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ON SPECIAL CASES OF OPIAL'S AND HARDY'S INEQUALITIES

[1]Anthonio Yisa Oluwatoyin, [1]Rauf Kamilu, [1]Ajisope Michael Oyelami and [2]Wahab Olalekan Taofeek

Abstract

In this paper, we establish the relationship between Opial-type and Hardy-type integral inequalities which extend Anthonio and Rauf Opial-type inequalities for convex function.

1. Introduction

The following interesting classical integral inequalities were stated and proved by Opial and Hardy respectively:

Theorem 1.1. *Let $x(t) \in C'[0,b]$ be such $x(0) = x(b) = 0$ and $x(t) > 0$ in (0, b), then*

$$\int_a^b |x(t)x'(t)|dt \leq \frac{b}{4}\int_a^b (x'(t))^2 dt \tag{1}$$

where $\frac{b}{4}$ in the best possible constant. (See [9])

Theorem 1.2. *For $f(x) \geq 0$ and $p > 1$,*

$$\int_0^\infty \left[\frac{1}{x}f(t)dt\right]^p dx \leq q^p \int_0^\infty f^p(t)dt \tag{2}$$

where $q = \frac{p}{p-1}$ is the best possible constant. (see [5])

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[1]Department of Mathematics, University of Ilorin, Ilorin, Nigeria; krauf@unilorin.edu.ng
[2]Department of Statistics and Mathematical Sciences, Kwara State University, Malete.

In view of the usefulness of these inequalities in analysis and its applications generally, many authors have established the necessary and sufficient conditions on p, q, v, w for the Hardy-type inequality:

$$\left[\int_a^b |u(x)|^q w(x)dx\right]^{\frac{1}{q}} \leq C\left[\int_a^b |u'(x)|^p v(x)dx\right]^{\frac{1}{p}} \tag{3}$$

to hold, where C is a constant depending on p and q. See [7] and the reference therein.

Theorem 1.3. *Let g be continuous and non-decreasing on $[a, b], 0 \leq a \leq b < \infty$, with $g(x) > 0$ for $x > 0$. Let $q \geq p \geq 1$ and $f(x)$ be non-negative and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a, b]$. Suppose δ is a real number such that $-\frac{p}{q} < \delta < 0$ then*

(4)

$$\left[\int_a^b g(x)^{\frac{\delta q}{p}}\left(\int_a^x f(t)dg(t)\right)^q dg(x)\right]^{\frac{1}{q}} \leq C(a,b,p,q,\delta)\left[\int_a^b g(x)^{(p-1)(1+\delta)} f(x)^p dg(x)\right]^{\frac{1}{p}}$$

where

(5)

$$C(a,b,p,q,\delta) = (-\delta)^{\frac{q(1-p)}{p}}\left(\frac{p}{q\delta+p}\right)^{\frac{p}{q}} g(b)^{\frac{q\delta+p}{p}}\left(g(b)^{-\delta} - g(a)^{-\delta}\right)^{\frac{q(p-1)}{p}} > 0$$

(See [2])

It was pointed out by Oguntuase (2009) that the constant $C(a, b, p, q, \delta)$ at the right hand side of (4) is wrong and stated the following:

Theorem 1.4. *Let g be a continuous and nondecreasing function on $[a, b], 0 \leq a < b < \infty$, with $g(x) > 0$ for $x > 0$. Let $q \geq p \geq 1$ and let $f(x)$ be nonnegative and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a, b]$. Suppose δ is a real number such that $-\frac{p}{q} < \delta < 0$ then,*

(6)

$$\left[\int_a^b g(x)^{\frac{q\delta}{p}}\left(\int_a^x f(t)dg(t)\right)^q dg(x)\right]^{\frac{1}{q}} \leq C(a,b,p,q,\delta)\left[\int_a^b g(x)^{(p-1)(1+\delta)} f^p(x)dg(x)\right]^{\frac{1}{p}}$$

where

$$C(a,b,p,q,\delta) = (-\delta)^{\frac{1-p}{p}}\left(\frac{p}{q\delta+p}\right)^{\frac{1}{q}}\left(g(b)^{-\delta} - g(a)^{-\delta}\right)^{\frac{p-1}{p}}\left(g(b)^{\frac{q\delta+p}{p}} - g(a)^{\frac{q\delta+p}{p}}\right)^{\frac{1}{q}}$$

(See [8])

Some special cases of the result were obtained. The purpose of the work is to extend the work of Anthonio and Rauf (2015) with a view to obtain the relationship between Opial-type and Hardy-type classical inequalities.

2. Main Results

Throughout this paper, we shall define: $h(x,t) = g(x)^{\zeta} f(t)^p g(t)^{p(1+\zeta)} d\mu(t)$ and $d\mu(t) = g(t)^{-(1+\zeta)} dg(t)$

The statement of the main results are as follows:

Lemma 2.1. *Let $h(x,t)$ be non negative, $x \geq 0, t \geq 0$ and $\mu \geq 0$ be non decreasing.*
Let$-\infty \leq 0 \leq x < \infty$, then the following holds:

$$\left[\int_a^x h(x,t)d\mu(t)\right]^{\frac{\nu}{p}} = g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t)^p g(x)^{(p-1)(1+\zeta)} dg(t)\right]^{\frac{\nu}{p}} \tag{7}$$

Proof:
Let $f(t)$ and $g(x)$ are absolutely continuous functions.

$$\left[\int_a^x h(x,t)d\mu(t)\right]^{\frac{\nu}{p}} = \left[\int_a^x g(x)^{\zeta} f(t)^p g(t)^{(p-1)} g(t)^{(1+\zeta)} dg(t)\right]^{\frac{\nu}{p}} \tag{8}$$

$$\left[\int_a^x h(x,t)d\mu(t)\right]^{\frac{\nu}{p}} = g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t)^p g(t)^{(p-1)(1+\zeta)} dg(t)\right]^{\frac{\nu}{p}} \tag{9}$$

which complete the proof of the Lemma.

Lemma 2.2. *Let $h(x,t)$ be non negative, $t \geq 0, x \geq 0$, $p \geq 1$ and $\mu \geq 0$ be non decreasing.*
Let$-\infty \leq 0 \leq x < \infty$, then the following holds:

$$\left[\int_a^x d\mu(t)\right]^{\frac{(1-p)\nu}{p}} = \left[\frac{g(t)^{-\zeta}}{-\zeta}\Bigg|_a^x\right]^{\frac{(1-p)\nu}{p}} = [g(x) - g(a)]^{\frac{(p-1)\zeta\nu}{p}} (-\zeta)^{-\frac{(p-1)\nu}{p}} \tag{10}$$

Proof:

$$\left[\int_a^x d\mu(t)\right]^{\frac{(1-p)\nu}{p}} = \left[\int_a^x g(t)^{-(1+\zeta)} dg(t)\right]^{\frac{(1-p)\nu}{p}} = \left[\frac{g(t)^{(-1-\zeta+1)}}{-1-\zeta+1}\Bigg|_a^x\right]^{\frac{(1-p)\nu}{p}}$$

$$= \left[\frac{g(t)^{-\zeta}}{-\zeta}\Bigg|_a^x\right]^{\frac{(1-p)\nu}{p}} = \left[g(x)^{\zeta} - g(a)^{\zeta}\right]^{\frac{(p-1)\nu}{p}} (-\zeta)^{-\frac{(p-1)\nu}{p}} \tag{11}$$

This completes the proof of the Lemma.

Lemma 2.3. *Suppose all the conditions of Lemma 2.2 hold, then we have:*

$$\left[\int_a^x h(x,t)^{\frac{1}{p}} d\mu(t)\right]^\nu = g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t)dg(t)\right]^\nu \tag{12}$$

Proof:

$$\left[\int_a^x h(x,t)^{\frac{1}{p}} d\mu(t)\right]^\nu = \left[\int_a^x \left(g(x)^\zeta f(t)^p g(t)^{p(1+\zeta)}\right)^{\frac{1}{p}} g(x)^{-\zeta+1} dg(x)\right]^\nu$$

$$= \left[\int_a^x \left(g(x)^{\frac{\zeta}{p}} f(t) g(t)^{(1+\zeta)}\right) g(x)^{-\zeta+1} dg(x)\right]^\nu$$

$$= \left[\int_a^x g(x)^{\frac{\zeta}{p}} f(t) g(t)^{(1+\zeta)} g(x)^{-\zeta+1} dg(x)\right]^\nu$$

$$= \left[\int_a^x g(x)^{\frac{\zeta}{p}} f(t) dg(x)\right]^\nu$$

$$\Rightarrow \left[\int_a^x h(x,t)^{\frac{1}{p}} d\mu(t)\right]^\nu \geq g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t)dg(t)\right]^\nu \tag{13}$$

The proof is completed.
Using the well known Jensen's inequality of the form: (see [3] and [4])

$$\int_a^x h(x,t)d\mu(s) \geq \left[\int_a^x d\mu(s)\right]^{1-p} \left[\int_a^x h(x,t)^{\frac{1}{p}} d\mu(s)\right]^p \tag{14}$$

Raising both sides of inequality (15) to power $\frac{\nu}{p}$ yields

$$\left[\int_a^x h(x,t)d\mu(s)\right]^{\frac{\nu}{p}} \geq \left[\int_a^x d\mu(s)\right]^{\frac{(1-p)\nu}{p}} \left[\int_a^x h(x,t)^{\frac{1}{p}} d\mu(s)\right]^\nu \tag{15}$$

Theorem 2.4. *Let $f(t)$ and $g(t)$ be a absolutely continuous function which is non-decreasing on $[a,b], 0 \leq a \leq b < \infty$. Suppose that $p \geq \nu \geq 1, \zeta > 0$ and $f(x)$ is Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a,b]$. Then,*

$$\left[\int_a^x f(t)dg(t)\right]^\nu \leq (-\zeta)^{\frac{(1-p)\nu}{p}} [g(x)-g(a)]^{\frac{(1-p)\zeta\nu}{p}} \left[\int_a^x f(t)^p g(t)^{(p-1)(1+\zeta)} dg(t)\right]^{\frac{\nu}{p}} \tag{16}$$

Proof :

Multiply both side of (16) with $g(x)^{\frac{\zeta\nu}{p}}$ and by combining the results of Lemma 2.1, 2.2 and 2.3 in inequality (17), we get

$$g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t)^p g(t)^{(p-1)(1+\zeta)} dg(t)\right]^{\frac{\nu}{p}} \geq (-\zeta)^{\frac{(p-1)\nu}{p}} [g(x)-g(a)]^{\frac{(p-1)\zeta\nu}{p}} g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^b f(t)dg(t)\right]^\nu \tag{17}$$

that is
(18)

$$(-\zeta)^{\frac{(p-1)\nu}{p}}[g(x)-g(a)]^{\frac{(p-1)\zeta\nu}{p}}g(x)^{\frac{\zeta\nu}{p}}\left[\int_a^x f(t)dg(t)\right]^{\nu}\leq g(x)^{\frac{\zeta\nu}{p}}\left[\int_a^x f(t)^p g(t)^{(p-1)(1+\zeta)}dg(t)\right]^{\frac{\nu}{p}}$$

which implies
(19)

$$g(x)^{\frac{\zeta\nu}{p}}\left[\int_a^x f(t)dg(t)\right]^{\nu}\leq (-\zeta)^{\frac{(1-p)\nu}{p}}[g(x)-g(a)]^{\frac{(1-p)\zeta\nu}{p}}g(x)^{\frac{\zeta\nu}{p}}\left[\int_a^x f(t)^p g(t)^{(p-1)(1+\zeta)}dg(t)\right]^{\frac{\nu}{p}}$$

Integrating both sides of (20) with respect to $g(x)$ on $[a,b]$ and then raising both sides to power $\frac{p}{\nu}$ to obtain the following inequality:

$$\left[\int_a^b g(x)^{\frac{\zeta\nu}{p}}\left[\int_a^x f(t)dg(t)\right]^{\nu}dg(x)\right]^{\frac{p}{\nu}}\leq(-\zeta)^{\frac{(1-p)\nu}{p}}\left[\int_a^b [g(x)-g(a)]^{\frac{(1-p)\zeta\nu}{p}}g(x)^{\frac{\zeta\nu}{p}}\right.$$

$$\left.\times\left[\int_a^x f(t)^p g(x)^{(p-1)(1+\zeta)}dg(t)\right]^{\frac{\nu}{p}}dg(x)\right]^{\frac{p}{\nu}} \tag{20}$$

which is the type of the result in [2] and [8] generalization.
This has successfully suggested that Hardy-type and Opial-type of the two classical inequalities can be found in Jensen's inequality for convex function.

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