is called a Ciric contractive condition if there exists a constant h, where $0 \le h < 1$, such that for each $x, y \in X$

$$G(Tx, Ty, Ty) \le h \max \Big\{ G(x, y, y), \frac{1}{2}[G(x, Tx, Tx) + G(y, Ty, Ty)],$$

$$\frac{1}{2}[G(x, Tx, Tx) + G(y, Tx, Tx)] \Big\} \tag{1}$$

Examples of G-metric space

Example 1: If X is a non-empty subset of $\mathbb{R}$, then the function $G : X^3 \to [0, \infty)$, given by $G(x, y, z) = |x - y| + |x - z| + |y - z|$ for all $x, y, z \in X$ is a G-metric on X.

Example 2: Every non-empty set X can be provided with the discreet G-metric, which is defined, for all $x, y, z \in X$ by

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z \\ 1, & \text{otherwise} \end{cases}$$

Example 3: Let $X = [0, \infty)$ be the interval of non-negative real number and let G be defined by

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z \\ \max[x, y, z], & \text{otherwise} \end{cases}$$

Then G is a complete G-metric space.

Some relationship between metric space and G-metric space. Every metric on X induces G-metric on X in different ways

- If (X, d) is a metric space, then the function $G_m^d, G_s^d : X^3 \to_+$ defined by

$$G_m^d(x, y, z) = \max \{d(x, y), d(y, z), d(z, x)\}$$

  and

$$G_s^d(x, y, z) = d(x, y), d(y, z), d(z, x)$$

  for $x, y, z \in X$ are G-metric on X.
- If (X, G) is a G-metric space, then the functions $d_m^G, d_s^G : X^2 \to_+$ defined by

$$d_m^G(x, y) = \max \{G(x, y, y), G(y, x, x)\}$$

  and

$$d_s^G(x, y) = G(x, y, y), G(y, x, x)$$

  for all $x, y \in X$ are metric space.

Example of a Geraghty contractive condition
Let f_{Ger} be the family of all Geraghty functions, that is, function $\beta : [0, \infty) \to [0, 1)$ satisfying the condition

$$\{\beta(t_n)\} \to 1 \ \text{ implies } \ \{t_n\} \to 0$$

and

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y), \ \forall \ x, y \in X$$

if and only if, there exists $\phi \in f_{Ger}$ such that

$$d(Tx, Ty) \leq \phi(d(x, y)) \ \forall \ x, y \in X.$$

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$$\beta_\phi(t) \ = \begin{cases} 0, & \text{if } t = 0 \\ \frac{\phi(t)}{t}, & \text{if } t > 0 \end{cases}$$

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$$\beta(t) \ = \begin{cases} 0, & \text{if } t = 0 \\ \frac{1}{t+1}, & \text{if } 0 < t \leq 1 \\ \frac{1}{2} + \frac{1}{4}\sin\left(\frac{1}{t-1}\right), & \text{if } t > 1. \end{cases}$$

Since $\frac{1}{4} \leq \beta(t) \leq \frac{3}{4}$ for all $t > 1$, it is clear that β is a Geraghty function.

3. Main Result

In this section, the existence and uniqueness of fixed point in G-metric space are proved.

Theorem 3.1 Let (X, G) be a complete G-metric space and $T : X \to X$ be a continuous mapping satisfying λ-generalized contraction of the form:

$$G(Tx, Ty, Ty) \leq q(x, y, y)G(x, y, y) + r(x, y, y)G(x, Tx, Tx) + s(x, y, y)G(y, Ty, Ty)$$

$$+t(x, y, y)\left[G(x, Ty, Ty) + G(y, Tx, Tx)\right] \tag{1}$$

for every $x, y, z \in X$ there exists non-negative number $q(x, y, y), r(x, y, y), s(x, y, y)$ and $t(x, y, y)$ such that

$$\sup_{x,y \in X} \{q(x, y, y) + r(x, y, y) + s(x, y, y) + t(x, y, y)\} = \lambda < 1$$

Then, there exists a unique fixed point p in T.

Proof: Let $q(x, y, y)$, $r(x, y, y)$, $s(x, y, y)$ and $t(x, y, y)$ be a contraction constant of the mapping T, let x_0 be arbitrary but fixed element in X. If the sequence of iteration is define as:

$$x_n = T^n x_0 \ \text{ for all } \ n \geq 1 \tag{2}$$

Considering two points to be equal i.e.
Let $x_{n+1} = y_n = z_n$
Since T is a contraction satisfying equation (1), then we have

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$$
$$\leq q(x,y,y)G(x_{n-1}, x_n, x_n) + r(x,y,y)G(x_{n-1}x_n, x_n) + s(x,y,y)G(x_n, x_{n+1}, x_{n+1})$$
$$+t(x,y,y)\left[G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n)\right]$$
$$\leq q(x,y,y)G(Tx_{n-2}, Tx_{n-1}, Tx_{n-1}) + r(x,y,y)G(Tx_{n-2}, Tx_{n-1}, Tx_{n-1})$$
$$+s(x,y,y)G(Tx_{n-1}, Tx_n, Tx_n) + t(x,y,y)\left[G(Tx_{n-2}, Tx_{n-1}, Tx_{n-1}) + G(Tx_{n-1}, Tx_n . Tx_n)\right]$$
$$\leq q^2(x,y,y)G(x_{n-2}, x_{n-1}, x_{n-1}) + r^2(x,y,y)G(x_{n-2}, x_{n-1}, x_{n-1}) + s^2(x,y,y)G(x_{n-1}, x_n, x_n)$$
$$+t^2(x,y,y)\left[G(x_{n-2}, x_{n-1}, x_{n-1}) + G(x_{n-1}, x_n, x_n)\right]$$
$$= q^2(x,y,y)G(Tx_{n-3}, Tx_{n-2}, Tx_{n-2}) + r^2(x,y,y)G(Tx_{n-3}, Tx_{n-2}, Tx_{n-2})$$
$$+s^2(x,y,y)G(Tx_{n-2}, Tx_{n-1}, Tx_{n-1}) + t^2(x,y,y)\left[G(Tx_{n-3}, Tx_{n-2}, Tx_{n-2})\right.$$
$$\left.+G(Tx_{n-2}, Tx_{n-1}, Tx_{n-1})\right]$$
$$\leq q^3(x,y,y)G(x_{n-3}, x_{n-2}, x_{n-2}) + r^3(x,y,y)G(x_{n-3}, x_{n-2}, x_{n-2})$$
$$+s^3(x,y,y)G(x_{n-2}, x_{n-1}, x_{n-1}) + t^3(x,y,y)\left[G(x_{n-3}, x_{n-2}, x_{n-2}) + G(x_{n-2}, x_{n-1}, x_{n-1})\right]$$

continue iteratively at n we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq q^n(x,y,y)G(x_0, x_1, x_1) + r^n(x,y,y)G(x_0, x_1, x_1)$$
$$+s^n(x,y,y)G(x_1, x_2, x_2) + t^n(x,y,y)\left[G(x_0, x_1, x_1 + G(x_1, x_2, x_2)\right]$$
$$\leq \left[q^n(x,y,y) + r^n(x,y,y) + t^n(x,y,y)\right]G(x_0, x_1, x_1) + \left[s^n(x,y,y) + t^n(x,y,y)\right]G(x_1, x_2, x_2)$$
$$G(x_n, x_{n+1}, x_{n+1}) - \left[s^n(x,y,y) + t^n(x,y,y)\right]G(x_n, x_{n+1}, x_{n+1})$$
$$\leq \left[q^n(x,y,y) + r^n(x,y,y) + t^n(x,y,y)\right]G(x_{n-1}, x_n, x_n)$$
$$\left[1 - (s^n(x,y,y) + t^n(x,y,y))\right]G(x_n, x_{n+1}, x_{n+1})$$
$$\leq \left[q^n(x,y,y) + r^n(x,y,y) + t^n(x,y,y)\right]G(x_{n-1}, x_n, x_n)$$
$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{\left[q^n(x,y,y) + r^n(x,y,y) + t^n(x,y,y)\right]}{\left[1 - \left(s^(x,y,y) + t^n(x,y,y)\right)\right]}G(x_{n-1}, x_n, x_n)$$

Let

$$\frac{\left[q^n(x,y,y) + r^n(x,y,y) + t^n(x,y,y)\right]}{\left[1 - \left(s^(x,y,y) + t^n(x,y,y)\right)\right]} = \lambda^n$$

Hence,
$G(x_n, x_{n+1}, x_{n+1}) \leq \lambda G(x_0, x_1, x_1)$.
Let $m, n \in$ where $m > n \in \mathbb{N}$

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m)$$
$$\leq \left[\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\right]G(x_0, x_1, x_1)$$
$$\leq \frac{\lambda^n}{1-\lambda}G(x_0, x_1, x_1)$$

The $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$, then the sequence $\{x_n\}$ is said to be G-Cauchy. By completeness of (X, G), there exist $\rho \in X$ such that $\{x_n\}$ is G-convergent to ρ.

Suppose $T\rho \neq \rho$, then

$$G(x_m, T\rho, T\rho) \leq q(x, y, y)G(x_{n-1}, T\rho, T\rho) + r(x, y, y)G(x_{n-1}, T\rho, T\rho)$$
$$+s(x, y, y)G(\rho, T\rho, T\rho) + t(x, y, y)\left[G(x_{n-1}, T\rho, T\rho) + G(\rho, T\rho, T\rho)\right]$$

Taking limit at $n \to \infty$, we have

$$G(\rho, T\rho, T\rho) \leq \left[q(x, y, y) + r(x, y, y) + s(x, y, y) + 2t(x, y, y)\right] G(\rho, T\rho, T\rho)$$

which is a contraction since

$$0 \leq q(x, y, y) + r(x, y, y) + s(x, y, y) + 2t(x, y, y) < \lambda$$

for uniqueness of ρ, suppose that $\rho \neq \mu$
such that $T\mu = \mu$, then our contraction gives

$$G(\rho, \mu, \mu) = G(T\rho, T\mu, T\mu)$$
$$\leq q(x, y, y)G(\rho, \mu, \mu) + r(x, y, y)G(\rho, \rho, \rho) + t(x, y, y)G(\mu, \rho, \rho)$$
$$+s(x, y, y)G(\mu, \mu, \mu) + t(x, y, y)\left[G(\rho, \mu, \mu) + G(\mu, \rho, \rho)\right]$$
$$\leq \left[q(x, y, y) + t(x, y, y)\right] G(\rho, \mu, \mu) + t(x, y, y)G(\mu, \rho, \rho)$$
$$G(\rho, \mu, \mu) - \left[q(x, y, y) + t(x, y, y)\right] G(\rho, \mu, \mu) \leq t(x, y, y)G(\mu, \rho, \rho)$$
$$\Rightarrow \ G(\rho, \mu, \mu) \leq \frac{t(x, y, y)}{1 - [q(x, y, y) + t(x, y, y)]} G(\mu, \rho, \rho)$$

By same argument,

$$G(\mu, \rho, \rho) \leq \frac{t(x, y, y)}{1 - [q(x, y, y) + t(x, y, y)]} G(\rho, \mu, \mu)$$

Thus we have

$$G(\rho, \mu, \mu) \leq \left(\frac{t(x, y, y)}{1 - [q(x, y, y) + t(x, y, y)]}\right)^2 G(\rho, \mu, \mu)$$

which shows that

$$\rho = \mu, \quad \text{since} \ \ 0 \leq \frac{t(x, y, y)}{1 - q(x, y, y) + t(x, y, y)} < \lambda < 1$$

Theorem 3.2

Let (X, G) be a complete G-metric space. $T : X \to X$ be a continuous mapping satisfying a Geraghty contraction formulated in a G-metric settings.

$$G(Tx, Ty, Ty) \leq \beta[G(x, y, y)]G(x, y, y) \tag{3}$$

Then there exists a fixed point ρ in T.

Proof: Let $y_n = z_n = x_{n+1}$ and x_0 be an arbitrary but fixed element in X. Then,

$$x_{n+1} = Tx_n \ \forall \ n \geq 0$$

$$x_n = T^n x_0 \ \forall \ n \geq 1$$

Then

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \beta \left[G(x_{n-1}, x_n, x_n)G(x_{n-1}, x_n, x_n)\right] \\ &= \beta \left[G(Tx_{n-2}, Tx_{n-1}, Tx_{n-1})G(Tx_{n-2}, Tx_{n-1}, Tx_{n-1})\right] \\ &\leq \beta^2 \left[G(x_{n-2}, x_{n-1}, x_{n-1})G(x_{n-2}, x_{n-1}, x_{n-1})\right] \end{aligned}$$

iteratively at n gives

$$G(x_n, x_{n+1}, x_{n+1}) \leq \beta^n \left[G(x_0, x_1, x_1)\right] G(x_0, x_1, x_1)$$

Let $m, n \in$

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &\leq \left[\beta^n + \beta^{n+1} + \cdots + \beta^{m-1}\right] G(x_0, x_1, x_1)G(x_0, x_1, x_1) \\ &\leq \frac{\beta^n}{1-\beta} G(x_0, x_1, x_1) \\ &\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0 \end{aligned}$$

Then the sequence $\{x_n\}$ is a Cauchy sequence which shows the completeness. By the completeness of the pair (X, G), it shows that the existence of $\rho \in X$ such that $\{x_n\}$ is G-convergent to ρ.
Suppose $T\rho \neq \rho$
then

$$G(x_n, T\rho, T\rho) \leq \beta \left[G(x_{n-1}, \rho, \rho)G(x_{n-1}, \rho, \rho)\right]$$

taking limit as $n \to \infty$

$$G(\rho, T\rho, T\rho) \leq \beta \left[G(\rho, \rho, \rho)G(\rho, \rho, \rho)\right]$$

which is a contraction since $\beta(t_n) \to 1 \ \forall \ t_n \to 0$, which can only be possible if $T\rho = \rho$.

Next is to obtain the uniqueness of the fixed point ρ
Suppose $T\rho \neq \mu$ i.e. $T\mu = \mu$
then,

$$G(\rho, \mu, \mu) = G(T\rho, T\mu, T\mu) \leq \beta \left[G(\rho, \mu, \mu)\right] G(\rho, \mu, \mu)$$

by the same argument

$$G(\mu,\rho,\rho) \leq \beta\,[G(\rho,\mu,\mu)]\,G(\rho,\mu,\mu)$$

Therefore

$$G(\rho,\mu,\mu) \leq \beta\,[G(\rho,\mu,\mu)]$$

which implies that $\rho = \mu$. Since $0 \leq \beta \leq 1$
Hence ρ is unique.

Theorem 3.3
Let (X, G) be a complete G-metric space, $T : X \to X$ be a continuous mapping satisfying a Ciric contractive condition (1) formulate in a G-metric settings.

$$G(Tx,Ty,Ty) \leq h \max\left\{G(x,y,y), \frac{1}{2}[G(x,Tx,Tx) + G(y,Ty,Ty)],\right.$$

$$\left.\frac{1}{2}[G(x,Tx,Tx) + G(y,Tx,Tx)]\right\} \tag{1}$$

There exist a unique fixed point u in X.

Proof: Let $x = x_n$, $y = z = x_{n+1}$
Let x_0 be an arbitrary but fixed in X.
If $x_n = x_{n+1}$
Then we define the iterant of the fixed point

$$x_{n+1} = Tx_n \quad \text{for all} \quad n \geq 0 \tag{2}$$

Therefore

$$G(x_n,x_{n+1},x_{n+1}) = G(Tx_{n-1},Tx_n,Tx_n) \leq h \max\{G(x_{n-1},x_n,x_n),\ \frac{1}{2}[G(x_{n-1},x_n,x_n){+}G(x_n,x_{n+1},x_{n+1}$$

$$\left.\frac{1}{2}[G(x_n,x_{n+1},x_{n+1}) + G(x_n,x_n,x_n)]\right\} \tag{3}$$

$$= h \max\left\{G(x_{n+1},x_n,x_n), \frac{1}{2}[G(x_{n-1},x_n,x_n) + G(x_n,x_{n+1},x_{n+1})],\right.$$

$$\left.\frac{1}{2}[G(x_n,x_{n+1},x_{n+1})]\right\}$$

$$\leq h\left[\frac{1}{2}[G(x_{n-1},x_n,x_n) + G(x_n,x_{n+1},x_{n+1})]\right] \tag{4}$$

$$= \frac{h}{2}\,[G(x_{n-1},x_n,x_n) + G(x_n,x_{n+1},x_{n+1})]$$

hence

$$G(x_n,x_{n+1},x_{n+1}) \leq \frac{h}{2-h}G(x_{n-1},x_n,x_n) \tag{5}$$

$$\leq hG(x_{n-1}, x_n, x_n) \tag{6}$$

since $0 \leq h < 1$

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq hG(x_{n-1}, x_n, x_n) \\ &= hG(Tx_{n-2}, Tx_{n-1}, Tx_{n-1}) \\ &\leq h^2 G(x_{n-2}, x_{n-1}, x_{n-1}) \\ &= h^2 G(Tx_{n-3}, Tx_{n-2}, Tx_{n-2}) \\ &\leq h^3 G(x_{n-3}, x_{n-2}, x_{n-2}) \end{aligned}$$

Iteratively at n, we have

$$G(x_n), x_{n+1}, x_{n+1} \leq h^n G(x_0, x_1, x_1) \tag{7}$$

which shows that $\{x_n\}$ is G-convergent.
For every $n, m \in$, $m > n$ using the rule $G5$ in the properties of G-metric space, we have

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m)$$

$$\leq \left[h^n + h^{n+1} + \cdots + h^{m-1}\right] G(x_0, x_1, x_1) \tag{8}$$

Applying geometric rule to equation (8), we have

$$G(x_n, x_m, x_m) \leq \frac{h^n}{1-h} G(x_0, x_1, x_1) \tag{9}$$

By equation (7) and (9) it shows that the sequence x_n is a G-convergent and G-Cauchy respectively, which prove the completeness of X.

To determine the fixed point, suppose u is the fixed point of T.
From equation (1) with $x = x_n, y = z = u$

$$G(x_n, Tu, Tu) \leq h \max\left\{G(x_n, Tu, Tu), \frac{1}{2}[G(x_n, x_{n+1}, x_{n+1}) + G(u, Tu, Tu)],\right.$$

$$\left.\frac{1}{2}[G(x_n, x_{n+1}, x_{n+1}) + G(u, x_{n+1}, x_{n+1})]\right\} \tag{10}$$

Taking the limit of both sides as $n \to \infty$

$$G(u, Tu, Tu) \leq \frac{h}{2} G(u, Tu, Tu)$$

which implies that

$$G(u, Tu, Tu) \leq hG(u, Tu, Tu), \quad \text{hence} \quad Tu = u$$

To show that the fixed point is unique.
Suppose that v is also a fixed point of T. Then from equation (1) with $x = u$, $y = v$

$$G(u, v, v) \leq h \max\left\{G(u, v, v), \frac{1}{2}[G(u, u, u) + G(v, v, v)],\right.$$

$$\left.\frac{1}{2}[G(u,u,u)+G(v,u,u)]\right\} \tag{11}$$

$$= h\max\left\{G(u,v,v), \frac{1}{2}G(v,u,u)\right\}$$

hence

$$G(u,v,v) \leq \frac{h}{2}G(v,u,u) \tag{12}$$

Using the same argument of (1) with $x = v$ and $y = u$, we have that

$$G(v,u,u) \leq h\max\left\{G(v,u,u), \frac{1}{2}[G(v,v,v)+G(u,u,u)],\right.$$

$$\left.\frac{1}{2}[G(v,v,v)+G(u,v,v)]\right\}$$

$$G(v,u,u) \leq \frac{h}{2}G(u,v,v) \tag{13}$$

combining equation (12) and (13) gives

$$G(u,v,v) \leq \left(\frac{h}{2}\right)^2 G(u,v,v) \tag{14}$$

Therefore, $u = v$ since $\frac{h}{2} < 1$.

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