



**THE COUNTERPART OF A RESULT OF WEBB FOR THE
BOUNDED DUAL E^b**

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ABSTRACT

Denote by E' , E^+ and E^b the continuous dual, the sequential dual and the bounded dual, respectively, of the $lcs(E, \tau)$. Well-known is that, [1, 22.2, p.87][2. Proposition 2.5.7, p.107], $f' \in E \iff ker f$ is closed.

John Webb added that, [5, Proposition 1.9., p.345], $f \in E^+ \iff ker f$ is sequentially closed.

I, hereby, add that $f \in E^b \iff ker f$ is locally closed.

The result substantially improves

[4, Proposition 6.2.15, p.177] A sequentially closed }
hyperplane of a bornological space is closed. }

to a locally closed hyperplane of a semibornological space is closed.

1. INTRODUCTION

Our notation and language shall be pretty standard as found for example in [1], [6], [2] and [3]. By a $lcs(E, \tau)$ shall be meant a separated locally convex space. All topologies are assumed separated unless otherwise stated or implied. The field of scalars is $K = R$ or C , the real field or the complex field.

For the $lcs(E, \tau)$ we refer to NOTATION and LANGUAGE and the first paragraph of NOTE 2 in [3] for the notations and language disc, locally converge, local limit point, local null, locally closed set, the algebraic dual $E^\#$, the bounded dual E^b , and the continuous dual E' . By 0 we denote the zero of K and by $ker f$, for

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linear functional $f : (E, \tau) \rightarrow K$, is meant the kernel, $f^{-1}(0)$, of f . If p is a seminorm on the vector space E , σp denotes the pseudometric topology of p on E . σp is locally convex. signifies the end or absence of a proof.

LEMMA A:

[[1], (17.2), p.68] Let E be a vector space over K and suppose $\phi \neq A \subseteq E$ a balanced subset of E . Then, $\lambda, \mu \in K$ and $|\lambda| \leq |\mu| \rightarrow \lambda A \subseteq \mu A$.

LEMMA B :

[2, Proof of **Proposition** 3.5.6, p.208 – 209] For a disc B of $lcs(E, \tau)$ the gauge q_B of B in E_B is a norm, and, the family $\{\lambda_B : \lambda > 0\}$ is a base of neighborhoods of zero of the normed space (E_B, q_B) .

LEMMA C :

Let (E, τ) be a lcs and $\{x_n\}_{n=1}^\infty$ a local null sequence. If $\beta > 0$, then $\{\beta x_n\}_{n=1}^\infty$ is also local null. If $\{\alpha_n\}_{n=1}^\infty$ is a bounded sequence of scalars, then $\{\alpha_n x_n\}_{n=1}^\infty$ is also local null.

Proof:

Let B be a disc of (E, τ) and suppose the sequence $\{x_n\}_{n=1}^\infty$ null in $(E_B, \sigma q_B)$. Let $\epsilon > 0$ and consider $\frac{x}{\beta}$. By **Lemma B**, therefore, for some positive integer $N(\epsilon)$, $x_n \in \frac{x}{\beta} B$, for all $n \geq N(\epsilon)$.

Hence $\beta x_n \in \beta \frac{x}{\beta} B$, for all $n \geq N(\epsilon)$.

By **Lemma B**, again, since ϵ was arbitrary, $\{x_n\}_{n=1}^\infty$ is local null in (E, τ) . This concludes the proof of the first part.

For the second part, suppose $\{x_n\}_{n=1}^\infty$ is a local null sequence. Let $\epsilon > 0$. Suppose $\{\alpha_n\}_{n=1}^\infty$ is a bounded sequence of scalars and $\beta > 0$ such that

$$|\alpha_n| < \beta \text{ for all } n \tag{4}$$

By **Lemma B**

$$x_n \in \frac{x}{\beta} B, \text{ for all } n \geq N'(\epsilon)$$

for some positive integer $N'(\epsilon)$. Hence,

$$\alpha_n x_n \in \alpha_n \left(\frac{x}{\beta} \right) B, \text{ for all } n \geq N'(\epsilon)$$

from which follows by (4) and **Lemma A** that

$$\alpha_n x_n \in \beta \left(\frac{x}{\beta} \right) B, \text{ for all } n \geq N'(\epsilon)$$

. That is,

$$\alpha_n x_n \in B, \text{ for all } n \geq N'(\epsilon)$$

Since ϵ was arbitrary, again by **Lemma B**, we have proved the second part.

LEMMA D:

Let (E, τ) be a *lcs*. (i) [4, **Proposition** 3.2.2, p. 82] Let D be a disc of (E, τ) . The topology, σq_B , of the norm q_B , is finer than the restriction $\tau|_{E_B}$. i.e., $\tau|_{E_B} \leq \sigma q_B$. And so a sequence that locally converges in (E, τ) to $x \in E$, say, also ordinarily τ -converges to x .

(ii) [4, **Proposition** 5.1.3(i), p. 151]. The sequence $\{x_n\}_{n=1}^{\infty}$ τ -locally converges to $x \in E \iff \{x_n - x\}_{n=1}^{\infty}$ is local null in (E, τ) . (iii) [4 **Lemma** 5.1.2, p.151] If (E, τ) is metrizable, then a τ -null sequence is local null.

We deduce the following well-known result from **Lemma D**.

THEOREM A :

Convergence is Mackey in a metrizable *lcs*. That is, if $lcs(E, \tau)$ is metrizable, then the sequence $\{x_n\}_{n=1}^{\infty}$ in E ordinarily converges to $x \in E \iff \{x_n\}_{n=1}^{\infty}$ local converge to x .

Proof \implies : Assume $\{x_n\}_{n=1}^{\infty}$ in E locally converges to $x \in E$. then by (ii) of **Lemma D**,

$\{x_n - x\}_{n=1}^{\infty}$ is local null. And so by (i) of **Lemma D**, $\{(x_n - x)\}_{n=1}^{\infty}$ is ordinarily null. By [6, **Problem** 4.1.1, p.39], therefore, $\{(x_n - x)\}_{n=1}^{\infty}$ ordinarily converges to x .

\Leftarrow : This is the second claim of **Lemma D**(i).

COROLLARY A:

In the scalar space $(K, \sigma|\cdot|)$ convergence is Mackey.

THEOREM B:

Let (E, τ) be a *lcs* and E^b its bounded dual (= the collection of all the bounded linear functionals $f : E \rightarrow K$ on E). Let $f \in E^{\#}$. Then, $f \in E^b \iff \ker f$ is locally closed.

Proof:

The proof is a modification of the proof of the theorem [5, **Proposition** 1.9, p. 345] whose counterpart for E^b is being established. For clarity we simply have to give it in detail (since different theorems have to be invoked).

\implies : So, let $f \in E^b$ and a sequence in $\ker f$ locally converging to $x \in E$. By **Lemma D**(ii), $\{x_n - x\}_{n=1}^{\infty}$ is local null. And so by **Theorem A** of **Note** 1 of [3], $\{f(x_n - x)\}_{n=1}^{\infty}$ is local null in $(K, \sigma|\cdot|)$. By **Theorem A**, therefore,

$\{f(x_n - x)\}_{n=1}^\infty$ is ordinarily null in the scalar space $(K, \rightarrow |\cdot|)$. Hence, using the linearity of f and [6, **Problem** 4.1.1, p.39] converges to $f(x)$ ordinarily. But $f(x_n) = 0$ for all n , and so $\{f(x_n - x)\}_{n=1}^\infty = (0, 0, \dots)$ which converges to 0. By uniqueness of limit in $(K, \sigma|\cdot|)$, $f(x) = 0$. Hence, $x \in \ker f$. This concludes the proof of the forward implication \implies .

\Leftarrow : Let $f \in E^\#$ and suppose $\ker f$ is locally closed. If f is the zero functional we have nothing to show. So suppose f is not the zero functional, and so there exist $x_0 \in E$ such that $f(x_0) = 1$. Let $\{x_n\}_{n=1}^\infty$ be a local null sequence in (E, τ) . For each n , $x_n = \alpha_n x_0 + z_n$ where α_n is a scalar and $z_n \in \ker f$, since $\ker f$ is a hyperplane to which x_0 does not belong. (Hyperplane are the kernels of non-zero linear functionals) [2, Second and third paragraphs of p.41][1, **Theorem** 21.3(a), (b), (e), p. 80]. We show that converges to 0 in $(K, \tau|\cdot|)$ to conclude from **Corollary A** and [3, **Theorem 1** of **Note 1**] that $f \in E^b$. Suppose the opposite that $\{f(x_n)\}_{n=1}^\infty$ does not converge to 0 in $(K, \sigma|\cdot|)$. And so, since $f(x_n) = \alpha_n$, for all n , the sequence of scalars $\{\alpha_n - x\}_{n=1}^\infty$ is not a null sequence. Extracting a subsequence, if necessary, the sequence $\{\frac{1}{\alpha_n}\}_{n=1}^\infty$ is a bounded sequence, and so by **LEMMA 3**, $\{\frac{1}{\alpha_n} x_n\}_{n=1}^\infty$ is local null. Thus,

$$\left\{ \frac{1}{\alpha_n} Z_n - (-x_0) \right\}_{n=1}^\infty = \left\{ \frac{1}{\alpha_n} Z_n + x_0 \right\}_{n=1}^\infty = \left\{ \frac{1}{\alpha_n} x_n \right\}_{n=1}^\infty$$

is local null, and so by **Lemma D** (ii), $\{\frac{1}{\alpha_n} x_n\}_{n=1}^\infty$ locally converges to x_0 in (E, τ) . But $\{\frac{1}{\alpha_n} x_n\}_{n=1}^\infty \in \ker f$, for all n , and $\ker f$ by hypothesis is locally closed. And so, $x_0 \in \ker f$, from which follows that $x_0 \in \ker f$. But this is incompatible with fact that $f(x_0) = 1$.

COROLLARY B: For $lcs(E, \tau)$,

$E' = E^b \iff$ every locally closed hyperplane in (E, τ) is closed.

Proof : Immediate from (1) of the abstract.

COROLLARY C : A locally closed hyperplane of a semibornological space is closed.

Proof A $lcs(E, \tau)$ for which $E' = E^b$ is called semibornological.

By **Lemma D** (i), a sequentially closed set is locally closed. A bornological space is also semibornological. So **Corollary C** substantially improves the result (4) of the abstract.

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