Nigerian Journal of Mathematics and Applications
Volume **24**, (2015), 176 − 183.

Functions Defined by Product of Geometric Expressions

F. M. JIMOH[1] and K. O. BABALOLA[2*]

Abstract

We introduce a new class, $J_n^\alpha(\beta)$, of analytic and univalent functions defined by product of geometric expressions. The univalence of functions in the new class as well as sufficient inclusion condition is established. The radius of convexity for the class $J_0^1(\beta)$ is also obtained.

1. Introduction

Let A denote the class of functions

$$f(z) = z + a_2z^2 + ...$$

which are analytic in $E = \{z \in \mathbb{C} : |z| < 1\}$ and let P be the class of functions

$$p(z) = 1 + c_1z + c_2z^2 + ...$$

which are also analytic in the unit disk E and satisfy $Re\ p(z) > 0$, $z \in E$. Furthermore, for $0 \leq \beta < 1$, let $P(\beta)$ denote the subclass of P satisfying the condition $Re\ p(z) > \beta$ and consisting of analytic functions of the form $p_\beta(z) = \beta + (1 - \beta)p(z)$, $p \in P$.

Received October 02, 2015. * Corresponding author.
2010 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Bazilevic functions, product of geometric expressions, bounded turning, starlike and convex functions, analytic and univalent functions.
[1]Department of Physical Sciences, Al-Hikmah University, Ilorin, Nigeria.
E-mail: folashittu@yahoo.com
[2]Department of Mathematics, University of Ilorin, Ilorin, Nigeria.
E-mail: kobabalola@gmail.com

A function $f(z)$ is said to be starlike of order β if and only if

$$Re\,\frac{zf'(z)}{f(z)} > \beta,\ z \in E$$

for some real number β such that $0 \le \beta < 1$. The class of starlike functions of order β is denoted by $S^*(\beta)$.

An analytic function is said to be convex of order β, that is $f(z) \in C(\beta)$ if and only if it satisfies the condition

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta,\ z \in E.$$

for $0 \le \beta < 1$.

The class $S_n(\beta)$ introduced in [6] generalizes the class of starlike and convex functions and it consists of functions satisfying

$$Re\,\frac{D^{n+1}f(z)}{D^n f(z)} > \beta,\ z \in E$$

for $0 \le \beta < 1$ and D^n, $(n = 0, 1, 2, ...)$ is the Salagean derivative defined by

$$D^n f(z) = D(D^{n-1}f(z)) = z[D^{n-1}f(z)]'.$$

In [5] Singh introduced a subclass of Bazilevic functions denoted by $B_1(\alpha)$ and satisfying the geometric expression

$$Re\,\frac{zf'(z)f(z)^{\alpha-1}}{z^\alpha} > 0,\ z \in E.$$

Opoola in [8] defined the class $T_n^\alpha(\beta)$ as the class consisting of functions that satisfy

$$Re\,\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta$$

where β is a real number such that $0 \le \beta < 1$, $\alpha > 0$ is real and D^n is the Salagean operator and proved that this class consists of univalent functions for $n \ge 1$.

In this paper, we present a new class of analytic functions denoted by $J_n^\alpha(\beta)$ which is defined by the product combination of the geometric expression for the classes $S_n(\beta)$ and $T_n^\alpha(\beta)$. We shall show that this class is a subclass of $T_n^\alpha(\beta)$ and so consists of univalent functions. Other properties investigated include a sufficient condition for functions to be in the class $J_n^\alpha(\beta)$ and the radius of convexity for a particular value of n and α.

Definition: A function $f \in A$ is said to be in the class $J_n^\alpha(\beta)$ if

$$\text{Re}\,\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1} f(z)}{D^n f(z)} > \beta$$

for non negative real number α, $0 \le \beta < 1$, $n \ge 0$ and D^n is the Salagean operator.

Remark 1. By varying the values of the parameter n, α and β, the class $J_n^\alpha(\beta)$ generates some existing classes of analytic and univalent functions. For instance if $n = 0$ and $\beta = 0$, the class $J_n^\alpha(\beta)$ reduces to the class $B_1(\alpha)$, if $n = 1$, we have the class of functions satisfying

$$Re\,\frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta$$

discussed in [4] which is the product combination of the geometric condition for the class $B_1(\alpha)$ and the class C of convex functions. If $\alpha = 0$ in the above, then we have

$$Re\,\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta,$$

which is a product combination of the geometric condition for starlike and convex functions. Furthermore, if $\beta = 0$, we have the class of functions studied by Lewandoski et al [9].

2. Preliminary Lemmas

The following lemmas are useful in the proving our main results.

Lemma 1. [3] *Let $f(z) \in A$, and $\alpha > 0$ be real. If $\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha}$ takes a value which is independent of n, then*

$$\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha} = \alpha \frac{D^{n+1} f(z)}{D^n f(z)}.$$

Lemma 2. [2] *Let $u = u_1 + u_2 i$, $v = v_1 + v_2 i$ and $\psi(u, v)$ a complex valued function satisfying:*

(a) $\psi(u, v)$ is continuous in a domain Ω of $\mathbb{C}^2$

(b) $(1, 0) \in \Omega$ and Re $\psi(1, 0) > 0$

(c) Re $\psi(\zeta + (1-\zeta)u_2 i, v_1) \le \zeta$ when $(\zeta + (1-\zeta)u_2 i, v_1) \in \Omega$ and $2v_1 \le -(1-\zeta)(1+u_2^2)$ for real $0 \le \zeta < 1$.

If $p \in P$ such that $(p(z), zp'(z)) \in \Omega$ and Re $\psi(p(z), zp'(z)) > \zeta$ for $z \in E$. Then, Re $p(z) > \zeta$ in E.

Lemma 3. [1] *Let $p(z)$ be holomorphic in E with $p(0) = 1$. Suppose that*

$$Re\,\left(1 + \frac{zp'(z)}{p(z)}\right) > \frac{3\beta - 1}{2\beta},\ z \in E$$

Then, Re $p(z) > 2^{1-\frac{1}{\beta}}$, $\frac{1}{2} \leq \beta < 1$, $z \in E$ *and the constant* $2^{1-\frac{1}{\beta}}$ *is the best possible.*

Lemma 4. [7] *Let* $p \in P$. *Then, for* $0 \leq \beta < 1$ *and* $|z| < 1$,

$$Re \left(\frac{zp'(z)}{\frac{\beta}{1-\beta} + p(z)} \right) \geq \begin{cases} -\frac{2(1-\beta)r}{(1+r)(1+(2\beta-1)r)}, & \text{for } R_1 \leq R_2, \\ -\frac{\beta}{1-\beta} + \frac{1}{1-\beta}\left(2R_1 - \frac{1-(2\beta-1)r^2}{1-r^2}\right), & \text{for } R_2 \leq R_1. \end{cases}$$

where $R_1 = \left(\frac{\beta-\beta(2\beta-1)r^2}{1-r^2}\right)^{\frac{1}{2}}$ *and* $R_2 = \frac{1+(2\beta-1)r}{1+r}$
The functions given by

$$p(z) = \begin{cases} \frac{1-z}{1+z}, & \text{for } R_1 \leq R_2, \\ \frac{1}{2}\left(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} + \frac{1+ze^{i\theta}}{1-ze^{i\theta}}\right), & \text{for } R_2 \leq R_1. \end{cases}$$

show that the inequalities are sharp, where $cos\theta$ *satisfies the equation*

$$\begin{aligned} &(2R_1 - a - \alpha) - 2cos\theta[(2R_1 - a - \alpha)(1 + \alpha) \\ &\quad + (1-\alpha)^2]r + [2\alpha(2R_1 - a - \alpha)(1 + 2cos^2\theta) + 4(1-\alpha)^2]r^2 \\ &\qquad - 2cos\theta[(2R_1 - a - \alpha)(3\alpha - 1) + (1-\alpha)^2]r^3 + (2\alpha - 1)r^4 = 0 \end{aligned}$$

with $a = \frac{1-(2\alpha-1)r^2}{1-r^2}$.

3. Main Results

Theorem 1.

$$J_n^\alpha(\beta) \subset T_n^\alpha(\beta).$$

Proof. Let

$$p(z) = \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} \tag{1}$$

By logarithmic differentiation we have,

$$\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha} = \alpha + \frac{zp'(z)}{p(z)}$$

using Lemma 1, we have

$$\frac{D^{n+1} f(z)}{D^n f(z)} = 1 + \frac{zp'(z)}{\alpha p(z)}.$$

Then, the geometric condition defining $J_n^\alpha(\beta)$ can now be written as

$$Re\ p(z)\left(1 + \frac{zp'(z)}{\alpha p(z)}\right) > \beta$$

That is

$$Re\ \left(p(z)+\frac{zp'(z)}{\alpha}\right)>\beta$$

Define

$$\psi(u,v)=u+\frac{v}{\alpha}$$

on a domain $\Omega=\mathbb{C}^2$. Then, clearly $\psi(u,v)$ satisfies the conditions (a) and (b) of Lemma 2. Also,

$$Re\ \psi(\beta+(1-\beta)u_2 i,v_1)=\beta+\frac{v_1}{\alpha}<\beta$$

whenever $v_1\leq-\frac{(1-\beta)(1+u_2^2)}{2}$. Therefore, ψ satisfies all the conditions of the Lemma 2 and so $Re\ p(z)>\beta$. That is

$$Re\ \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}>\beta.$$

Hence $f\in T_n^\alpha(\beta)$. □

Since the class $T_n^\alpha(\beta)$ is known to consist of univalent functions only for $n\geq 1$ [2], then we have:

Corollary 1. *For $n\geq 1$, the class $J_n^\alpha(\beta)$ consists only of univalent functions in E.*

Theorem 2. *Let $f(z)\in A$. If*

$$Re\ \left(\frac{D^{n+1}f(z)^\alpha}{D^n f(z)^\alpha}+\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-\frac{D^{n+1}f(z)}{D^n f(z)}\right)>\frac{\beta+2\alpha\beta-1}{2\beta}$$

Then,

$$Re\ \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1f(z)}}{D^n f(z)}>2^{1-\frac{1}{\beta}},\ \frac{1}{2}\leq\beta<1.$$

Proof. Let

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1}f(z)}{D^n f(z)}=p(z), \tag{2}$$

then, $p(z)$ is analytic with $p(0)=1$. Differentiating (2) logarithmically we have

$$\frac{zp'(z)}{p(z)}=\frac{D^{n+1}f(z)^\alpha}{D^n f(z)^\alpha}+\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-\frac{D^{n+1}f(z)}{D^n f(z)}-\alpha$$

so that

$$1+\frac{zp'(z)}{p(z)}=\frac{D^{n+1}f(z)^\alpha}{D^n f(z)^\alpha}+\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-\frac{D^{n+1}f(z)}{D^n f(z)}+1-\alpha.$$

Thus, by the hypothesis of the theorem

$$Re\left(1+\frac{zp'(z)}{p(z)}\right)$$
$$=Re\left(\frac{D^{n+1}f(z)^{\alpha}}{D^{n}f(z)^{\alpha}}+\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-\frac{D^{n+1}f(z)}{D^{n}f(z)}+1-\alpha\right)>\frac{3\beta-1}{2\beta}$$

This last expression is equivalent to the expression

$$Re\left(\frac{D^{n+1}f(z)^{\alpha}}{D^{n}f(z)^{\alpha}}+\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-\frac{D^{n+1}f(z)}{D^{n}f(z)}\right)>\frac{\beta+2\alpha\beta-1}{2\beta}.$$

Thus, by Lemma 3, we have

$$Re\,\frac{D^{n}f(z)^{\alpha}}{\alpha^{n}z^{\alpha}}\frac{D^{n+1f(z)}}{D^{n}f(z)}>2^{1-\frac{1}{\beta}},\ \frac{1}{2}\leq\beta<1$$

as required. □

Corollary 2. *Let $f(z)\in A$. If*

$$Re\left(\frac{D^{n+1}f(z)^{\alpha}}{D^{n}f(z)^{\alpha}}+\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-\frac{D^{n+1}f(z)}{D^{n}f(z)}\right)>\frac{\beta+2\alpha\beta-1}{2\beta},$$

then,

$$f(z)\in J_{n}^{\alpha}(\beta).$$

Corollary 3. *Let $f(z)\in A$ and*

$$Re\left((\alpha-1)\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right)>-\frac{\beta-2\alpha\beta+1}{2\beta},$$

then

$$Re\,\frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}}>2^{1-\frac{1}{\beta}}.$$

If $\alpha=0$ and $\beta=\frac{1}{2}$ in the above, we have a result obtained earlier in [1].

Corollary 4. *Let $f(z)\in A$. If*

$$Re\left(\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)>-\frac{3}{2},$$

then

$$Re\,\frac{zf'(z)}{f(z)}>\frac{1}{2}.$$

This next corollary is obtained by putting $\alpha=1$ and $\beta=\frac{1}{2}$ in Corollary 2.

Corollary 5. *Let $f(z) \in A$. If*

$$Re\ \left(\frac{zf''(z)}{f'(z)}+1\right) > \frac{1}{2},$$

then,

$$Re\ f'(z) > \frac{1}{2}.$$

Theorem 3. *Let $f(z) \in J_n^\alpha(\beta)$. Then,*

$$Re\ \left(\frac{D^{n+1}f(z)^\alpha}{D^n f(z)^\alpha}+\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-\frac{D^{n+1}f(z)}{D^n f(z)}\right) > 0$$

for $|z| < r_0(\alpha, \beta)$ where $r_0(\alpha, \beta)$ is given by

$$r_0(\alpha,\beta) = \begin{cases} \frac{A}{\alpha(1-2\beta)}, & \text{if } 0 \le \beta \le \beta_0(\alpha), \\ \left(\frac{B}{(1+\alpha^2)(1-\beta)+2\alpha(1-3\beta)}\right)^{\frac{1}{2}}, & \text{if } \beta_0(\alpha) \le \beta < 1, \beta \neq \frac{(1+\alpha)^2}{1+6\alpha+\alpha^2}, \\ \left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{2}}, & \text{if } \beta = \frac{(1+\alpha)^2}{1+6\alpha+\alpha^2}. \end{cases} \tag{3.14}$$

where

$$A = [(1+\alpha^2)(1-\beta)^2 - 2\alpha\beta(1-\beta)]^{\frac{1}{2}} + \beta(1+\alpha) - 1$$

and

$$B = 4[\alpha\beta(1-\beta)]^{\frac{1}{2}} - [(1-\alpha^2)(1-\beta)+4\alpha\beta]$$

Proof. Let $f \in J_n^\alpha(\beta)$. Then, there exists $p(z) \in P$ such that

$$\frac{D^n f(z)^\alpha D^{n+1} f(z)}{D^n f(z)} = \alpha^n z^\alpha[\beta + (1-\beta)p(z)]$$

Differentiating logarithmically we obtain

$$\frac{D^{n+1}f(z)^\alpha}{D^n f(z)^\alpha}+\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-\frac{D^{n+1}f(z)}{D^n f(z)} = \alpha + \frac{(1-\beta)p'(z)}{\beta+(1-\beta)p(z)}$$

Let R_1 and R_2 be as in Lemma 4, by applying that Lemma we have that

$$Re\ \left(\frac{D^{n+1}f(z)^\alpha}{D^n f(z)^\alpha}+\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-\frac{D^{n+1}f(z)}{D^n f(z)}\right) \ge \alpha - \frac{2(1-\beta)r}{(1+r)(1+(2\beta-1))}$$

The right hand side of the above equation corresponds to that obtained in Theorem 1.1 of [3], and so the rest of the proof follows similarly. □

Corollary 6. *If $f \in J_0^1(\beta)$, then the radius of convexity $r_0(1,\beta)$ of $f(z)$ is given by*

$$r_0(1,\beta) = \begin{cases} \left(\frac{2(1-\beta)}{(1-2\beta)}\right)^{\frac{1}{2}} - 1, & \text{if } 0 \le \beta \le \frac{1}{10}, \\ \left(\frac{-\beta+[\beta(1-\beta)]^{\frac{1}{2}}}{(1-2\beta)}\right)^{\frac{1}{2}}, & \text{if } \frac{1}{10} \le \beta < 1, \beta \ne \frac{1}{2}, \\ 2^{-\frac{1}{2}}, & \text{if } \beta = \frac{1}{2}. \end{cases}$$

This corresponds to a result obtained in [7].

References

[1] Babalola, K. O.(2013). *On λ-Pseudo-Starlike functions.* Journal of classical Analysis, *3*(2), 137–147

[2] Babalola, K. O. and Opoola, T. O. (2006). *Iterated integral transforms of Caratheodory functions and their applications to analytic and Univalent functions.* Tamkang Journal of Mathematics, **37**(4), 355–366.

[3] Babalola, K. O. and Opoola, T. O. (2008). *Radius problems for certain classes of analytic functions involving Salagean derivative.* Advances in Inequalities for series, 15-21.

[4] Ganiyu, M. A., Jimoh, F. M., Ejieji, C. N. and Babalola, K. O.(2014).*Coefficients Estimates for certain classes of Analytic Functions* Journal of the Nigerian Mathematical Society,**33**, 175-183.

[5] Singh, R. (1973). *On Bazilevic functions* Proc. Amer. Math. Soc. **38**, 261–271.

[6] Salagean, G. S.(1983). *Subclasses of univalent functions.* Lecture notes in Mathematics. Springer-Verlag, Berlin, Heidelberg and Newyork. **1013** 362-372.

[7] Tuan, P. D. and Ahn, V. V.(1983). *Radii of starlikeness and convexity of certain classes of analytic functions.* J. Math. Anal. App., **64**, 146-148.

[8] Opoola, T. O. (1994). *On a new subclass of univalent functions.* Matematica Tome, **36**(59), No. 2, 195–200.

[9] Lewandowski, Z., Miller, S. and Zlotkiewicz, E. (1976). *Generating functions for some classes of univalent functions*, Proc. Amer. Math. Soc. **56**, 111–117.