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A Two Step Lo- Stable Second Derivative Hybrid Block Method for Solution of Stiff Initial Value Problems

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Abstract

We present a two step Lo-stable second derivative hybrid block method of order eight for the direct solution of stiff Initial Value Problems (IVPs). The main method and additional methods are obtained from the same continuous scheme derived using interpolation and collocation technique to form the block method. The stability properties of the block method is discussed via a single matrix equation. The methods simultaneously integrate the stiff IVPs over non-overlapping intervals. Numerical results obtained using the proposed second derivative hybrid block method reveal that it compares favorably well with existing methods in the literature.

1. Introduction

Consider the stiff IVP of the form

$$y' = f(t, y),\ y(t_0) = y_0\ ,\ t \in [a, b] \tag{1}$$

.
where f satisfies the Lipschitz condition as given in Henrici [17]).

Equation (1) occur in the mathematical formulation of physical situations in a number of areas particularly in chemical kinetics, control theory, electrical circuit, robotics, aeronautics etc. Stiffness in most ordinary differential equations

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(ODEs) has posed a lot of computational difficulties in many practical application model of ODEs and therefore affects the efficiency of numerical methods.
In the literature, several authors have proposed various methods including hybrid method for the solution of (1), see([3], [4], [6], [7],[8], [19]), and their references therein. Hybrid method is the modified form of the k-step linear multistep method (LMM) obtained by incorporating off-step points in the derivation process in order to overcome the Dahlquist barrier theorem. Second derivative methods proposed by Enright ([9],[10]), were shown to be of order up to $k+2$ and implemented in a variable order, variable-step mode. In this paper we proposed a second derivative hybrid method of order $3k+2$ in the form (2)

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h(\sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{j=1}^{v} \beta_{\eta_j} f_{n+\eta_j}) + h^2 \gamma_k g_{n+k} \tag{2}$$

where $\alpha_k = 1$ and $\beta_k = 0, k = 2$ is the step number, $\beta_k = 0$ and $\alpha_k = 1$ do not vanish, $\alpha_j, \beta_j, \beta_{\eta_j} and \gamma_k$ are unknown constants, $v = 4$ is the number of off-step points and η_j are rational numbers. Equation (2) is derived through interpolation and collocation, (see Lie and Norsett [21], Onumanyi et al. [24],[25],[26] and Gladwell and Sayers [14]). The continuous representation generates two main discrete hybrid second derivative method and four additional method which are combined and used as a block method to simultaneously produce approximations $\{y_{\frac{1}{4}}, y_{\frac{3}{4}}, y_1, y_{\frac{5}{4}}, y_{\frac{7}{4}}, y_2\}$ for the solution of (1) at a block of points $\{t_{\frac{1}{4}}, t_{\frac{3}{4}}, t_1, t_{\frac{5}{4}}, t_{\frac{7}{4}}, t_2\}$ $h = t_{n+1} - t_n$, $n = 0, 2..., N-2$ on a partition [a, b], where $a, b \epsilon \Re$, h is the constant step-size, n is a grid index and $N > 0$ is the number of steps.

The rest of the paper is presented as follows: In section 2, we discuss the basic idea behind the algorithm and obtain a continuous representation $Y(t)$ for the exact solution $y(t)$ which is used to generate members of the block method for solving (1). In section 3, we present the analysis of our Two step second derivative hybrid block method. The accuracy of the new method is shown via some standard problems in section 4. Finally, in section 5 we present some concluding remarks.

2. Derivation of the method

We derive a continuous representation of the second derivative hybrid method which is used to generate the main discrete method by seeking an approximation of the exact solution $y(t)$ by assuming a continuous solution $Y(t)$ of the form

$$Y(t) = \sum_{j=0}^{p+2q-1} b_j \varphi_j(t) \tag{3}$$

where $t \in [t_0, T_n]$, b_j are unknown coefficients to be determined, $\varphi_j(t)$ are polynomial basis functions of degree $p+2q-1$, such that the number of interpolation points p and the number of distinct collocation points $2q$ are respectively chosen to satisfy $1 \leq p < k$ and $q > 0$.

The new proposed method is thus constructed by specifying the following parameters: $\eta_i = (1/3, 2/3, 4/3, 5/3), i = 1 \ldots 4, \varphi_(t) = t^j, j = 0, 1 \ldots p+2q-1, p = 1, q = 4, andk = 2$ and assuming that $y_{n+\frac{\imath}{3}}$ denote numerical solution of the exact solution $y(t_{n+\frac{\imath}{3}}), f_{n+\frac{\imath}{3}} = f(t_{n+\frac{\imath}{3}})$, n is the grid index, $\imath = 0, \ldots 6$. While $g_n + k = \frac{df(t_{n+k}, y_(n+k))}{dt}$.
By imposing the following conditions:

$$\sum_{j=0}^{8} b_j t_{n+i}^{j} = y_{n+i}, \; i = 1 \tag{4}$$

$$\sum_{j=0}^{8} b_j t_{n+\frac{i}{3}}^{j-1} = f_{n+\frac{i}{3}}, \; i = 0, \ldots 6, \tag{5}$$

$$\sum_{j=0}^{8} b_j j(j-1) t_{n+2}^{j-2} = g_{n+2}. \tag{6}$$

Equations (4), (5), and (6) lead to a system of nine equations which is solved to obtain the coefficient b_j.

The two step continuous second derivative hybrid method is obtained by substituting these values of b_j into (3). After some algebraic computation, our method yields the expression in the form:

$$Y(t) = \alpha_1(t) y_{n+1} + h \sum_{j=0}^{2} \beta_j(t) f_{n+j} + h \sum_{j=1}^{4} \beta_{\eta_j}(t) f_{n+\eta_j} + h^2 \gamma_2(t) g_{n+2} \tag{7}$$

Where $\alpha_1(t), \beta_j(t)$, $\beta_{\eta_j}(t)$ and $\gamma_2(t)$ are continuous coefficients. Equation (7) is then used to generate the main discrete second derivative hybrid method (8) and (9) by evaluating at point $t = (t_{n+2})$.

(8)

$$y_{n+2} = y_{n+1} - \frac{11h}{26880} f_n + \frac{9h}{2240} f_{n+\frac{1}{3}} - \frac{369}{17920} f_{n+\frac{2}{3}} + \frac{563h}{3360} f_{n+1} + \frac{3123h}{8960} f_{n+\frac{4}{3}} + \frac{153}{448} f_{n+\frac{5}{3}} + \frac{8567h}{53760} f_{n+2} - \frac{h^2}{128} g_{n+2}$$

And evaluating at points
$t = t_n, t_{n+\frac{1}{3}}, t_{n+\frac{2}{3}}, t_{n+\frac{4}{3}}, t_{n+\frac{5}{3}}$ we obtained the additional methods (9),(10), (11),(12) and (14).

(9)
$$y_n = y_{n+1}-\frac{527h}{5376}f_n-\frac{1143h}{2240}f_{n+\frac{1}{3}}-\frac{1521}{17920}f_{n+\frac{2}{3}}-\frac{1613h}{3360}f_{n+1}+\frac{2547h}{8960}f_{n+\frac{4}{3}}-\frac{387}{2240}f_{n+\frac{5}{3}}+\frac{3319h}{53760}f_{n+2}-\frac{h^2}{128}g_{n+2}$$

(10)
$$y_{n+\frac{1}{3}} = y_{n+1}+\frac{187h}{68040}f_n-\frac{131h}{945}f_{n+\frac{1}{3}}-\frac{6659}{15120}f_{n+\frac{2}{3}}-\frac{821h}{8505}f_{n+1}-\frac{197h}{7560}f_{n+\frac{4}{3}}+\frac{4h}{189}f_{n+\frac{5}{3}}-\frac{121h}{15120}f_{n+2}-\frac{h^2}{972}g_{n+2}$$

(11)
$$y_{n+\frac{2}{3}} = y_{n+1}-\frac{383h}{435456}f_n+\frac{3263}{302400}f_{n+\frac{1}{3}}-\frac{75499}{483840}f_{n+\frac{2}{3}}-\frac{62941h}{272160}f_{n+1}+\frac{15713h}{241920}f_{n+\frac{4}{3}}-\frac{1873}{60480}f_{n+\frac{5}{3}}+\frac{73153h}{7257600}f_{n+2}-\frac{191h^2}{155520}g_{n+2}$$

(12)
$$y_{n+\frac{4}{3}} = y_{n+1}-\frac{191h}{435456}f_n+\frac{1343h}{302400}f_{n+\frac{1}{3}}-\frac{11371}{483840}f_{n+\frac{2}{3}}+\frac{49571h}{272160}f_{n+1}+\frac{47777h}{241920}f_{n+\frac{4}{3}}-\frac{2257}{60480}f_{n+\frac{5}{3}}+\frac{25451h}{2419200}f_{n+2}-\frac{191h^2}{155520}g_{n+2}$$

(13)
$$y_{n+\frac{5}{3}} = y_{n+1}-\frac{h}{13608}f_n+\frac{h}{945}f_{n+\frac{1}{3}}-\frac{131h}{15120}f_{n+\frac{2}{3}}+\frac{1171h}{8505}f_{n+1}+\frac{3067h}{7560}f_{n+\frac{4}{3}}+\frac{134h}{945}f_{n+\frac{5}{3}}-\frac{491h}{45360}f_{n+2}+\frac{h^2}{972}g_{n+2}$$

The main and additional Methods are combined and implemented simultaneously as single block integrators to provide the approximate solutions $y_{n+\frac{1}{3}}, y_{n+\frac{2}{3}}, y_{n+1}, y_{n+\frac{4}{3}}, y_{n+\frac{5}{3}}, y_{n+2}$ for equation (1) at discrete block points $t = t_{n+\frac{1}{3}}, t_{n+\frac{2}{3}}, t_{n+1}t_{n+\frac{4}{3}}, t_{n+\frac{5}{3}}, t_{n+2}$, $n = 0, 2, \ldots, N-2$ on a partition $[t_0, T_n]$.

3. Analysis of the two step second derivative hybrid block method

In this section, we discuss the local truncation error and order, consistency, zero-stability, and convergence of the two step second derivative hybrid block method.
The combined methods (8-12) can be represented as a matrix finite difference equation in block form given as

$$A^{(1)}Y_{\varpi+1} = A^{(0)}Y_{\varpi} + h[B^{(1)}F_{\varpi+1} + B^{(0)}F_{\varpi}] + h^2C^(1)G_{\varpi+1}, \tag{14}$$

where
$Y_{\varpi+1} = (y_{n+\frac{1}{3}}, y_{n+\frac{2}{3}}, y_{n+1}, y_{n+\frac{4}{3}}, y_{n+\frac{5}{3}}, y_{n+2})^T$, $Y_{\varpi} = (y_{n-\frac{5}{3}}, y_{n-\frac{4}{3}}, y_{n-1}, y_{n-\frac{2}{3}}, y_{n-\frac{1}{3}}, y_n)^T$
$F_{\varpi+1} = (f_{n+\frac{1}{3}}, f_{n+\frac{2}{3}}, f_{n+1}, f_{n+\frac{4}{3}}, f_{n+\frac{5}{3}}, f_{n+2})^T$, $F_{\varpi} = (f_{n-\frac{5}{3}}, f_{n-\frac{4}{3}}, f_{n-1}, f_{n-\frac{2}{3}}, f_{n-\frac{1}{3}}, f_n)^T$
$G_{\varpi} = (g_{n+\frac{1}{3}}, g_{n+\frac{2}{3}}, g_{n+1}, g_{n+\frac{4}{3}}, g_{n+\frac{5}{3}}, g_{n+2})^T$
$\varpi = 0, 1, 2, \ldots$ and $n = 0, 2, \ldots$ and the matrices $A^{(1)}, A^{(0)}, B^{(1)}, B^{(0)}$ and $C^(1)$ are defined as follow

$$A^{(1)} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$A^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B^{(1)} = \begin{pmatrix} -\frac{113}{945} & -\frac{6659}{15120} & \frac{821}{8505} & -\frac{197}{7560} & \frac{4}{189} & -\frac{121}{15120} \\ \frac{3263}{302400} & -\frac{75499}{483840} & -\frac{62941}{272160} & \frac{15713}{241920} & -\frac{1873}{60480} & \frac{73153}{7257600} \\ -\frac{1143}{2240} & -\frac{1521}{17920} & -\frac{1613}{3360} & \frac{2547}{8960} & -\frac{387}{2240} & \frac{3319}{53760} \\ \frac{1343}{302400} & -\frac{11371}{483840} & \frac{49571}{272160} & \frac{4777}{241920} & -\frac{2257}{60480} & \frac{25451}{2419200} \\ \frac{1}{945} & -\frac{131}{15120} & \frac{1171}{8505} & \frac{3067}{7560} & \frac{134}{945} & -\frac{491}{45360} \\ \frac{9}{2240} & -\frac{369}{17920} & \frac{563}{3360} & \frac{3123}{8960} & \frac{153}{448} & \frac{8567}{53760} \end{pmatrix}$$

$$B^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{187}{68040} \\ 0 & 0 & 0 & 0 & 0 & -\frac{383}{435456} \\ 0 & 0 & 0 & 0 & 0 & -\frac{527}{5376} \\ 0 & 0 & 0 & 0 & 0 & -\frac{191}{435456} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{13608} \\ 0 & 0 & 0 & 0 & 0 & -\frac{11}{26880} \end{pmatrix}$$

$$C^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{972} \\ 0 & 0 & 0 & 0 & 0 & -\frac{191}{155520} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{128} \\ 0 & 0 & 0 & 0 & 0 & -\frac{191}{155520} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{972} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{128} \end{pmatrix}$$

3.1. **Local truncation error and order.** Following Fatunla [11] and Lambert [19] we define the local truncation error associated with (2) to be the linear difference operator

$$L[y(t,h] = \sum_{j=0}^{k}\{\alpha_j y(t+jh) - h\beta_j y't + jh - h\sum_{j=1}^{v}\beta_{\eta_j} y't + \eta_j h\} - h^2\gamma_k y''(t+kh) \tag{15}$$

Assuming that $y(t)$ is sufficiently differentiable, we can expand the terms in (10) as a Taylor series about the point t to obtain the expression

$$L[y(t,h] = C_0 y(t) + C_1 y'(t) + \ldots + C_s h^s y^(s)(t) + \ldots,$$

where the constant coefficients $C_s, s = 0, 1, \ldots$ are given as follows:

$$C_0 = \sum_{j=0}^{k}\alpha_j, C_1 = \sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k} j\beta_j - \sum_{j=1}^{v}\beta_{\eta_j} \vdots C_s = \frac{1}{s!}[\sum_{j=0}^{k} j^s\alpha_j - s(\sum_{j=0}^{k} j^{s-1}\beta_j + \sum_{j=0}^{v}\eta_j^{s-1}\beta_{\eta_j}) + s(s-1)k^{s-2}\gamma_k].$$

According to [17], we say that the method (3) has order m if

$C_0 = C_1 = \ldots = C_s = 0, \ C_{s+1} \neq 0$

therefore, C_{s+1} is the error constant and $C_{s+1}h^{s+1}y^{(s+1)}(t_n)$ the principal local truncation error at the point t_n. Thus, we can write the local truncation error (LTE) of the method of order m as

$LTE = C_{s+1}h^{s+1}y^{(s+1)}(t_n) + \bigcirc(h^{m+2}).$

It is established from our calculations that the second derivative block hybrid methods are of order s$=(8,8,8,8,8,8)^T$ and relatively small error constants

$$C_9=[-\frac{139}{391910400},-\frac{433}{8928208800},\frac{10067}{285702681600},\frac{7123}{285702681600},-\frac{17}{8928208800},\frac{11}{391910400}]^T$$

.

3.2. **Zero-stability.** The zero stability of the methods in (13) are determined as the limit h tends to zero. Thus as $h\to 0$ the method (13) tends to the difference system

$$A^{(1)}Y_{\varpi+1}=A^{(0)}Y_{\varpi}$$

to obtain the first characteristic polynomial $\rho(R)$ given by

$$\rho(R)=\det(RA^{(1)}-A^{(0)})=R^5(R-1) \tag{16}$$

Following Fatunla [11], the block method (13) is zero-stable, since from (15), $\rho(R)=0$ satisfies $|R_j|\leq=1$, $j=1,\ldots,6$, and for those roots with $|R_j|=1$, the multiplicity does not exceed 1.We note that the single members of the block method are not zero-stable, but this property is gained when the methods are combined as numerical integrators in the block form (13).

3.3. **Consistency and Convergence.** The block method (13) is consistent since each of the integrators has order $s>1$. According to Henrici [17], convergence = consistency + zero-stability. Hence the two step second derivative hybrid block method is convergent.

3.4. **stability analysis. Definition 3.3.1:** A numerical method is said to be A_0-Stable if $|\xi(-z)|<1$ for all $z>0$

Definition 3.3.2 A numerical method is said to be Lo-Stable if $|\xi(-z)|<1$ for all $z>0$ and

$$\lim_{z\to\infty}\xi(-z)=0, where\ z=h\lambda$$

By applying the method (13) to the test equation $y'=-\lambda y$ $y'=\lambda y,\ \lambda\in R$ to yield

$$Y_{\varpi+1}=D(z)Y_{\varpi}\ ,\ z=\lambda h,$$

where the matrix $D(z)$ is given by

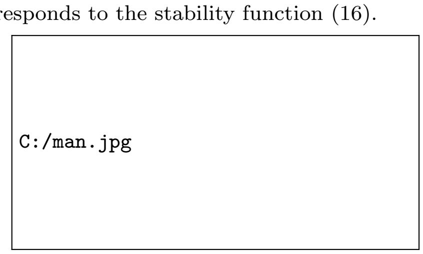
$[D(z)=(A^{(1)}-zB^{(1)}-z^2C^{(1)})^{-1}(A^{(0)}+zB^{(0)})]$ is the amplification matrix.

The stability function $\xi(z):\mathbb{C}\to\mathbb{C}$ is obtained from the eigenvalues of $D(z)$ which is a rational function with real coefficients given by

$$\xi_6=\frac{-AB}{C+D} \tag{17}$$

where $A=91922$, $B=3.9281*10^{17}+3.4361*10^{17}z+1.3586*10^{17}z^2+3.2475*10^{16}z^3+5.0061*10^{15}z^4+4.6889*10^{14}z^5+2.0273*10^{13}z^6$, $C=-3.6108*10^{22}+4.0631*10^{22}z-2.1394*10^{22}z^2+1.5466*10^{21}z^3-1.6503*10^{21}z^4$ and $D=2.4306*10^{20}z^5-6.3227*10^{19}z^6+3.3189*10^{18}z^7$.

In the spirit of Hairer and Wanner [15], the stability region S is presented in white colour which is drawn using the equations (21) as shown in Fig. 1. In Figures below,t the rectangles represent the zeros and plus signs represent the poles of (16). The plots in white on the left half of the complex plane represent the stability region which corresponds to the stability function (16).



Clearly, from Figure 1 above, it is obvious that the method is not A- stable, since it has at least one pole of the stability function (16) represented by the plus sign in the left half complex plane. However the method is said to be L_o- Stable as it satisfies the definition 3.3.2

4. Numerical examples

In this section some numerical example are considered with all computations carried out with our written code in Mapple 17.

Example 4.1. We consider the system of initial value problem which has been solved by Jackson and Kanue [18] and Sahi etal. [27].

$$y' = -y + 95z, \; y(0) = 1$$

$$z' = -y - 97z, \; y(0) = 1$$

With exact solution of the system given by
$y(t) = \frac{95}{47}exp^(-2t) - \frac{48}{47}exp(-96t)$
$z(t) = \frac{48}{47}exp^(-96t) - \frac{1}{47}exp(-2t)$
We compare the new block method with related results obtained by Jackson and Kanue [18] and Sahi etal. [27] and reproduced in Table 1. As expected the result shows twice as accurate as that Jackson and Kanue [18] and gained at least five digit more than those of Sahi etal [27]

Table 1. Computed values of $error = |y(t) - y|$, $error = |z(t) - z|$ for Example 4.1

h	Jackson and Kanue [18] $\lvert y(t)-y\rvert$ $\lvert z(t)-z\rvert$	Sahi etal.citeSJ $\lvert y(t)-y\rvert$ $\lvert z(t)-z\rvert$	New method $\lvert y(t)-y\rvert$ $\lvert z(t)-z\rvert$
0.0625	3×10^{-7}	9×10^{-11}	1×10^{-16}
	4×10^{-7}	1×10^{-8}	1×10^{-17}
0.03125	1×10^{-8}	4×10^{-12}	5×10^{-19}
	1×10^{-8}	4×10^{-12}	5×10^{-20}

Example 4.2. Next, we consider stiff system (see [5]), in the range $0 \leq t \leq 10$

$$y' = 998y + 1998z, \; y(0) = 1$$

$$z' = -999y - 1999z, \; z(0) = 1$$

Its exact solution is given by the sum of two decaying exponentials components.

$$y_1 = 4e^{-t} - 3e^{-1000t} \;, \; y_2 = -2e^{-t} + 3e^{-1000t}$$

The stiffness ratio is 1:1000. In Table 3, the comparison of the result of new method with that in [5] at the end point $t = 10$ is presented.

Example 4.3. Lastly, we considered the second order ordinary differential equation given by,

$$y'' + 1001y' + 1000y = 0$$

and reduced to a system of first order equation as,

$$y' = z, \; y(0) = 1$$

$$z' = -1000y - 1001z, \; z(0) = 0$$

Table 2. A comparison of methods for Example 4.2 at h=0.1

t	Exact $y(t) \times 10^{-3}$ $z(t) \times 10^{-3}$	$BBDF_8$ $y \times 10^{-3}$ $z \times 10^{-3}$	New method $y \times 10^{-3}$ $z \times 10^{-3}$	$BBDF_8$Absolute error $\lvert y(t) - y\rvert$ $\lvert z(t) - z\rvert$	New method Absolute error $\lvert y(t) - y\rvert$ $\lvert z(t) - z\rvert$
10	0.18159971904994	0.18159971946833	0.18159971904994	4.183×10^{-13}	2.650×10^{-18}
	-0.09079985952497	-0.09079985973416	0.09079985952597	2.092×10^{-13}	1.324×10^{-18}

This problem has also been considered by Abhulimen [1], Abhulimen and Okunuga [2] and Okunuga [23]. The stiff system has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1000$. For the purpose of comparison, we solve the problem on the interval $0 < x < 1$. Numerical results is compared with that of Abhulimen [1], Abhulimen and Okunuga [2] and [23]. The results in table 3 showed that the new method is more accurate than those ([1],[2], and [23].

Table 3. A comparison of methods for Example 4.3 at h=0.1

Method	t	$error = \lvert y_{10} - y(1)\rvert$
Abhulimen [1]	1	1.8×10^{-7}
Okunuga [23]	1	5.26×10^{-8}
Abhulimen and Okunuga [2]	1	5.29×10^{-9}
New method	1	1.56×10^{-14}

It is clear from table 3 that the new method yields a more accurate result than that derived in [1], [2] and [23].

5. Conclusion

A two step second derivative hybrid method which is used together with additional methods in the block form (13) to simultaneously solve (1) has been proposed. The block method is found to be Lo-stable and implemented without the need for starting values or predictors and hence it is selfstarting. We have demonstrated the efficiency of the methods on three numerical examples. Details of the numerical results are displayed in Tables 1-3.

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