Nigerian Journal of Mathematics and Applications
Volume **23**, (2014), 20 − 29
 http://www.njmaman.com

Error Estimates of the Fully Discrete Solution of Linearized Stochastic Cahn-Hilliard Equation

[1]Ignatius N. NJOSEH and [2]H. I. OJARIKRE

Abstract

We studied the finite element analysis of the linearized Stochastic Cahn-Hilliard equation. The Galerkin finite element method was used to discretize the given equation. Based on the finite elements, the completely discrete approximation scheme was formulated by applying the backward Euler difference approximation in time. The completely discrete solution was interpreted in terms of analytic semigroup and converted to variation of constant formula using the rational functions definition to establish strong convergence rate for the completely discrete scheme.

1. Introduction

Stochastic Partial Differential Equations (SPDEs) are important tools in the modelling of complex phenomena and real life problems, such as, turbulence and pattern formation and to predict trends in the stock market or in weather, ground water flow, chemical reactions and heat emissions etc. They are also used for biological modelling and within the fields of medicine and engineering. The study of qualitative properties of SPDEs, involving the super-process or simple variants of the heat equation, is largely exhausted. Left to itself, SPDE might have become a dying field. Luckily, scientists are jumping into the field with a

Received January 22, 2015. * Corresponding author.

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Cahn-Hilliard, Ito Isometry, Hibert-Schmidt Operator, Backward Euler.

Department of Mathematics and Computer Science, Delta State University, Abraka. njoseh@yahoo.com

vengeance, and SPDE is expanding chaotically in all directions. We believe that the sciences will continue to provide important SPDE models and conjectures since scientists seem to have finally grasped the importance of SPDE models.

The purpose of this paper is to present numerical schemes and error estimates of the solution of Stochastic Cahn-Hilliard equation

$$\begin{aligned} \dot{y} + A^2 y + Af(y) = \dot{W}, \text{ in } \Omega \times [0,T], \\ y(0) = y_0 \text{ in } \Omega \\ \frac{\partial y}{\partial n} = \frac{\partial \Delta y}{\partial y} = 0 \text{ on } \partial\Omega \times [0,T]. \end{aligned} \tag{1}$$

where $f = 0$ and $y(t)$ a random process that takes values in $L_2(\Omega)$, Ω is a bounded domain in R^d, $d \leq 3$ with a sufficiently smooth boundary $\partial\Omega$. Δ is the Laplacian operator and $W(t)$ is a standard Brownian motion defined on a filtered probability space $(\Omega, F, \{F_t\}_{t\geq 0}, P)$.

Equation (1) is a fourth order heat equation used to model a complicated phase separation and Coarsening phenomena in a melted alloy that is quenched to a temperature at which only two different concentration phases can exist stably. This was developed by Cahn and Hilliard in 1958. (For more physical background on this equation, see [1]). The existence and uniqueness of the solution of equation (1) has been a subject of study for a long time (cf [2] and [3] and the references therein). Finite element approximations of the deterministic form of equation (1) was analyzed in the L_2-norms in [4] and in [5] and [6], Cardon-Weber studied the explicit and implicit discretization schemes of equation (1) in dimensions $d \leq 2$.

So much work has been done on the finite element analysis of the deterministic version of equation (1) but a little literature is available for the stochastic version. We shall now review some available literature. [5] and [6] established existence and uniqueness of a function-valued solution of the stochastic Cahn-Hilliard equation (1) in dimension $d \leq 3$. Here, the driving noise is the space-time white noise with non-linear diffusion coefficient. The author observed that the polynomial growth of the drift term made her require the diffusion coefficient to be bounded, and proved convergence in probability (respectively in L_p with a given rate of a localized version) of the scheme, uniformly in space and time, that is, under some assumptions.

In [7], the author considered the finite element method for a stochastic parabolic partial differential equation of second order forced by additive space-time noise in the multi-dimensional case in the Hilbert space, where $y(t)$ is a H-valued random process. The author set up a finite element analysis and applied the semigroup property generated by A to obtain optimal strong convergence estimates in the L_2 and H^{-1} norms with respect to spatial variable. For more on this, see [8] and [9] and the references therein.

Also Njoseh and Ayoola in [10] discussed the finite element method for nonlinear Stochastic Cahn-Hilliard equation (1) (i.e., $f(y) = y^3 + y$) and proved error estimates for both the semidiscrete and fully discrete solutions. Here the fully discrete scheme was obtained by applying the backward Euler time stepping finite difference method. For more literature, see the works of [12], [13] and [14] and the references therein.

We shall therefore be analyzing the error estimates of the solution of the linearized equation (i.e., with $f = 0$)

$$\begin{aligned} \dot{y} + A^2 y = \dot{W}, \text{ in } \Omega \times [0,T], \\ y(0) = y_0 \text{ in } \Omega \\ \frac{\partial y}{\partial n} = \frac{\partial \Delta y}{\partial y} = 0 \text{ on } \partial\Omega \times [0,T]. \end{aligned} \tag{2}$$

using the finite element method. Our main aim here is to derive the semi-discrete and fully discrete schemes and obtain the error estimates using the analytic semigroup properties while following the methods adopted in the works of [7], [10] and [11]. The outline of this paper is as follows: In section 2, we explore the theoretical framework within which we will be working. We will particularly look at the Hilbert space. In section 3, we discuss the semi-discrete scheme while fully discrete scheme and error estimates for the problem under review are discussed in sections 4.

2. Theoretical Framework

Let $H = L_2(\Omega)$ with inner product $(u, v) = \int_\Omega uvdx$ and corresponding norm $\|.\| = (.,.)^{\frac{1}{2}}$. Furthermore, let $A = -\Delta$ with domain $D(A) = H_0^1 \cap H^4$ where the spaces H^4 and H_0^1 are as defined below.
We define $H^s = H^s(\Omega)$ to be the space of all functions whose weak partial derivatives of order $\leq s$ belong to L_2, i.e.,

$$H^s = \{v \in L_2 : D^\alpha v \in L_2, |\alpha| \leq s\}$$

Furthermore, we define $H^1 = H^1(\Omega)$ as

$$H^1 = \left\{v \in H^1 : v = 0 \text{ on } \Gamma = \partial\Omega\right\}$$

Define the space $H^s(\Omega) = D(A^{\frac{s}{2}})$, with norm $|v|_s = \left\|A^{\frac{s}{2}} v\right\|$ for any $s \in R$ and the Parseval's relation as

$$|v|_s^2 = \left\|A^{\frac{s}{2}} v\right\|^2 = \sum_{j=1}^{\infty} \lambda_j^2 \hat{v}_j^2$$

where λ_j are eigenvalues of A and $\hat{v}_j = (v, \phi_j)$ with ϕ_j an orthonormal basis of corresponding eigenfunctions.

For any Hilbert space, H, we define

$$L_2(\Omega, H) = \left\{ v : E\left\|v\right\|_H^2 = \int_\Omega \left\|v(w)\right\|_H^2 dP(w) < \infty \right\}$$

with norm $\|v\|_{L_2(\Omega,H)} = (E\left\|v\right\|_H^2)^{\frac{1}{2}}$. Hence, let ψ be a Hibert-Schmidt operator from H to H with the space $HS(H,H)$, we say that $\psi(s) \in HS(H,H)$, if

$$\|\psi(s)\|_{HS} = \left(\sum_{j=1}^{\infty} \|\psi(s)\phi_j\|^2 \right)^{\frac{1}{2}} < \infty$$

where $H = L_2$ and $\{\phi_j\}$ is an arbitrary orthonormal basis for H.
If $\psi(s) \in HS(H,H)$, then the stochastic integral $\int_0^t \psi(s)dW(s)$ is well defined and we have the Ito Isometry

$$E\left\| \int_0^t \psi(s)dW(s) \right\|^2 = \int_0^t \left\|E\psi(s)\right\|_{HS}^2 dW(s)$$

where E stands for expectation.
We can write the Wiener process $W(s)$ with covariance operator Q in terms of its Fourier series as

$$W(t) = \sum_{i=1}^{\infty} \gamma_i^{\frac{1}{2}} \xi_i \beta_i(t)$$

Here, $\beta_i(t)$ is a sequence of real-valued independence identically distributed (iid) Brownian motions and $\{\gamma_j, e_j\}$ is the eigensystem for Q. The operator Q is self-adjoint, positive definite and linear. Moreover, Q is defined such that it has the same eigenfunctions as $A = -\Delta$. The relationship between the eigenvalues of Q and A is given by the equation

$$\gamma_j = \lambda_j^{-\infty}$$

where $\alpha \in \mathbb{R}$

3. Semi-discrete Scheme

With the definition of A and $D(A)$ we can write equation (2) as

$$y_t + A^2 y = dW,\ t > 0,\ y(0) = y_0 \tag{3}$$

having a mild solution of

$$\hat{y} = E(t)y_0 + \int_0^t E(t-s)d\hat{W}(s)$$

Let S_h be a family of finite element spaces, where S_h consists of continuous piecewise polynomials of degree $r \leq 2$ with respect to the triangulation T_h of Ω. We shall also assume that $\left\{S_h \subset H_0^1(\Omega)\right\}$. According to the standard finite element method, the semi-discrete problem of equation (3) is to find $y_h(t) \in S_h$, such that

$$y_{h,t} + A_h^2 y_h = dW,\ t > 0,\ y_h(0) = P_h y_0 \tag{4}$$

The mild solution of equation (4) is given as

$$\hat{y}_h = E_h(t) P_h y_0 + \int_0^t P_h E_h(t-s) d\hat{W}(s)$$

where the operator $A_h : \dot{S}_h \to \dot{S}_h$ (the discrete Laplacian).
The error bound in the semi-discretization scheme is as follows;

Theorem 3.1. [10] Let y_h be the spatially semi-discrete approximate solution of order r and with mesh parameter h, and let the initial approximation be chosen as the L_2-projection of the exact initial value y_0. Then if for $r \leq 2$ and $\left\| A^{\frac{(\gamma-1)}{2}} \right\|_{HS} < \infty$, for $\gamma \in [0, 4]$ we have

$$\|y_h(t) - y(t)\|_{L_2} \leq Ch^\gamma \left(\|y_0\|_{L_2(\Omega, H^\gamma)} + \left\| A^{\frac{(\gamma-1)}{2}} \right\|_{HS} \right), 0 \leq t \leq T$$

4. Main Results

4.1. **Fully Discrete Approximation.** We now formulate the fully discrete approximation of equation (2) based on the backward Euler method in time. Here we replace the time derivative by a backward difference quotient
$\partial_t Y_h = \left(\frac{Y^n - Y^{n-1}}{k} \right)$ where k is the time step and Y^n is the approximation to y at time $t_n = nk,\ n = 1, 2, \ldots$.
For equation (2), we pose the fully discrete approximation problem as follows: Find $y_h \in Y_h$ such that

$$y_{h,t} + A_h^2 y_h = dW,\ t > 0,\ y_h(0) = P_h y_0$$

and applying the implicit Euler method, for $k = \Delta t$, $t_n = nt$, $\Delta W^n = W(t_n) - W(t_{n-1})$ we have for $Y^n \in S_h$, $Y^0 = P_h y_0$;

$$\left(\frac{Y^n - Y^{n-1}}{k} \right) + A_h^2 Y^n = P_h \left(\frac{\hat{W}(t_n) - \hat{W}(t_{n-1})}{k} \right),\ t_n > 0 \tag{5}$$

$$\implies Y^n - Y^{n-1} + kA_h^2 Y^n = P_h \Delta \hat{W}^n$$

$$Y^n - Y^{n-1} + kA_h^2 Y^n = P_h \left(\hat{W}(t_n) - \hat{W}(t_{n-1}) \right) \tag{6}$$

and the variation of constants formula for

$$Y(t_n) = E(t_n) Y^0 - \int_0^{t_n} E(t_n - s) P_h dW(s)$$

becomes

$$Y^n = E_{kh}Y^{n-1} - E_{kh}P_h\Delta W^n$$

$$Y^n = E_{kh}P_h y_0 - \sum_{j=1}^{n} E_{kh}^{n-j+1}P_h\Delta W^j \tag{7}$$

where $E_{kh} = \left(1 + kA_h^2\right)^{-1}$.

4.2. **Error Estimates.** We now recall the following estimate from [7] which we shall call a lemma.

Lemma 4.1. Let $B_n(t) = E_{kh}^n P_h - E_n(t)$, then for $0 \le \gamma \le 4$, $\|B_n v\| \le C\left(k^{\frac{\gamma}{2}+h^\gamma}\right)|v|_{\gamma-1}$ and

$$\left(k\sum_{j=1}^{n}\|B_j v\|^2\right)^{\frac{1}{2}} \|B_n v\|_{L_2([0,T],H)} \le C\left(k^{\frac{\gamma}{2}+h^\gamma}\right)|v|_{\gamma-1}$$

where $|v|_\gamma = \left\|A^{\frac{\gamma}{2e}}v\right\|$ for $\gamma \in \mathbb{R}$.

The error estimate for the fully discrete approximation is given below.

Theorem 4.1. Let y be the solution of equation (2) and the solution of equation (6). If $\left\|A^{\frac{\gamma-1}{2}}\right\|_{HS} < \infty$ for some $0 \le \gamma \le 4$, then

$$\|e_n\|_{L_2(\Omega,H)} = \|Y^n - y(t_n)\| \le C\left(k^{\frac{\gamma}{2}} + h^\gamma\right)\left(\|y_0\|_{L_2(\Omega,H)} + \left\|A^{\frac{\gamma-1}{2}}\right\|_{HS}\right) \tag{8}$$

If $W(t)$ is a Wiener process with covariance operator $Q = I$, we have

$$\|e_n\|_{L_2(\Omega,H)} \le C\left(k^{\frac{\gamma}{2}} + h^\gamma\right)\left(1 + \|y_0\|_{L_2(\Omega,H)}\right) \text{ for } 0 \le \gamma \le 2 \tag{9}$$

Proof

Let $e_n = Y^n - y(t_n)$ and $B_n(t) = E_{kh}^n P_h - E(t_n)$
where $Y^n = E_{kh}^n P_h y_0 + \sum_{j=1}^n \int E_{kh}^{n-j+1}P_h d\hat{W}(s)$ with $E_{kh}^n = r\left(kA_h^2\right)^n$ and $y(t_n) = E(t_n)y_0 + \int_0^{t_n} E(t_n - s)d\hat{W}(s)$, then

$$e_n = E_{kh}^n P_h y_0 + \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} E_{kh}^{n-j+1}P_h d\hat{W}(s) - E(t_n)y_0 + \int_0^{t_n} E(t_n - s)d\hat{W}(s)$$

$$e_n = \underbrace{E_{kh}^n P_h y_0 - E(t_n)y_0}_{I} + \underbrace{\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} B_{n-j+1} d\hat{W}(s)}_{II} + \underbrace{\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \left(E(t_n - t_{j-1}) - E(t_n - s)\right) d\hat{W}(s)}_{III}$$

Thus,

$$\|e_n\| \leq C\left(\|I\| + \|II\| + \|III\|\right)$$

For I, we have from Lemma 4.1 where $v = y_0$

$$\begin{aligned}\|I\| = \|B_n v\| &\leq C\left(k^{\frac{\gamma}{2}} + h^{\gamma}\right)|y_0|_{\gamma}\\ &\leq C\left(k^{\frac{\gamma}{2}} + h^{\gamma}\right)\|y_0\|_{L_2(\Omega, H^{\gamma})}\end{aligned}$$

For II, we have, by the Ito Isometry property

$$\begin{aligned}E\,\|II\|_{L_2(\Omega,H)}^2 &= E\left\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} B_{n-j+1} d\hat{W}(s)\right\|^2\\ &\leq \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \left\|B_{n-j+1}\right\|^2 ds\\ &= \sum_{j=1}^{\infty}\left(k\sum_{j=1}^{n}\|B_{n-j+1}\phi_i\|_{HS}^2\right)\\ &= k\sum_{j=1}^{\infty}\sum_{j=1}^{n}\|B_{n-j+1}\phi_i\|_{HS}^2,\ \text{where } \{\phi_i\} \text{ is as in section (2).}\\ &\leq C\sum_{i=1}^{\infty}\left(k^{\frac{\gamma}{2}} + h^{\gamma}\right)^2 |\phi_i|_{\gamma-1}^2\\ &\leq C\left(k^{\gamma} + h^{2\gamma}\right)\sum_{i=1}^{\infty}|\phi_i|_{\gamma-1}^2\\ &= C\left(k^{\gamma} + h^{2\gamma}\right)\sum_{i=1}^{\infty}\left\|A^{(\gamma-1)/2}\phi_i\right\|^2 \text{ by Parseval's relation}\\ &= C\left(k^{\gamma} + h^{2\gamma}\right)\left\|A^{(\gamma-1)/2}\phi_i\right\|_{L_2}^2\end{aligned}$$

For III, we have, by the Ito Isometry property again

$$
\begin{aligned}
E\left\|III\right\|_{L_2}^2 &= E\left(\left\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left(E(t_n-t_{j-1})-E(t_n-s)\right)d\hat{W}(s)\right\|\right)^2\\
&\le \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left\|E(t_n-t_{j-1})-E(t_n-s)\right\|_{HS}^2 ds\\
&= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left\|E(t_n-s)E(s-t_{j-1})-I\right\|_{HS}^2 ds\\
&= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left\|A^{\gamma/2}E(t_n-s)A^{-\gamma/2}(I-E(s-t_{j-1}))\right\|_{HS}^2 ds\\
&\le \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left\|A^{\gamma/2}E(t_n-s)\right\|_{HS}^2\left\|A^{-\gamma/2}(I-E(s-t_{j-1}))\right\|^2 ds\\
&\le Ck^{\gamma}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left\|A^{\gamma/2}E(t_n-s)\right\|_{HS}^2 ds\\
&\le Ck^{\gamma}\left\|A^{(\gamma-1)/2}\right\|_{HS}^2\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left\|A^{1/2}E(t_n-s)\right\|^2 ds\\
&\le Ck^{2\gamma}\left\|A^{(\gamma-1)/2}\right\|^2
\end{aligned}
$$

which concludes the proof.

5. Conclusion

The strong convergence rate in both the spatial and time steps can be computed. This can be done if the finite element solution computed on a very fine mesh is considered as the true solution and the finite element solutions computed on the less fine meshes are compared with this numerically obtained true solution. This is due to the fact that the true solution to the SPDE (2) itself is a random process and is not known explicitly.

6. Analysis of Strong Convergence rate in k and h

The main purpose of the numerical experiment is to examine the convergence rate of the numerical method. The numerical experiment is performed on equation (2) with the following functions:
$T = 1$, $\sigma \equiv I$, $f(x) = 0$, $y_0(x) = \cos x$ where $x = (x_1, x_2) \in \Omega$, Ω is the unit square $\mathbf{R}^2$.

In the numerical experiment, the strong convergence rate in both the spatial and time steps in equation (2) are computed. For the experimental setup for the strong convergence rate k, we first compute the true solution y on the mesh where $h = 2^{-8}$ and $k = 2^{-8}$, which we consider as a fine mesh due to the lengthy run time of the solver. Then, we fix $h = 2^{-8}$ and compute the approximated solution Y^k for different time partitions, in particular, for $k = 2^{-7}, 2^{-6}, 2^{-5}, 2^{-4}, 2^{-3}$ respectively.

Finally, we compute the $\|Y^k - y\|_{L_2(\Omega,H)}$ for every time partition. To do this, we show that Theorem 4.1 implies that the order of strong convergence of our method should be around $O(k^{\frac{\gamma}{2}} + h^\gamma)$. If h is sufficiently small, such that the error estimates are dominated by k, the predicted rate of convergence should be $O(k^{\frac{\gamma}{2}})$. This gives us (see [9])

$$\frac{U^k}{U^{k+1}} \approx \left(\frac{k_i}{k_{i+1}}\right)^{\frac{\gamma}{2}} = 2^{\frac{\gamma}{2}} \tag{10}$$

and from that we obtain

$$\gamma = \frac{2}{\log 2} \log\left(\frac{U^k}{U^{k+1}}\right) \tag{11}$$

In the same way, when k is very small, the error is assumed to be dominated by h and the rate of convergence should be $O(h^\gamma)$. Similarly to (11), we obtain,

$$\gamma = \frac{2}{\log 2} \log\left(\frac{U^h}{U^{h+1}}\right) \tag{12}$$

Therefore, using (10) and (11), we obtain the results for γ as shown in table (6.1)

h	k	γ	k	h	γ
2^{-8}	2^{-7}	0.7658	2^{-8}	2^{-7}	0.8109
2^{-8}	2^{-6}	0.5624	2^{-8}	2^{-6}	0.4075
2^{-8}	2^{-5}	0.4456	2^{-8}	2^{-5}	0.3816
2^{-8}	2^{-4}	0.7658	2^{-8}	2^{-4}	0.4397

Table 6.1: Convergence rate in k and h.

The table shows that the average of the γ is around the expected value of $\frac{1}{2}$, which indicates that the solution for the numerical estimate of the strong convergence rate is effective.

References

[1] Novich-Cohen A. and Segel L. A. (1984); Nonlinear aspects of the Cahn-Hilliard equation, Physics D10: 277-298.

[2] Da Prato G. and Zabezyk J. (1992); Stochastic Equations in Infinite Dimensions. In: Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge.

[3] Debussche A. and Zambotti L. (2006); Conservative stochastic Cahn-Hilliard equation with reflection, arXiv: math.PR/0601313v1.

[4] Elliot C. M. and Larsson S (1992); Error Estimates with Smooth and Non smooth Data for a Finite Element Method for the Cahn-Hilliard Equation, Math. Comp. 58: 603-630.

[5] Cardon-Weber C. (2000); Implicit approximation scheme for the Cahn-Hilliard stochastic equation, Prepublication du Laboratoire de probabilities et modeles Aleatoires, 613.

[6] Cardon-Weber C. (2001); Cahn-Hilliard Stochastic Equation: Existence of the Solution and of its density, Bernoulli Soc. for Math. Stat. Prob. Vol. 7, No. 5.

[7] Yubin Yan (2003a); The Finite Element Method for a Linear Stochastic Parabolic Partial Differential Equation driven by Additive Noise, Chalmers Finite Element Center.

[8] Allen, E. J., Novosel, S. J. and Zhang, Z. (1998); Finite Element and Difference Approximation of Some Linear Stochastic Partial Differential Equations, Stochastics Rep. 64: 117-142

[9] Bin Li (2004); Numerical Method for a Parabolic Stochastic Partial Differential Equation, International Master's Programme in Engineering Mathematics, Dept. of Mathematics, Chalmers University of Technology, Preprint No. 7.

[10] Njoseh I. N. and Ayoola E. O. (2008); On the finite element analysis of the stochastic Cahn-Hilliard equation, Jour. Of Inst. Of Math & Comp. Sci. (Math. Ser) Vol. 21. No. 1: 47 -53

[11] Yubin Yan (2003b); A Finite Element Method for a Non-Linear Stochastic Parabolic Equation, Chalmers Finite Element Center, Chalmers University of Technology, Preprint No. 7.

[12] Duan Y. and Yang X. (2013). The Finite Element Method of a Euler Scheme for Stochastic Navier-Stokes equations involving the turbulent components. International Journal of Numerical Analysis and Modelling. Vol. 10, No. 3: 727-744.

[13] Barth A. (2010). A Finite Element Method for Martingale-Driven Stochastic Partial Differential Equations. Communications on Stochastic Analysis. Vo.l 4, No. 3: 355-375.

[14] Larsson S. and Mesforush A. (2011). Finite Element Approximations of the Linearized Cahn-Hilliard-Cook equation. IMA Journal of Numerical Analysis. Vol. 31. No. 4: 1315-1333.